Calculus 12

Guide

NOVA SCOTIA
NOUVELLE-ÉCOSSE

2004

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Calculus Guide 12

Prepared by the Department of Education and Early Childhood Development

This is the most recent version of the current curriculum materials as used by teachers in Nova Scotia.

Atlantic Canada Mathematics Curriculum: Calculus 12

DRAFT 2004

Section 1

LIMITS AND CONTINUITY

15-20 Hours

Outcomes

SCO: By the end of this course, students will be expected to

B1 calculate and interpret average and instantaneous rate of change.

Elaboration – Instructional Strategies/Suggestions

B1 In a previous course, students calculated average rate of change for a variety of situations. Students should be able to look at a graph and determine how fast the dependent variable is changing with respect to the independent variable. Using the function $y = f(x)$, students should interpret average speed as an average of *y* with respect to *x*. This would mean the change in *y* per unit change in *x*.

Average rate of change of *y* with respect to *x* is:

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$$

The average rate of change over *y* with respect to *x* over the interval *a* to *b* is:

$$\frac{f(b) - f(a)}{b - a}$$

In a previous course, students calculated instantaneous rate of change using secants where the distance between the endpoints approached zero.

Use the definition of average rate of change to calculate the instantaneous rate of change of $f(x)$ at $x = a$, using the interval *a* to $a + h$.

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{f(a+h) - f(a)}{h}$$

Stress to students that instantaneous speed is a limit.

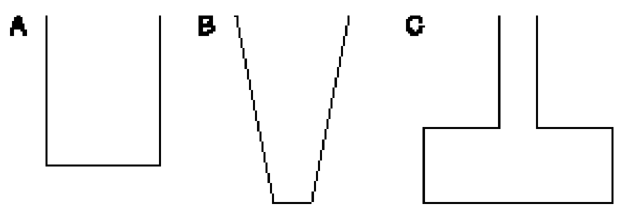
Students also used graphing technology in a previous course to find the instantaneous rate of change. A graphing calculator or computer program can be used to check results and examine instantaneous rates of change over several spots on a curve.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B1**)

1. A tennis ball is hit vertically into the air. The ball's height *h* in metres, *t* seconds after it is hit can be found by using the formula $h = -4.9t^2 + 32t + 1$.
   a. Find the average velocity of the ball from the time it is hit by the racket until it hits the ground.
   b. Find the average velocity of the ball between 3 and 4 seconds.
   c. Find the instantaneous velocity of the ball at 2 seconds.
   d. When is the instantaneous velocity of the ball equal to zero? Explain how you know this.
   e. When is the ball moving the fastest? Explain how you know this.

2. A container is filled from a tap running at a constant rate. Sketch the graphs for each container that shows the height of the water in the container over time.



3. Assuming that the containers in question 2 were already filled and started to leak out of the bottom at a constant rate.
   a. Sketch the graphs for each container that shows the height of the water in the container over time.
   b. Describe how the water height-time graphs compare in this question and the previous question.

Suggested Resources

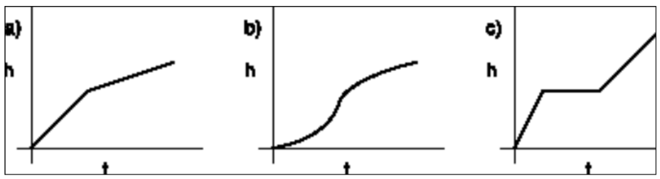
Section 2.1

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to **B1** calculate and interpret average and instantaneous rate of change.	At this point it is important to take the time to ask students to look at graphs with very little description and to interpret what they are seeing. To discuss what the various slopes mean. It may be possible to get students to take a graph and write a little story about what they see happening in the picture at various points along the graph. Also ensure that the students are exposed to a variety real world examples and that they have enough practice to be able to interpret the various situations. Pay close attention to the units of the graphs. Students need to be able to express the rates of change with the appropriate units attached.

Worthwhile Tasks for Instruction and/or Assessment

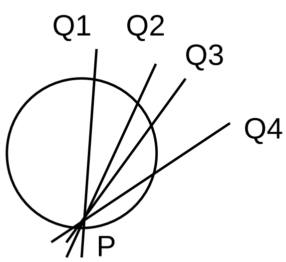
Performance (**B1**)

4. The following are water height-time graphs for three containers. Sketch the possible shape of each container.

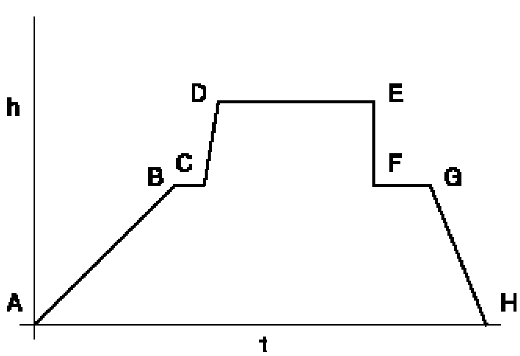


Journal (**B1**)

5. This diagram shows a circle and a fixed point P on the circle. Lines PQ are drawn from P to points Q on the circle and are extended in both directions. Such lines across a circle are called secants, and some examples are shown in the diagram. As Q gets closer and closer to P, explain what happening to the secant?



6. Below is a graph of the water height in a bathtub versus time. Write a story that explains what is going on in the tub. Make sure that all the points labeled are discussed and identified in your story.



Suggested Resources

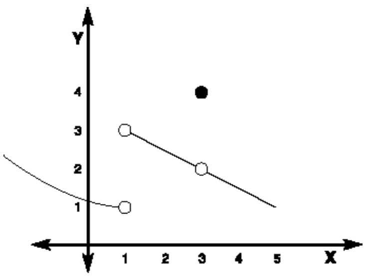
Section 2.1

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **B2** calculate limits for function values and apply limit properties with and without technology.	**B2** In a previous course, students were introduced to limits using technology and algebraic manipulation. Students are expected to understand and be able to calculate limits using the properties of limits. They are also expected to be able to deal with one-sided and two-sided limits. An introduction to this topic that would be very useful for the students' understanding of a limit is to look at the following definition of a limit: A limit is the specific number that $\frac{\Delta y}{\Delta x}$ approaches as Δx approaches zero. In other words, find $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Use this definition and a table of values to find the limit of a simple function, like, $y = x^3$ at $x = 1$. Use graphing technology as it was used in example 4 (p. 59) of the text to explore limits to help further the students understanding. Students will know the properties of limits and be able to calculate limits algebraically using those properties. One and two sided limits: Limits can be approached from both sides of the x value that you are looking at. In many cases, the left hand limit and the right hand limit will be the same. If they are the same, then the limit exists. If they are not the same then the limit does not exist. That is, if $\lim_{\Delta x \to a^-} f(x) = L$ and $\lim_{\Delta x \to a^+} f(x) = L$, then $\lim_{\Delta x \to a} f(x) = L$ Remember to show students that sometimes limits don't exist. Make sure that students understand question 63 (p. 64), it will be important later on. Teachers may want to develop $\lim_{\Delta x \to 0} \frac{\sin x}{x}$ and $\lim_{\Delta x \to 0} \frac{1 - \cos x}{x}$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B2**)

1. The graph of the function f is below. Use it to state the values, if they exist, of the following:



a. $\lim_{x\to 1^-} f(x)$ b. $\lim_{x\to 1^+} f(x)$ c. $\lim_{x\to 1} f(x)$

d. $\lim_{x\to 3^-} f(x)$ e. $\lim_{x\to 3^+} f(x)$ f. $\lim_{x\to 3} f(x)$

2. Evaluate the limits if they exist:

a. $\lim_{x\to 5}\left[x^3(x-1)\right]$

b. $\lim_{x\to -2}\dfrac{x^2+3x-2}{x+1}$

c. $\lim_{x\to -3^+}\dfrac{x^2-x-12}{x+3}$

d. $\lim_{x\to -3^-}\dfrac{x^2-x-12}{x+3}$

e. $\lim_{x\to -3}\dfrac{x^2-x-12}{x+3}$

f. $\lim_{x\to -2}\dfrac{\frac{1}{2}+\frac{1}{x}}{x+2}$

g. $\lim_{x\to 2}\dfrac{x}{\sqrt{x-2}}$

h. $\lim_{x\to -3}\dfrac{\sqrt{x+3}}{\sqrt{x}-2}$

Suggested Resources

Section 2.1

Outcomes	Elaboration – Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B2**)

3. Let $f(x)=\begin{cases}(x+1)^2 & x<-1\\ x & \text{if } -1\le x\le 1\\ 2x-x^2 & x>1\end{cases}$

   a. Find the following limits, if they exist:

      i. $\lim_{x\to -1^-} f(x)$ ii. $\lim_{x\to -1^+} f(x)$ iii. $\lim_{x\to -1} f(x)$

      iv. $\lim_{x\to 1^-} f(x)$ v. $\lim_{x\to 1^+} f(x)$ vi. $\lim_{x\to 1} f(x)$

   b. Sketch the graph of *f*(*x*).

   c. Where is *f*(*x*) discontinuous?

   d. Which part(s) of piecewise function is(are) decreasing?

Journal (**B2**)

4. Explain how both a left-hand and right-hand limit can exist, but the limit at that point may not.

Suggested Resources

Section 2.1

Outcomes	Elaboration – Instructional Strategies/Suggestions

Outcomes

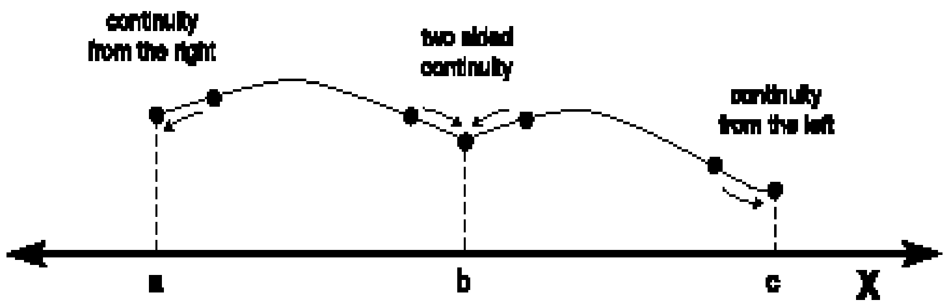
SCO: By the end of this course, students will be expected to

C1 identify the intervals upon which a given function is continuous and understand the meaning of a continuous function

Elaboration – Instructional Strategies/Suggestions

C1 Students should understand that a continuous function is a function that is continuous at every point of its domain. That is, it is a function in which the curve does not have a break over the domain. A discontinuous function has at least one break in it. A simplistic way to demonstrate this concept is to draw a curve on the overhead from point *a* to point *b* without lifting the marker up. To demonstrate a discontinuous curve from the same points, draw it again, but lifting at one point, or by erasing a point in the curve.

Students need to understand the difference between continuity at an interior point, and at an endpoint.



A function is discontinuous at point *c*, if it is not continuous there. Point *c* is a point of discontinuity.

In previous courses students discovered the domains of polynomial, rational, absolute value, exponential, logarithmic, trigonometric, and radical functions. Time should be spent on a brief review of the domains of these types of functions. Students should understand that all of these functions are continuous on their domains.

Worthwhile Tasks for Instruction and/or Assessment

Journal (**C1**)

1. Have students explain their understanding of continuity.

2. Distinguish between continuity of an interior point in the domain, and continuity of an endpoint.

3. What do you understand by "$f(x)$ is discontinuous at $x = c$".

Suggested Resources

Section 2.3

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to **B3** remove removable discontinuities by extending or modifying a function **C1** identify the intervals upon which a given function is continuous and understand the meaning of a continuous function.	**B3/C1** Students are responsible for the identifying the four types of discontinuities from the classification scheme used by the text. They are: 1) removable discontinuity; 2) jump discontinuity; 3) infinite discontinuity; and 4) oscillating discontinuity. The textbook (p. 76) shows examples of each type of discontinuity. Students are responsible to remove reasonable discontinuities. There are two main ways to accomplish this: 1) Look at the rational function $f(x)=\frac{x^2-5x+6}{x-3}$. It has a point of discontinuity at $x=3$. This function factors to be $f(x)=\frac{(x-3)(x-2)}{x-3}$. By removing the $x-3$ in each part, the function becomes $f(x)=x-2$ which is continuous everywhere. The limit as *x* goes to 3 in the new *f(x)* is 1. 2) Look at the function $f(x)=\begin{cases}3x, & x\neq 1\\ 2, & x=1\end{cases}$. At $x=1$, the function has a removable discontinuity. This can be removed and replaced with $\lim_{x\to 1}3x$, which is 3.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B3/C1**)

1. Give a formula for the extended function that is continuous at the indicated point:

   a. $f(x) = \dfrac{x^2 - 16}{x - 4}, x = 4$

   b. $f(x) = \dfrac{x^3 - 2x^2 - 5x + 6}{x^2 - 9}, x = 3$

2. For each of the following functions, find each point of discontinuity. Which of the discontinuities are removable? Which are not? Explain.

   a. $f(x) = \begin{cases} -1 + x & x < 3 \\ x + 5 & x > 3 \end{cases}$

   b. $f(x) = \begin{cases} x^2 & x < 1 \\ 2 & x = 1 \\ x & x > 1 \end{cases}$

3. Sketch a possible graph of a function f that has the stated properties:

   a. *f*(5) exists, but $\lim_{x \to 5} f(x)$ does not.

   b. *f*(-2) exists, $\lim_{x \to -2} f(x)$ exists, but f is not continuous atv $x = -2$.

Journal (**B3/C1**)

4. Have students explain the different types of discontinuity in words and with a sketch.

Suggested Resources

Section 2.3

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to **B4** apply the properties of algebraic combinations and composites of continuous functions.	**B4** a) Students should understand that algebraic combinations of continuous functions are continuous at that same point. They should see that sums, differences, products, constant multipliers, and quotients (provided that the denominator $\neq 0$) of continuous functions at a point *c* are also continuous at *c*. For example: If $f(x) = \sin x$ is continuous at $x = c$, and $g(x) = \tan x$ is continuous at $x = c$, then $f(x) \cdot g(x)$ is also continuous at $x = c$. b) Students will be able to apply the property of the composite of continuous functions to show continuity of functions. The theorem states that, if *f* is continuous at *c* and *g* is continuous at *f(c)*, then the composite $g \circ f$ is continuous at *c*. For example: Verify that $y = \lvert x^2 + 3x \rvert$ is continuous, and find its domain. A look at a graphing utility suggests that this function is continuous. If it is expressed as a composition of two functions, where: $g(x) = \lvert x \rvert$ and $f(x) = x^2 + 3x$ we see that y is the composite $g \circ f$. ✓ $y = x$ and $g(x) = \lvert x \rvert$ are continuous. ✓ By the product theorem, $y = x^2 = x \cdot x$ is continuous. ✓ By the constant multiplier theorem, $y = 3x$ is continuous. ✓ By the sum theorem, $f(x) = x^2 + 3x$ is continuous. ✓ By the composite theorem, $y = \lvert x^2 + 3x \rvert$ is continuous. ✓ Domain $(-\infty, \infty)$.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B4**)

1. Determine where each of the following functions is continuous and state the domain of each:

   a. $y = x^3 + \sqrt{x^2 + 2x + 1}$

   b. $y = (\sin x)(\cos x)$

   c. $y = \begin{cases} \sin 3x & x \neq 0 \\ 3 & x = 0 \end{cases}$

   d. $y = |\sin 2x|$

Suggested Resources

Section 2.3

Outcomes	Elaboration – Instructional Strategies/Suggestions

Outcomes

SCO: By the end of this course, students will be expected to

A1 apply, understand and explain average and instantaneous rates of change and extend these concepts to secant line and tangent line slopes.

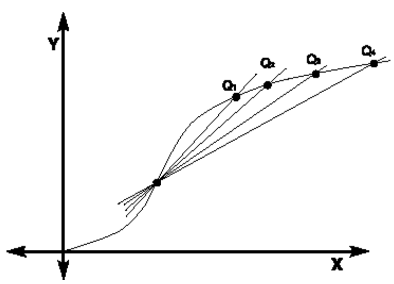
C2 understand the development of the slope of a tangent line from the slope of a secant line.

Elaboration – Instructional Strategies/Suggestions

A1/C2 Students understand that average rate of change is the rate of change (slope) from one point to another on a curve. Slope has been defined in previous courses as rise over run. The slope of the secant is also the slope from one point to another on a curve. The secant line slope is the average rate of change of a function between two points.

The instantaneous rate of change is the rate of change (slope) at one point on the curve. The tangent line slope is the slope of the curve at one point. Therefore, the slope of the tangent line and the instantaneous rate of change are equivalent at $x = c$ for the function *f(x).*

The slope of the secant line becomes the slope of the tangent line as the distance between the two points of the secant approaches zero. In the diagram below, *q* is considered to be the floating point because it floats closer to point *p*, and *p* is the point where we are evaluating the tangent slope.



It is a good idea to revisit average rate of change here. The average rate of change between two points $P(a, f(a))$ and $Q(b, f(b))$ can be found as follows:

$$M_{avg} = \frac{f(b) - f(a)}{b - a}$$

Set $b = a + h$, then:

$$M_{avg} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$M_{avg} = \frac{f(a+h) - f(a)}{h}$$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C2/A1**)

1. Find the slope of the curve for each of the following:
   a. $y = -x^2 + 6x - 5$ at $x = 4$

   b. $y = 4$ at $x = 10$

   c. $y = (x+1)^2$ at $x = -3$

2. Find the tangent slope of $f(x) = x^2 + 4x - 3$ at $x = a$. Explain what is happening to the tangent at $x = a$ as a changes.

3. At what point is the tangent to $f(x) = x^2 - 3x + 7$ horizontal?

Journal (**C2/A1**)

4. Explain how the slope of a tangent is related to the slopes of certain secants.

Suggested Resources

Section 2.4

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **A1** apply, understand and explain average and instantaneous rates of change and extend these concepts to secant line and tangent line slopes. **C2** understand the development of the slope of a tangent line from the slope of a secant line.	The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is the number: $$m = \lim_{h\to 0}\frac{f(a+h) - f(a)}{(a+h)-(a)} = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h},$$ where *h* is the distance between the two points of the secant, and provided the limit exists. For example: If $f(x) = 5x^2 + 2$, find the tangent slope at the point where $x = c$. $$m = \lim_{h\to 0}\frac{(5(c+h)^2 + 2) - (5c^2 + 2)}{h}$$ $$m = \lim_{h\to 0}\frac{5c^2 + 10ch + 5h^2 + 2 - 5c^2 - 2}{h}$$ $$= \lim_{h\to 0}\frac{10ch + 5h^2}{h} = \lim_{h\to 0} 10c + 5h$$ $$= 10c$$

Worthwhile Tasks for Instruction and/or Assessment	**Suggested Resources**

Outcomes

SCO: By the end of this course, students will be expected to

C3 find the equations of the tangent and normal lines at a given point.

Elaboration – Instructional Strategies/Suggestions

C3 Students will use the definition of the slope of a curve to find the slope of the tangent line, and will find the equation of the tangent line using skills developed in previous courses.

Recall that:

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is the number:

$$m = f'(x) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-(a)} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

where *h* is the distance between the two points of the secant, and provided the limit exists.

For example: Find the equation of the tangent to the curve $f(x) = 5x^2 + 2$ at $x = 1$.

Recall from the previous page, that the slope of this curve at $x = c$ is 10*c*. So, substituting $x = 1$ in *f*(*x*) gives:

$$m = 10c = 10(1) = 10$$

Substituting $x = 1$ in $f(x) = 5x^2 + 2$:

$$f(1) = 5(1)^2 + 2 = 5 + 2 = 7$$

Using $y - y_1 = m(x - x_1)$ solve for the tangent line.

$$y - 7 = 10(x - 1)$$

$$y = 10x - 10 + 7$$

$$y = 10x - 3$$

The normal line to a curve at a point is the line that is perpendicular to the tangent at that point. From a previous course, students understand that lines that are perpendicular have slopes that are negative reciprocals of each other.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C3**)

1. Find:
   a. An equation for each tangent to the curve $y = \frac{1}{x-1}$ that has slope $-\frac{1}{4}$.
   b. An equation for each normal to the curve $y = \frac{1}{x-1}$ that has slope 4.
   c. The equation of the tangent to $y = x^2 + 6x$ that has slope zero.
   d. The equation of the tangent to $y = x^2 - 3x$ at $(a, a^2 - 3a)$.

Suggested Resources

Section 2.4

Outcomes

SCO: By the end of this course, students will be expected to

C3 find the equations of the tangent and normal lines at a given point.

Elaboration – Instructional Strategies/Suggestions

C3 In the previous example, the slope of the tangent line was 10. Find the equation of the normal line for the previous example.

The normal slope would equal $-\frac{1}{10}$ because that is the negative reciprocal of the tangent slope. The point is still (1, 7).

Using $y - y_1 = m(x - x_1)$ solve for the normal line.

$$y - 7 = -\frac{1}{10}(x - 1)$$

$$y = -\frac{1}{10}x + \frac{1}{10} + 7$$

$$y = -\frac{1}{10}x + \frac{71}{10}$$

Worthwhile Tasks for Instruction and/or Assessment | **Suggested Resources**

Section 2

DERIVATIVES

30-35 Hours

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **A2** demonstrate an understanding of the definition of the derivative.	**A2** Students should have a good understanding of the relations between average rate of change, secant slope, instantaneous rate of change, and tangent slope. The definition of a derivative should be built on this understanding.

The derivative of a function *f* with respect to the variable x is the function f' whose value at x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Explain that *f* has a derivative at *x*, that is, *f* is differentiable at *x*.

For example: Differentiate $f(x) = x^2 - 3$ at $x = c$.

$$f'(c) = \lim_{h \to 0} \frac{((c+h)^2 - 3) - (c^2 - 3)}{h}$$

$$f'(c) = \lim_{h \to 0} \frac{c^2 + 2ch + h^2 - 3 - c^2 + 3}{h}$$

$$= \lim_{h \to 0} \frac{2ch + h^2}{h} = \lim_{h \to 0} 2c + h = 2c$$

At this point, students should be introduced to the different types of notation for the derivative. Make sure that you mention that "prime" notations f' and y' are from Lagrange, and that the *d/dx* notations are from Leibniz. Newton used dots over the letter representing the function. Students' attention should be directed to page 97 of their textbook, where all the different notations are shown and explained.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**A2**)

1. Differentiate $y = \sqrt{x-3}$ at the point $(7,2)$.

2. Find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$ at (1,1).

3. Use the hyperbola $xy = 1$ at the point $\left(-2, -\frac{1}{2}\right)$ to do the following:
   a. Find the tangent slope.
   b. Find the equation of the tangent line.
   c. Find the equation of the normal line.

Journal (**A2**)

4. Explain what a derivative is and how it is related to secant and tangent slopes.

Suggested Resources

Section 3.1

Web enabled "conic section"

Outcomes

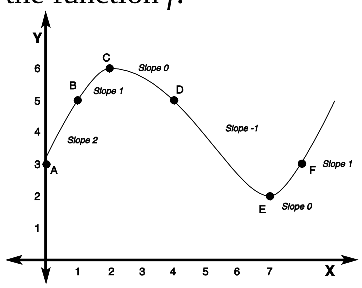
SCO: By the end of this course, students will be expected to

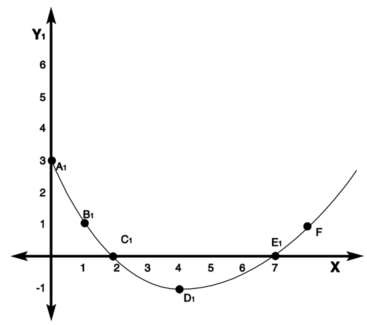
C4 demonstrate an understanding of the connection between the graphs of *f* and *f ′*.

Elaboration – Instructional Strategies/Suggestions

C4 Students need to explore the relationships between *f* and *f ′*. Students should be able to look at a graph of *f* and be able to use that graph, without knowing the function, to graph *f ′*. They should also be able to graph a suitable *f* if they have the graph of *f′*, for simple functions.

For example: Using the graph below, graph the derivative of the function *f*:





To graph *f ′*, first you label the axis *x* and *y ′*. Next you take the slope off the graph of *f* from various points and graph them as values on the *f ′* graph. Once the scatter plot is drawn, connect the dots with a smooth curve.

Even though we do not have the formula for either *f* or *f ′*, the graph of each function reveals important information about the behavior of the other. Make sure that you take the time to have students look at the relationships between the graphs in small groups followed by a class discussion. Pay close attention to the roots. Discuss maximum and minimum, increasing and decreasing - this will give students a taste of what is to come. You may also want to touch on concavity and points of inflection at this point. It will better help the students' understanding when you start to analyze graphs more intensely later on.

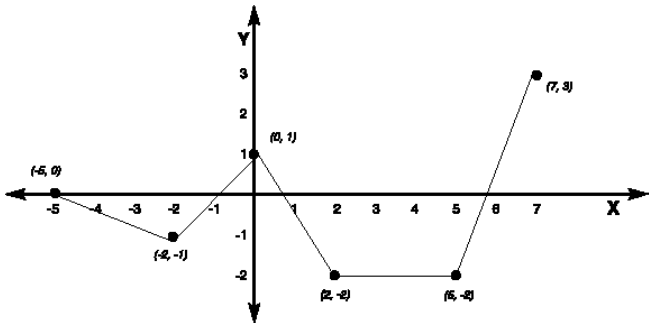
At this point you should pick a function and have students graph it using graphing technology. They can use the calculator to find various tangent slopes and then should graph the derivative using paper or scatter plots on their calculators to verify what they just discovered in the previous exercise.

When doing questions of multiple solutions (like p. 27 #2) have class share to see that there are many possible answers.

Worthwhile Tasks for Instruction and/or Assessment

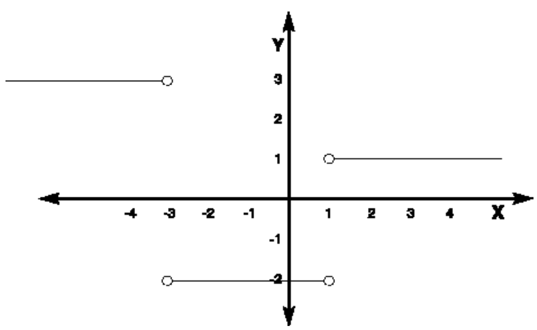
Performance (**C4**)

1. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end.



   a. Graph this function's derivative.
   b. At what values of x between –5 and 7 is the function not differentiable?

2. The graph of the function below is the derivative of $y = f(x)$. Graph a possible function *f*(*x*).



Journal (**C4**)

3. Sketch $y = \sin x$ and $y = \cos x$. Which function could be the derivative of the other? Explain your answer with information that you know about the graphs.

Suggested Resources

Section 3.1

Outcomes

SCO: By the end of this course, students will be expected to

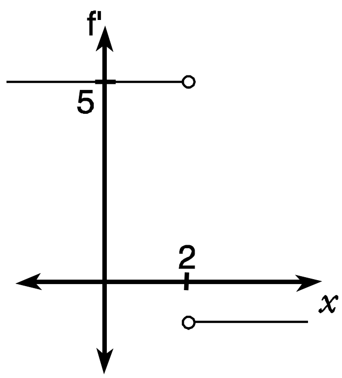
C4 demonstrate an understanding of the connection between the graphs of *f* and *f ′*.

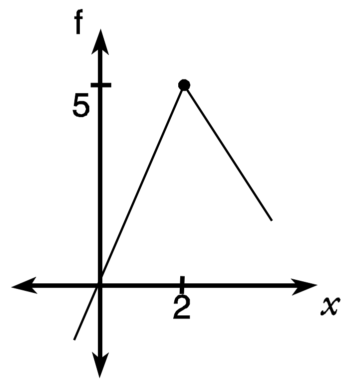
Elaboration – Instructional Strategies/Suggestions

C4 They should also be able to graph *f* if they have the graph of *f ′*, for simple functions. Students need to learn that if they have the graph of *f ′* it represents the family of curves of *f*. You get a specific curve if you know a value of *f*(*x*) to use as a reference.

You may wish to touch on how a derivative can fail to exist visually. It will help prepare the students for **B5**, which is an integral part of this section.

For example: Use the graph below of *f ′* to construct *f*. Also note that $f(0) = 0$, and that *f* is continuous for all *x*.





The absolute value function is one of the easier ways to explore this graphically. Graph the origin point. Then the students should notice that the slope on the left of $x = 2$ is 3 and the slope on the right of $x = 2$ is –1. These are both straight lines and can easily be graphed as shown above.

Example 5 on page 99 of the text is a wonderful way to investigate graphing the derivative using data. It is well worth the time to get the students to do that example or one similar to practice estimating slopes of the tangent. It would be interesting for them to graph the data in a scatter plot, which can be done quickly using technology, and get them to predict the graph of the derivative before they start their calculations. This will help identify how successful the introduction to this unit has been.

Worthwhile Tasks for Instruction and/or Assessment	**Suggested Resources**

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **B5** find where a function is not differentiable and distinguish between corners, cusps, discontinuities, and vertical tangents.	**B5** Students should be introduced to the four ways that a derivative may fail to exist that are the topic of this section. A function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines fail to approach a limit as *x* approaches *a*. A good way to investigate this is to use graphing technology with the class. Put them in groups and have them investigate the following four equations: a. $f(x) = \lvert x+1 \rvert$ b. $f(x) = x^{\frac{4}{5}}$ c. $f(x) = \sqrt[5]{x}$ d. $f(x) = \begin{cases} -2, & x < 1 \\ 1, & x \geq 1 \end{cases}$ Function A has a corner at $x = -1$. A corner (node) is where the one-sided derivatives differ. Function B has a cusp at $x = 0$. For a function a cusp is an extreme case of a corner. In this case, the limit of the slope of one side approaches infinity, while the other side approaches negative infinity. Function C has a vertical tangent at $x = 0$. A vertical occurs when the limits approach infinity or negative infinity from both sides. Students should have been exposed to this concept when they discussed slopes of vertical lines. Function D has a discontinuity at $x = 1$. This will cause one or both sides of the limit to be nonexistent. Graphing calculators can help with the understanding of differentiability. Differentiability implies local near-linearity. The graph of any function f that is differentiable at x has a tangent line at $(x, f(x))$; and if we examine the graph of $f(x)$ at increasingly large scales near this point, the graph increasingly resembles the tangent line.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B5**)

Using technology, show that the following functions contain discontinuities and identify the type:

1. $f(x) = x^{\frac{2}{7}}$

2. $f(x) = \begin{cases} \tan^{-1} x & x \neq 0 \\ 1 & x = 0 \end{cases}$

3. $f(x) = 2 - \sqrt[3]{x}$

4. $f(x) = 3x - |x| - 2$

Suggested Resources

Section 3.2

Outcomes

SCO: By the end of this course, students will be expected to

B5 find where a function is not differentiable and distinguish between corners, cusps, discontinuities, and vertical tangents.

Elaboration – Instructional Strategies/Suggestions

B5 Technology will allow the students to see the curve become linear by using the zoom function.

The discussion in this section leads to the following theorem:

> If *f* has a derivative at $x = a$, then *f* is continuous at $x = a$.

A word of warning, the converse of that theorem is false, as the students have seen in this section.

Worthwhile Tasks for Instruction and/or Assessment	Suggested Resources

Outcomes

SCO: By the end of this course, students will be expected to

B6 derive, apply, and explain power, sum, difference, product and quotient rules.

Elaboration – Instructional Strategies/Suggestions

B6 Students will use everything that is learned in this section for the rest of their mathematical career. It is important that they spend enough time working in this section to have a good understanding of rules, why they work and how they work.

A great way to introduce some of these rules is to have students discover them. For example, the derivative of a constant can be discussed as a whole class quickly as a way to get the class's interest. If you graph two or more horizontal lines, students should readily see that the slopes of the tangent anywhere along those lines are zero. With that knowledge, they should be able to come up with Rule 1: Derivative of a Constant Function.

Rule 1: Derivative of a Constant Function.

If *f* is a function with the constant value *c*, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

Another rule that the students should be able to discover is Rule 2: Power Rule for Positive Integer Powers of *x*. Using the definition of limits on some simple powers can discover this.

Rule 2: Power Rule for Positive Integer Powers of *x*.

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Rule 3: The Constant Multiple Rule is another good candidate for discovery. By this time the students should be able to predict the derivative before they do the work.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B6**)

1. Find $f'(x)$ for each of the following:

   a. $f(x) = 7$

   b. $f(x) = x^4 - \frac{3}{2}x^2 + 7x - 15$

   c. $f(x) = \sqrt{x}(a + bx)$, where *a* and *b* are constants

   d. $f(x) = \frac{x^2 + 2x - 1}{3x - 2}$

   e. $f(x) = \frac{x}{ax^2 + bx + c}$, where *a*, *b* and *c* are constants.

2. If trig derivatives have been studied, then find $f'(x)$ for each of the following:

   a. $f(x) = x^3 \sin x$

   b. $f(x) = \frac{x \cos x}{x + 3}$

3. a. Using first principles employed by binomial distribution prove the Power rule for positive integer exponents.
   b. Use mathematical induction and the product rule to prove the Power rule for positive integer exponents.

4. Use the quotient rule to prove the power rule for negative integer exponents.

5. Find the tangent of the curve $y = -x^2 + x$ where the slope is 6. What is the smallest slope of the curve? At what value of x does the curve have this slope?

6. Find the tangents of the curve $y = x^3 + 4$ where the slope is 3. What is the smallest slope of the curve? At what value of x does the curve have this slope?

Suggested Resources

Section 3.3

Outcomes	Elaboration – Instructional Strategies/Suggestions

Outcomes

SCO: By the end of this course, students will be expected to

B6 derive, apply, and explain power, product and quotient rules.

Elaboration – Instructional Strategies/Suggestions

B6 Rule 3: The Constant Multiple Rule.

> If *u* is a differentiable function of *x* and *c* is a constant, then
>
> $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Rule 4: The Sum and Difference Rule should come to the students easily since they have seen similar rules (for limits) in the previous unit of this course.

Rule 4: The Sum and Difference Rule

> If *u* and *v* are differentiable functions of *x*, then their sum and difference are differentiable at every point where both *u* and *v* are differentiable. At such points,
>
> $$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

At this point students should be allowed time to apply the four rules. Also, teachers may want to teach the basic trigonometric derivatives (see outcome B8) before this section is completed. Introducing the trigonometric functions at this time, or at least sine and cosine, will make the introduction of the product rule less artificial than when they are used simply on polynomials. This is only a suggestion; teachers are to proceed in the best interest of their students.

Rule 5: The Product Rule is easier to justify if polynomial and trigonometric functions are used together. Make sure that students realize that the derivative of the product of two functions is not the product of their derivatives.

For example: $\frac{d}{dx}(2 \cdot x) = \frac{d}{dx}2x = 2$,

But: $\frac{d}{dx}2 \cdot \frac{d}{dx}x = 0 \cdot 1 = 0$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B6**)

7. Let $y=(x^5+1)^4$.

   a. Find $\frac{d^2y}{dx^2}$.

   b. What is the smallest *n* for which $\frac{d^ny}{dx^n}=0$, for all x?

Suggested Resources

Section 3.3

Outcomes

SCO: By the end of this course, students will be expected to

B6 derive, apply, and explain power, product and quotient rules.

Elaboration – Instructional Strategies/Suggestions

B6 Rule 5: The Product Rule:

The product of two differentiable functions *u* and *v* is differentiable, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Again, Rule 6: The Quotient Rule is easier to justify if polynomial and trigonometric functions are used together.

Rule 6: The Quotient Rule

At a point where $v \neq 0$, the quotient $y = \frac{u}{v}$ of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Let students know that it is usually beneficial to leave the denominator of the quotient rule in factored form.

Up to this point, students have been exposed to first order derivatives, y'. Students need to practice second, y'', third, y''', fourth, $y^{(4)}$, and nth, $y^{(n)}$ order derivatives.

Although students should not be held accountable for the proofs of all the rules in this section, it is reasonable that they be able to prove some of the relationships. For example: Proving the constant multiplier rule, or the sum/difference rule, or deriving the quotient rule from the product rule.

Worthwhile Tasks for Instruction and/or Assessment	Suggested Resources

Outcomes

SCO: By the end of this course, students will be expected to

B7 apply the chain rule to composite functions.

Elaboration – Instructional Strategies/Suggestions

B7 The chain rule requires lots of practice to ensure that students truly can apply it. It is extremely important throughout calculus; therefore, it is time well spent to linger here to ensure that students have the concept well in hand.

A brief review of composite functions would greatly help students with the study of the chain rule.

The two notations of the chain rule should be introduced to students. They should be expected to use both notations competently and to make an intelligent choice of which notation to use in specific circumstances.

The Chain Rule:

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

For example: Find the derivative of $y = \cos(2x)$.

Using the chain rule, $u = 2x$ so $\frac{du}{dx} = 2$.

Also $y = \cos u$ which makes $\frac{dy}{du} = -\sin u$.

Therefore, $\frac{dy}{dx} = -\sin u \cdot 2 = -\sin(2x) \cdot 2 = -2\sin(2x)$.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B7**)

1. Differentiate the following:

   a. $y = (3x - 7)^8$
   b. $y = \sqrt{x^3 - 4x + 2}$
   c. $y = \sin 6x$
   d. $y = \tan x^2$
   e. $y = (x + \cos^2 x)^4$
   f. $y = \dfrac{\csc^2 4x}{2x}$

2. What is the largest possible value for the slope of the curve $y = \cos\left(\dfrac{x}{2}\right)$?

3. Let g be differentiable on $[0, \infty)$ and $H(x) = g(\sqrt{x})$.
   a. Find $H'(x)$.
   b. Where is H differentiable?

4. Let $y = (x^5 + 1)^4$.
   a. Find $\dfrac{d^2y}{dx^2}$.
   b. What is the smallest *n* for which $\dfrac{d^ny}{dx^n} = 0$, for all x?

Journal (**B7**)

5. Explain how the chain rule works to a classmate that missed the lesson. Feel free to include well-explained examples.

6. Why did Leibniz create this notation? What mathematical purpose does it serve? (answer…it follows the rules of algebra).

Suggested Resources

Section 3.6

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to **B8** use derivatives to analyze and solve problems involving rates of change.	**B8** Rates of change is one of the most important real world applications of derivatives. Do not just limit discussion to velocity and volume questions. Make sure that the treatment of this topic includes questions that relate to economics, biology, chemistry and other fields of study. It is important to show students that mathematics is important in other fields; it is not an isolated field of study with no practical applications. Be careful to pick topics that students will understand. It should be made clear that speed and velocity are not the same things. Velocity includes direction from a point of reference whereas speed does not. If an object travels along a path, and displacement, $s = f(t)$, is measured over time *t*, then the first derivative, $s' = f'(t)$, equals the velocity of that object, $v(t) = f'(t)$. The second derivative, $s'' = f''(t)$, equals the acceleration of the object, $a(t) = f''(t)$. Students should realize that the first derivative of velocity is acceleration $v'(t) = a(t)$. It should also become clear to students that one of the applications of higher order derivatives is to find acceleration. Some teachers like to look at implicit differentiation, max/min problems, and related rates at this point in the curriculum. Do what is in the best interest of your students.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B8**)

1. A ball is thrown vertically upward from the roof of a building 84.5 m high, with an initial velocity of 21.5 m/s. Its height above the ground at time t is given by $h(t) = -4.9t^2 + 21.5t + 84.5$.
   a. Find the velocity at time t.
   b. Find the acceleration at time t.
   c. When does the velocity equal 0?
   d. Find the maximum height reached by the ball.
   e. When will the ball hit the ground?
   f. What is the velocity of the ball the instant it hits the ground?
   g. What is the acceleration of the ball the instant it hits the ground?

2. An apple farmer can sell her apples today for $0.50 a kilogram. Each week she waits her apples are worth $0.02 less per kilogram. If she picks her trees now, they would yield 60 kilograms each. Each week she waits to harvest the apples, she gains 3 kilograms per tree.
   a. Write a function representing revenue.
   b. When should she harvest her apples to make the maximum revenue?
   c. What is the maximum revenue?
   d. Estimate period of time this remains plausible.

3. A warm bottle of water is placed in a cold refrigerator.
   a. Sketch the graph of the temperature of the water as a function of time.
   b. Is the initial rate of change of temperature greater or less than the rate of change after a half hour?

4. One of Joshua's favorite things to do is to drop stones into a pond and watch the water rings get larger and larger. If Joshua watches the radius of a ring increase at a rate of 6 mm/sec when the radius = 3 mm, what is the rate at which the area is expanding?

Suggested Resources

Section 3.4

Outcomes

SCO: By the end of this course, students will be expected to

B9 apply the rules for differentiating the six trigonometric functions.

Elaboration – Instructional Strategies/Suggestions

B9 Students should complete a review of the previously learned trigonometric identities before proceeding with this section. The exception is if part of this section is being completed before the product and quotient rule. In that case look at the simple trigonometric derivatives, then go back to the product and quotient rules. After that is completed, then the trigonometric review and the remainder of this section should be completed.

Students can make plausible conjectures about a few of the trigonometric derivatives quite easily through exploring the tangent slope of their graphs. Students should graph the function $y = \sin x$ using technology, and then they should find the tangent slopes at various points along the curve. Once they have the slopes at points like $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, they should be instructed to graph the derivative. They should easily discover that the derivative of the function $y = \sin x$ is $y' = \cos x$. The same can be done for $y = \cos x$.

In a previous course, students were introduced to the very important limits $\lim_{x \to 0} \frac{\sin x}{x}$ and $\lim_{x \to 0} \frac{1 - \cos x}{x}$. They should be able to use Cauchy's limit definition of the derivative to find the derivatives of the sine and cosine functions.   Students should be able to build on these to discover the derivatives of the other 4 trigonometric functions.
The six trigonometric derivatives are:

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \csc x = -\csc x \cot x$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \cot x = -\csc^2 x$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B9**)

1. Find the derivatives of the following:

   a. $f(x) = \frac{\sin x}{x^2}$

   b. $f(x) = 3\sin x + \cos x - 1$

   c. $f(x) = (\sec x)(\csc x)$

2. Determine the slope of the tangent to the curve $y = \sin x$ at the points $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$. Graph the results.

3. Use the quotient rule and the definition of $\sec x$, $\sec x = \frac{1}{\cos x}$, to show that the derivative of $\sec x$ is $\sec x \tan x$.

4. Show that the graphs of $\sin x$ and $\csc x$ have horizontal tangents at $x = \frac{\pi}{2}$.

5. Find the equation of the tangent line of the curve $y = \sqrt{2}\sin x$ at $x = \frac{3\pi}{4}$.

Suggested Resources

Section 3.5

Outcomes

SCO: By the end of this course, students will be expected to

A3 demonstrate understanding of implicit differentiation and identify situations that require implicit differentiation.

Elaboration – Instructional Strategies/Suggestions

A3 Student should understand that when the dependent variable is not explicitly written in terms of the independent variable, then implicit differentiation should be used. Teachers should remember that algebra is a significant issue when separating $\frac{dy}{dx}$ from the rest of the expression.

Ask students to find and compare:

$$\frac{d}{dx}(y^2), \frac{d}{dx}(\cos y), \frac{d}{dx}(x^2), \frac{d}{dx}(\cos x)$$

Students should explore the equation $x^2 + y^2 = 16$. Get them to graph it and discuss the slope of the tangent at $(1, -\sqrt{15})$. Students know that the circle is not a single function, it is the combination of two curves, $y_1 = \sqrt{16 - x^2}$ and $y_2 = -\sqrt{16 - x^2}$. Both of these can be differentiated and then the proper derivative can be used to find the slope required.

A much more efficient approach in a situation like this is implicit differentiation. This approach requires that we find only one derivative which will be expressed in terms of *x* and *y*.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(16)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Now the point can be substituted in and the slope of the tangent line at $(1, -\sqrt{15})$ is:

$$\frac{dy}{dx} = -\frac{(1)}{\left(-\sqrt{15}\right)} = \frac{1}{\sqrt{15}}\left(\frac{\sqrt{15}}{\sqrt{15}}\right) = \frac{\sqrt{15}}{15}.$$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**A3**)

1. Find $\frac{dy}{dx}$ for the following:

   a. $x^2 + 3xy = 2$

   b. $x^4 + (xy)^3 + y = 3$

   c. $x\cos 3y = y\sin 3x$

2. Find the equation of the tangent line of $x^2 y + xy^2 = 6$ at $(2,1)$.

3. Show that an equation of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

4. Show that the tangents to the graphs of $y = \frac{x^2}{\sqrt{2}} + \frac{1}{2\sqrt{2}}$ and $x^2 + y^2 = 1$ are perpendicular at their points of intersection.

5. Show that the sum of the *x*- and *y*-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to *c*.

6. Use the results of question 5 to graph the astroid as the envelop of all line segments joining $(\pm x, 0)$ and $(0, \pm y)$ with $x + y = 1$.

Suggested Resources

Section 3.7

Outcomes

SCO: By the end of this course, students will be expected to

A3 demonstrate understanding of implicit differentiation and identify situations that require implicit differentiation.

Elaboration – Instructional Strategies/Suggestions

A3 Now is another good time to point out the usefulness of Leibniz notation.

Students should spend some time here mastering this concept, as it is useful throughout the course. Make sure students get practice with all of the major algebraic manipulations needed to isolate $\frac{dy}{dx}$.

Worthwhile Tasks for Instruction and/or Assessment	Suggested Resources

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to	**B10** Students have a good understanding of inverse functions from another course. This understanding should be tapped into with an opening discussion of figure 3.50 (p. 157).
B10 apply the rules for differentiating the six inverse trigonometric functions. **(Optional)**	Exploration 1 (p. 158) is extremely important for the students' understanding of the concepts. This is an ideal activity for small groups to work on and discuss, followed by a class discussion.

The class should find the derivatives of the first few inverse functions, do the rest individually, and then the class should have a summary of all six:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, |x|<1, \qquad \frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, |u|<1$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, |x|>1, \qquad \frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, |u|>1$$

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, |x|<1, \qquad \frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, |u|<1$$

$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}, |x|>1, \qquad \frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, |u|>1$$

Note: You may not wish to give the derivatives of the last three functions because they are questions 24 to 26 (p. 163).

It should be pointed out that the most useful are $\frac{d}{dx}\sin^{-1}u$ and $\frac{d}{dx}\tan^{-1}u$.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B10**)

1. Find f' of the following:

   a. $f(x) = \cos^{-1}\sqrt{2x-1}$

   b. $f(x) = (\tan^{-1} x)^{-1}$

   c. $f(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$

   d. $f(x) = x \sin x \csc^{-1} x$

   e. $f(x) = \cos^{-1}\left(\dfrac{b + a\cos x}{a + b\cos x}\right)$

2. Find the equation of the tangent line to
   $f(x) = x\sin^{-1}\left(\dfrac{x}{4}\right) + \sqrt{16 - x^2}$ at $x = 2$.

3. Differentiate with respect to *x*, $y^2 \sin x = \tan^{-1} x - y$.

Suggested Resources

Section 3.8

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **B11** calculate and apply derivatives of exponential and logarithmic functions.	**B11** Although students have studied exponential and logarithmic functions in previous courses, a review of these functions and their graphs is strongly recommended before proceeding with the calculus involved. Sections 1.3 and 1.5 of *Calculus: Graphical, Numerical, Algebraic* contain the reviews recommended. Students have dealt with Euler's number, *e*, in a previous course. At this point they should be shown that *e* can be defined as the unique *x* such that: $$\lim_{h\to 0}\frac{x^h-1}{h}=1$$ With this definition in mind students should be able to find $\frac{d}{dx}e^x$. Remember to point out why the power rule cannot be used to find this derivative. Also look at 2^x and 3^x to help the understanding. The rest of the derivatives may be discovered algebraically using implicit differentiation, and they are: ❖ $\frac{d}{dx}(a^u)=a^u\ln a\frac{du}{dx}, a>0, a\neq 1$ ❖ $\frac{d}{dx}\ln u=\frac{1}{u}\frac{du}{dx}$ ❖ $\frac{d}{dx}\log_a u=\frac{1}{u\ln a}\frac{du}{dx}, a>0, a\neq 1$ You may want to point out that these types of derivatives are particularly important in the real world. A lot of science and business models rely on exponential and logarithmic growth. Logarithmic differentiation can be used in some cases to help simplify the differentiation process. The students should practice this method and look at questions where it does and does not help the process so that they get a "feel" for when it should be used.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B11**)

1. Find $\frac{dy}{dx}$ or $f'(x)$ of the following:

   a. $f(x) = x^2 e^{-x}$

   b. $f(x) = \sin\left(e^x\right)$

   c. $f(x) = \frac{e^x}{\cos x}$

   d. $f(x) = 7^{3x}$

   e. $f(x) = \log_2(\sec x)$

   f. $f(x) = \ln^3(2x+1)$

   g. $f(x) = x \log_6 x$

   h. $f(x) = \sin x \log_3(\sin x)$

2. Use logarithmic differentiation to find the derivative of the following:

   a. $y = x^{x^2}$

   b. $y = x^{\left(x^x\right)}$

Suggested Resources

Section 3.9

Outcomes

SCO: By the end of this course, students will be expected to

B12 apply Newton's method to approximate zeros of a function. **(Optional)**

Elaboration – Instructional Strategies/Suggestions

B12 Newton's method is a powerful tool that is used to estimate zeros of a function. This can be very helpful when looking for zeroes, critical points (zeroes of the first derivative), and points of inflections (zeroes of the second derivative).

Students should be able to use a graphing utility, or table of values to help them with this activity. That can allow the students to get values that are close to the zero to start with. Students need to be reminded that you have to do this activity several times for each question in order to find a good approximation to the correct value of the zero. It is important that students carry at least 6 decimal places to ensure accuracy.

The first step is to guess a reasonable approximation of the zero. Once students have this, Newton's method is employed to find a better approximation. Then this approximation is used to find a better one, etc. The formula that is used is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For example: Find the real root of $x^3 + x - 1 = 0$. A graph of this function shows that $x_1 = 0.5$ is a good approximation of the zero.

$$x_{n+1} = x_n - \frac{x_n^{\ 3} + x_n - 1}{3x_n^{\ 2} + 1}$$

$$x_2 = 0.5 - \frac{(0.5)^3 + (0.5) - 1}{3(0.5)^2 + 1} = 0.7142857$$

$$x_3 = 0.6831797$$
$$x_4 = 0.6823284$$
$$x_5 = 0.6823278$$
$$x_6 = 0.6823278$$

The root is approximately 0.6824278 since x_5 and x_6 are equal.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B12**)

1. Find all of the roots of $2 - x^2 = 0$.

2. Use Newton's Method to find the points of inflection of $y = x^7 + 35x^3 - 63x^2$.

3. A group activity that can be useful is to consider why this procedure works and when it can fail to work.

4. Use Newton's Method to approximate $\sqrt[3]{7}$.

Suggested Resources

Section 4.5

Outcomes

SCO: By the end of this course, students will be expected to

B13 estimate the change in a function using differentials and apply them to real world situations.

Elaboration – Instructional Strategies/Suggestions

B13 Students need to be aware that the *dy* and *dx* appearing in $\frac{dy}{dx}$ are the same as *dy* and *dx* used in differentials, however, the horizontal bar in $\frac{dy}{dx}$ does not represent division.

Differentials: Let $y = f(x)$ be a differentiable function. The differential *dx* is an independent variable. The differential *dy* is

$$dy = f'(x)dx$$

For example: Find *dy* if $y = 7\cos x$.

$$dy = (-7\sin x)dx$$

Students need to be able to use differentials to find the estimate of change in a function at a given point.

Differential Estimate of Change: Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of *f* when *x* changes from *a* to $a + dx$ is:

$$df = f'(a)dx$$

Section 4.5 in *Calculus: Graphical, Numerical, Algebraic* has 6 wonderful examples of how to apply this method. Students must see a number of examples in order to grasp this concept. As always, it is important to look at problems from different fields of study so that student understand that calculus is not independent of other disciplines.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B13**)

1. Find the differential *dy* of the following functions:

   a. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

   b. $y = x - \tan^2 x$

2. The radius of a sphere is 10cm, with a possible error of 0.05cm.
   a. Use differentials to approximate the maximum possible error in calculating the volume of the sphere.
   b. Use differentials to approximate the maximum possible error in calculating the surface area of the sphere.

3. The measurements of the base and altitude of a triangle are 54cm and 62cm respectively. The possible error of each measurement is 0.15cm. Using differentials, calculate the possible error in computing the area of the triangle.

4. A current of *I* amps passes through a resistor of *R* ohms. Ohm's Law is $E = IR$, where *E* is voltage. If the voltage is constant, show that the magnitude of the relative error in *R* caused by a change in *I* is equal in magnitude to the relative error in *I*.

Suggested Resources

Section 4.5

Outcomes

SCO: By the end of this course, students will be expected to

B14 solve and interpret related rate problems.

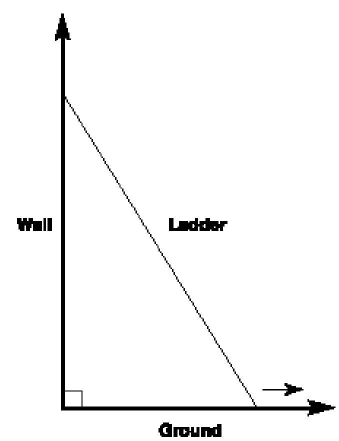
Elaboration – Instructional Strategies/Suggestions

B14 Students will use their knowledge of calculus to solve related rates questions. It is important that students are able to articulate what the information found represents, or the power of this outcome will be lost. Use implicit differentiation to solve these problems.

Calculus: Graphical, Numerical, Algebraic suggests a six-step strategy (p. 234) to solving these types of questions. A seventh step should be added, which would be: Interpret the information found.

It is probably best not to give students the strategy without working through an example to discover it. A class discussion at each step should help them discover the strategy. If they can discover it as a group, as opposed to being given it, they should have a better understanding of what is involved in solving related rate problems, and that will serve them better in the long run.

For example: Suppose that a ladder 6 m long is leaning against a wall. The bottom of the ladder is sliding away from the wall at 0.5 m/s. At what rate is the top of the ladder sliding down the wall at the instant when the bottom of the ladder is 3 m from the wall?



Step 1 (Picture and variable):
Let t = time (seconds)
x = the horizontal distance from the wall to the ladder at time t
y = the vertical distance the ladder is up the wall at time t

Step 2 (Numerical information):

6 m = length of ladder

$$\frac{dx}{dt} = 0.5$$

$$x = 3$$

Step 3 (To find): We need to find $\frac{dy}{dt}$.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B14**)

1. An ice cube is melting in such a way that its shape is always cubic. Its side length is decreasing by 1cm/min.
   a. How fast is the volume decreasing at time *t*?

   b. What is $\frac{dV}{dt}$ at the instant the side length is 5cm?

   c. What is $\frac{dV}{dt}$ at the instant the volume is $64cm^3$?

   d. How fast is the surface area *A* decreasing at time *t*?

   e. What is $\frac{dA}{dt}$ at the instant the side length is 5cm?

   f. What is $\frac{dA}{dt}$ at the instant the surface area is $96cm^2$?

2. A rectangular swimming pool has length 30m, width 20m, and depth 2m.
   a. If it is being filled at $10m^3$/min, at what rate is the depth of the water increasing?
   b. If the depth is increasing at 0.1m/min, at what rate is the pool being filled?

3. Two sailboats leave port one hour apart. The first to leave, boat M, travels west at 10km/hour, and boat N heads north at 15km/hour an hour later. At what rate are the ships separating after boat N has been traveling for 2 hours?

Suggested Resources

Section 4.6

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **B14** solve and interpret related rate problems.	**B14**

Step 4 (How the variables relate):

From the diagram we see that $x^2 + y^2 = 36$.

Step 5 (Differentiate with respect to *t*):

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{2x\frac{dx}{dt}}{2y} = -\frac{x\frac{dx}{dt}}{y}$$

Step 6 (Evaluate):

So, $\frac{dx}{dt} = 0.5$, $x = 3$, and $y = \sqrt{36 - 3^2} = \sqrt{27}$

$$\frac{dy}{dt} = -\frac{(3)(0.5)}{\sqrt{27}} \approx -0.289 \text{ m/s}$$

Step 7 (Interpret):

The top of the ladder is sliding down the wall at approximately 0.289 m/s.

Again, select problems from a range of fields of study to underscore the importance of calculus.

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

Section 3

MORE APPLICATIONS OF DERIVATIVES

20 Hours

Outcomes

SCO: By the end of this course, students will be expected to

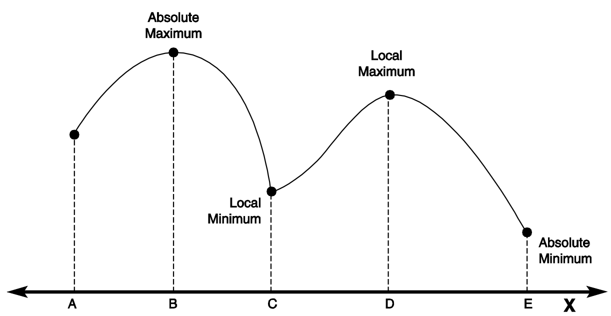
B15 demonstrate an understanding of critical points and absolute extreme values of a function.

Elaboration – Instructional Strategies/Suggestions

B15 Students should understand the difference between absolute (global) extreme values and local extreme values. They should also understand how critical points relate to extreme values. They need to be able to identify extreme values both graphically and algebraically.

An absolute maximum is a highest point on the curve. An absolute minimum is a lowest point. A local maximum is the highest point in that section of the curve, but there may be higher points elsewhere. A local minimum is the lowest point on that section of the curve. An absolute maximum is also a local maximum and the same is true for minimums.

A random graph, like this one, can be used to start the lesson to help clarify between absolute and local extreme values.



Point A is just an endpoint. The text suggests that it is a local minimum but that contradicts the definition with respect to open interval. Point B is an absolute maximum; point C is an local minimum; point D is a local maximum: and point E is an absolute minimum.

A critical point is a point in the interior of the domain of a function *f* at which point $f'=0$ or f' does not exist.

For example: Find the extreme value of $y=x^2-3x+5$.

So, $y'=2x-3$

And $2x-3=0$

$x=\frac{3}{2}$ is a critical point.

Since this is a quadratic equation with no reflection, $x=\frac{3}{2}$ is the absolute minimum.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B15**)

1. Identify the local max/min(s), and the absolute max/min(s) in each of the following:

   a. $y = x^3 - 9x^2 + 24x$

   b. $y = 3x^4 - 16x^3 + 18x^2$, $-2 \le x \le 6$

   c. $y = |2x - 1|$

   d. $y = 2x^3 - 3x^2 - 12x + 13$

   e. $y = \dfrac{x-4}{x+1}$

   f. $y = \dfrac{x^2 + 2x - 4}{x^2}$

   g. $y = \dfrac{x^2}{\sqrt{x+1}}$

2. Find the maximum product of two positive real numbers whose sum is 20.

3. Five kennels for five small dogs are to be of equal dimensions and in a row. If 30 m of fencing is available, what are the dimensions of the pens that maximize the total area in each of the following cases:
   a. When the kennels are paced against a barn so that on of the sides is the wall of the barn.
   b. When the kennels are away from the barn.

Journal (**B15**)

4. Explain the difference between absolute and local extreme values. Include a sketch.

5. Can a function have more than one absolute maximum? Explain using an example or sketch.

Suggested Resources

Section 4.1

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to **B16** find the intervals on which a function is increasing or decreasing **B15** demonstrate an understanding of critical points and absolute extreme values of a function.	**B16/B15** Students understand from a previous course when a function is increasing or when it is decreasing. Students build on this knowledge and their knowledge of calculus to find an algebraic method to discover where a function is increasing or decreasing. A function is increasing on the interval $[a,b]$ if wherever $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ and $f(x_1) < f(x_2)$. If $f' > 0$ at each point in the interval $[x_1, x_2]$, then *f* increases on $[x_1, x_2]$. A function is decreasing on the interval $[a,b]$ if whenever $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ and $f(x_1) > f(x_2)$. If $f' < 0$ at each point in the interval $[x_1, x_2]$, then *f* decreases on $[x_1, x_2]$. Use this newfound knowledge along with the ability to find critical points to determine where functions increase or decrease. For example: Find where the function $y = -2x^3 - 3x^2 + 36x$ increases and decreases. $$y' = -6x^2 - 6x + 36 = -6(x^2 + x - 6) = -6(x+3)(x-2)$$ So, the extreme values occur at $x = -3, x = 2$. Now test three points to see where it increases or decreases: At $x = -4, y' = -36$. Since $y' < 0$, *y* decreases on $(-\infty, -3)$. At $x = 0, y' = 36$. Since $y' > 0$, *y* increases on $(-3,2)$. At $x = 3, y' = -36$. Since $y' < 0$, *y* decreases on $(2, \infty)$. Students should be encouraged to graph this function using technology to check their work. At this point a class discussion should take place around why the “test point” technique actually works. That is, why knowledge of the critical points allows us to chose test points to determine increasing and decreasing.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B16/B15**)

1. Find (i) the local extrema, (ii) the intervals in which the function is increasing, and (iii) the intervals in which the function is decreasing for each of the following:

   a. $y = 2x^3 - 3x^2 - 12x + 7$

   b. $y = 2x^3 + 3x^2 - 36x + 10$, $-5 \le x \le 5$

   c. $y = x^2(3 - x)^2$

   d. $y = x\sqrt{9 - x}$

   e. $y = \dfrac{x-1}{x+1}$

   f. $y = -x^3 + 48x - 10$

2. The temperature in Liverpool *t* hours after noon on a certain day is *T* degrees Celsius. This can be modeled by the function $T = \frac{1}{3}t^3 - 4t^2 + 3t + 12$ for $0 \le t \le 6$. When is the temperature falling?

3. The height *h*, in metres, t seconds after a stone thrown upward from a bridge is given by $y = -4.9x^2 + 10x + 20$.
   a. When is the stone rising?
   b. When is it falling?
   c. How long is it in the air?
   d. At what instant is the stone's velocity 0m/s?

4. Hayden decides to open a t-shirt company. He produces t-shirts for $5.00 each and sells them for $15.00 and can sell 500 shirts at that price. Hayden finds that for every $0.15 the price is lowered, 20 more shirts can be sold.
   a. When is the profit increasing?
   b. What price maximizes profit?
   c. How many shirts are sold at that price?

Suggested Resources

Section 4.2

Outcomes

SCO: By the end of this course, students will be expected to

C5 apply the First and Second Derivative Tests to determine the local extreme values of a function.

B15 demonstrate an understanding of critical points and absolute extreme values of a function.

Elaboration – Instructional Strategies/Suggestions

C5/B15 A simpler way, in some cases to find whether or not a critical point is a minimum or a maximum is to use the second derivative test.

Once a critical point has been identified using the first derivative test, the second derivative test is used to see if it is a maximum, minimum, or neither. This test cannot apply if $f'(a)$ does not exist, it only works where $f'(a)$ is actually 0.

The Second Derivative Test states that:

- ❖ If $x = a$ is a critical point, and $f''(a) > 0$, then a local minimum occurs at $x = a$.
- ❖ If $x = a$ is a critical point, and $f''(a) < 0$, then a local maximum occurs at $x = a$.
- ❖ If $x = a$ is a critical point, and $f''(a) = 0$, then it is not an extreme value, but it may be a point of inflection.

Continued from the example for **B16/B15** (p. 66): Find the local extreme values of the function $y = -2x^3 - 3x^2 + 36x$.

$$y' = -6x^2 - 6x + 36 = -6(x^2 + x - 6) = -6(x+3)(x-2)$$

So, the extreme values occur at $x = -3, x = 2$.

$$y'' = -12x - 6$$

At $x = -3$, $y'' = 30$, therefore a local minimum occurs at that point.
At $x = 2$, $y'' = -30$, therefore a local maximum occurs at that point.

Students should check their work with a graphing utility.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C5**/**C6**/**B15**)

1. Use the analytic method to find the intervals over which each of the following functions are: (i) increasing, (ii) decreasing, (iii) concave up, (iv) concave down, and then find any (v) local extrema, and (vi) points of inflection.

   a. $y = 2x^3 - 3x^2 - 9x + 27$

   b. $y = 2\sin x + \cos 2x$, $0 \le x \le 2\pi$

   c. $y = -x^5$

   d. $y = x^3 - 8x^2 + 16x$

   e. $y = xe^{2x}$

   f. $y = \sqrt{x^2 + 1}$

2. Use the second derivative test to show that $y = ax^2 + bx + c$:
   a. Has a single maximum point if $a < 0$.
   b. Has a single minimum point if $a > 0$.

3. Show that the function $y = \ln x$ has no points of inflection and that its graph is concave down for $x > 0$.

4. The predicted population P, in millions, of a small country is modeled by $P = \frac{x^2}{900} - \frac{x}{100} + 7$ over a short period of time, where *x* is the number of years after 2005.
   a. When will the population be increasing?
   b. When will it be decreasing?
   c. When will the population reach its minimum?
   d. What is the minimum population?

Suggested Resources

Section 4.3

Outcomes

SCO: By the end of this course, students will be expected to

C6 determine the concavity of a function and locate the points of inflection by analyzing the second derivative.

Elaboration – Instructional Strategies/Suggestions

C6 By this time students should have a good handle on looking at derivatives to find out various things about the function and its graph. Points of inflection and concavity are the last step.

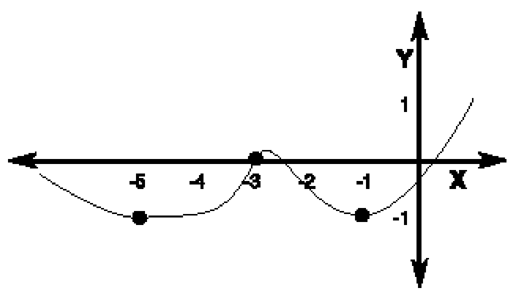
Concavity is tested much like increasing and decreasing intervals. However, it is usually tested in the second derivative. The exception is if the curve does not have a second derivative.

The curve of a function that has a second derivative is concave up on any interval where $y'' > 0$, and concave down where $y'' < 0$.

A point of inflection is a point where the graph has a tangent line and where the concavity changes. What this means in most cases is that $y'' = 0$ and that the sign of y'' changes around it. It is important to note that if $y' \neq 0$, and $y'' = 0$ at $x = a$, then $(a, f(a))$ is a point of inflection.

It is a good idea to recap the types of tests and what they are used to determine at this point. This should crystallize any problems that are present and help to clarify them.

A nice activity at this point is to get students to make rough sketches of functions when some information is given, but not the equation of the function itself.



For example: Sketch a graph that satisfies the following:

- ❖ $f(-3) = 0$
- ❖ $f'(-1) = 0$
- ❖ $f'(-5) = 0$
- ❖ $f'(-3) = 3$
- ❖ $f''(-3) = 0$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C5**/**C6/B15**)

5. For each situation below, sketch a possible graph that satisfies it. If no sketch can be drawn, explain why not.
   a. The first and second derivatives are always positive.
   b. The function is always positive and the first and second derivatives are always negative.
   c. The first derivative is always negative and the second derivative is always positive.
   d. The first derivative is always positive and the second derivative is always negative.
   e. The first derivative is always positive and the second derivative is both at various intervals of the graph.
   f. The function is always negative and the first and second derivatives are always negative.

6. As a whole class discussion or group activity, discuss why the second derivative test works the way it does. Specifically, why does it allow us to determine the concavity of a function?

Suggested Resources

Section 4.3

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to **B17** solve application problems involving maximum or minimum values of a function.	**B17** Students will use what they learned earlier in this section to solve maximum and minimum problems. Students have solved these types of questions many different ways in the past. Having said that, students traditionally have difficulties setting up these types of problems. Be sure to spend time here looking at examples from different fields of study. In most cases the maximum or minimum value can be found easily once an equation is set up. If you are trying to maximize volume, for example, you take the first derivative of the function and find its maximum point. If you are trying to maximize profit, it is advisable to find the profit equation first where profit equals revenue minus cost and then find the maximum point. Again, these questions are familiar to students, but take the time to get them assimilating their previous knowledge of optimization questions with the tools that they have learned in calculus.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B17**)

1. Hana, a lifeguard at a beach has 500m of rope to make a rectangular swimming area. One side of the rectangle is the straight edge of the beach.
   a. If she wants to maximize the swimming area, what should the dimensions of the area be?
   b. To ensure safety of the swimmers, no one should be more than 60m from the shore. What should the dimensions of the new rectangle be?

2. The canned noodle company estimates that the cost of making x cans of canned noodles is $C(x) = 30000 + 0.21x - 0.0002x^2$ and the revenue is $P(x) = 0.72x - 0.0002x^2$. How many cans of noodles should be made to maximize profit?

3. A ladder is being carried down a hallway that is 3m wide. At the end of the hall there is a right-angled turn. The new hallway is 2m wide. What is the longest ladder that can be carried around the corner assuming that the ceiling is 3m high, and:
   a. The ladder has to be horizontal.
   b. The ladder can be diagonal.

Suggested Resources

Section 4.4

Section 4

THE DEFINITE INTEGRAL AND ITS APPLICATIONS

30 Hours

Outcomes

SCO: By the end of this course, students will be expected to

D1 apply and understand how Riemann sums can be used to determine the area under a polynomial curve.

D2 demonstrate an understanding of the meaning of area under a curve.

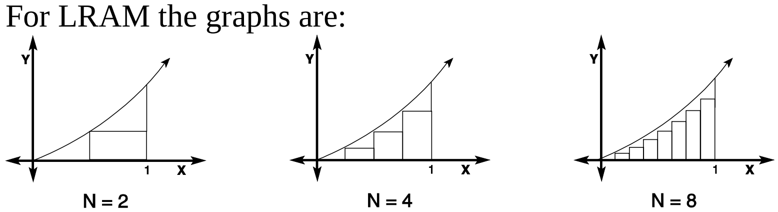
Elaboration – Instructional Strategies/Suggestions

D1/D2 Students should be introduced to this topic with a relatively simple example using the rectangular approximation method (RAM). It is extremely important that they understand that as the number of intervals, n, increases over the domain that LRAM (left hand rectangular approximation method) and RRAM (right hand rectangular approximation method) converge and approach the area under the curve.

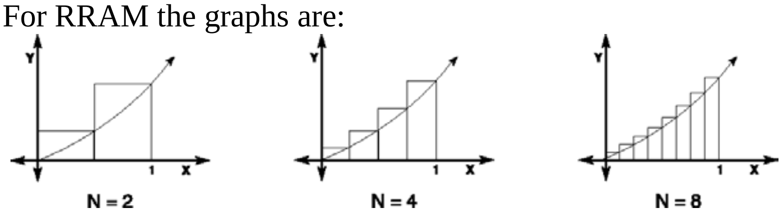
For example: Find the area under the curve of $y = x^2$ for the interval $0 \le x \le 1$.

To start set $n = 2$, then $n = 4$, and finally $n = 8$, as shown in the diagrams below.

For LRAM the graphs are:



For RRAM the graphs are:



n	LRAM	RRAM
2	0.125	0.625
4	0.219	0.469
8	0.273	0.398

Calculate $n = 10$. If you have access to a computer program that will allow you to go further with this example, then it is worthwhile to complete the task. If not this is an appropriate place to stop and discuss where the LRAM and RRAM seem to be converging. It seems to be converging to an area of 0.333 units^2.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**D1/D2**)

Use the rectangular approximation method (RAM), with a reasonable number of rectangles, to estimate the area of the region enclosed between the graph of $f(x)$, the x-axis, and $a \le x \le b$ for the following:

1. $y = x^2$, $a = -2$, $b = 2$

2. $y = \cos x$, $a = 0$, $b = \pi$

3. $y = e^{2x}$, $a = 0$, $b = 4$

4. $y = x^3 + 1$, $a = -4$, $b = 2$

Suggested Resources

Section 5.1

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **D3** express the area under the curve as a definite integral. **D2** demonstrate an understanding of the meaning of area under the curve. **D4** compute the area under a curve using numerical integration procedures	**D3/D2/D4** Students studied sigma notation in a previous course, but it is worthwhile reviewing this briefly before continuing on with this outcome. Look at $\sum_{i=1}^{5}(2i+2)^2$. Discuss as a class how the definite integral is defined as a limit of Riemann Sums. This is clearly laid out on pages 258 to 262 of *Calculus: Graphical, Numerical, Algebraic*. You may wish to assign the reading of those pages the day before you talk about them. Area Under the Curve is defined as: If $y = f(x)$ is nonnegative and continuous over a closed interval $[a,b]$, then the area under the curve $y = f(x)$ from *a* to *b* is the integral of *f* from *a* to *b*: $$A = \int_a^b f(x)dx$$ Students continue to use RAM to compute the Area Under the Curve, but they use the integral notation. Teachers Note: Omit Discontinuous Integrable Functions. Point out that in questions like $\int_0^2 (3x-9)dx$, parenthesis are not necessary. Most calculators have a numerical integration capability (referred to as NINT in the text). This technology should be used to help students calculate area under the curve.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**D3/D2/D4**)

1. Use areas to evaluate the following integrals. Check your answers with NINT:

   a. $\int_{-3}^{4} 7\,dx$

   b. $\int_{0}^{2} (3x-9)\,dx$

   c. $\int_{-2}^{2} \left(2-\sqrt{4-x^2}\right)dx$

   d. $\int_{-5}^{5} x^3\,dx$

2. The speed of a runner increased steadily during the first 4 seconds of a race. The following table gives half-second intervals. Find the lower and upper estimates for the distance traveled in the first 4 seconds.

t(s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
v(m/s)	0	2.1	3.3	4.1	4.7	5.2	5.6	5.9	6.1

Suggested Resources

Section 5.2

Outcomes

SCO: By the end of this course, students will be expected to

C7 solve initial value problems of the form $dy/dx = f(x)$, $y_0 = f(x_0)$, where $f(x)$ is a function that students can recognize as a derivative.

Elaboration – Instructional Strategies/Suggestions

C7 Students should be introduced to this in a manner that helps them to see that they are finding the **original equation**. They are working "backwards" if you will, trying to find the equation that was differentiated. Let the students know that they are solving differential equations at this point.

Example: Solve the initial value problem $\frac{dy}{dx} = 3x - 2$, $y(3) = 1.5$.

$$\int \frac{dy}{dx} dx = \int (3x - 2) dx$$

$$y = \frac{3}{2}x^2 - 2x + C$$

The solution above represents the **family of curves**. To pin down the specific solution (below), the initial condition must be used.

Substitute in the initial condition $y(3) = 1.5$

$$(1.5) = \frac{3}{2}(3)^2 - 2(3) + C$$

$$1.5 = 7.5 + C$$

$$-6 = C$$

Therefore, $y = \frac{3}{2}x^2 - 2x - 6$

Students should be shown that second derivatives represent a family of curves with 2 unknowns and therefore they need 2 additional conditions to find the specific solution.

It is advisable to do several examples of antiderivatives at this point.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C7**)

Solve the initial value problems:

1. $y' = x^2 + 3$, $y(0) = 5$

2. $y' = \sin x + \cos x$, $y(0) = 2$

3. $y'' = 2$, $y'(0) = 3$ and $y(0) = 5$

4. A family of curves has slope defined by the equation $y' = 4x - 1$. Find the equation for each of the following:
   a. x-intercept = 9.
   b. Passes through (3, 7)

5. An object starts from rest and accelerates down an incline at 5 cm/s^2.
   a. How fast is the object moving after 4s?
   b. When will the object be moving at 50 cm/s?
   c. How far down the incline did the object travel after 4s?

Suggested Resources

Section 6.1

Outcomes

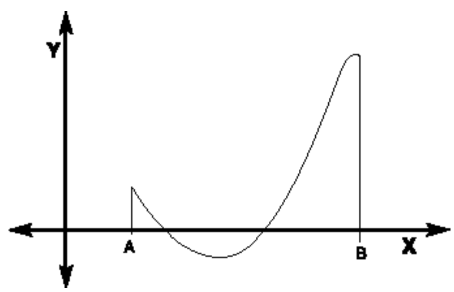
SCO: By the end of this course, students will be expected to

B18 apply rules for definite integrals.

Elaboration – Instructional Strategies/Suggestions

B18 Students should be introduced to the rules for the definite integral in table 5.3 on p. 269 of the text. Students will be familiar with similar rules for other functions so they should acquire this knowledge easily.

Exploration 2, page 273, should be carried out as a class activity using the graph below, instead of the graph in the book.



Students need to understand that the antiderivative of a function can be evaluated as follows:

$\int_a^b f(x)dx = F(b) - F(a)$, where *F* is any antiderivative of *f*.

De-emphasize the Mean Value Theorem for Definite integrals.

Teachers need to be aware that section 5.3 goes fairly heavily into antiderivatives without showing students what the antiderivatives of common functions are. Teachers should introduce some of the antiderivatives at this point. Table 6.2 on page 307 shows many of the antiderivatives that will be used in this course. Feel free to introduce the ones that you feel are pertinent at this point.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B18**)

1. Suppose that *f* and *g* are continuous functions and that $\int_1^3 f(x)dx = -2$, $\int_1^7 f(x)dx = 12$, and $\int_1^7 g(x)dx = 9$. Find:

   a. $\int_7^1 f(x)dx$

   b. $\int_3^7 f(x)dx$

   c. $\int_1^7 5g(x)dx$

   d. $\int_1^7 [f(x) - 6g(x)]dx$

2. Evaluate the following integrals:

   a. $\int_0^3 3x^2 dx$

   b. $\int_0^3 (3x - 1)dx$

   c. $\int_0^{\sqrt{3}} \left(x + \sqrt{3}\right)dx$

   d. $\int_0^4 e^{-x} dx$

Suggested Resources

Section 5.3

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **B19** apply the Fundamental Theorem of Calculus. **C8** understand the relationship between the derivative and the definite integral as expressed in both parts of the Fundamental Theorem of Calculus.	**B19/C8** Students saw the Fundamental Theorem of Calculus in section 5.3 but they were unaware of its name, and possibly its significance. At this point a formal discussion of the theorem and its importance should be conducted. The Fundamental Theorem of Calculus is used to calculate definite integrals. It has two parts: Part 1: If *f* is continuous on $[a,b]$, then the function $F(x)=\int_a^x f(t)dt$ has a derivative at every point *x* in $[a,b]$, and $\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)dt = f(x)$ Part 2: If *f* is continuous at every point of $[a,b]$, and if *F* is any antiderivative of *f* on $[a,b]$, then $\int_a^b f(x)dx = F(b)-F(a)$ Students should be encouraged to guess and check in order to help their understanding of the relationships involved in this outcome. Example 1: Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $F(x)=\int_0^x \sqrt{1+3t}\,dt$. So $f(x)=\sqrt{1+3x}$ Example 2: Use Part 2 of the Fundamental Theorem of Calculus to evaluate $\int_{-1}^2 x^3 dx$. $\int_{-1}^2 x^3 dx = \left.\frac{x^4}{4}\right\vert_{-1}^{2} = \frac{(2)^4}{4}-\frac{(-1)^4}{4} = 4-\frac{1}{4}=\frac{15}{4}$ Do exploration 1 and find the integral numerically using a calculator with a numerical integration capability (referred to as NINT in the text).

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B19/C8**)

Use the Fundamental Theorem of Calculus to find the derivatives of the following functions:

1. $g(x)=\int_x^2 \cos(t^2)\,dt$
2. $g(x)=\int_{-1}^x \sqrt{t^3+1}\,dt$
3. $g(x)=\int_{3x}^{4x} \frac{t-1}{t+1}\,dt$

Evaluate the following integrals using; a) NINT; and b) the Fundamental Theorem of Calculus.

4. $\int_0^2 \frac{7}{2}x^{\frac{5}{2}}\,dx$
5. $\int_0^2 x^{\frac{5}{2}}\,dx$
6. $\int_0^{\frac{\pi}{6}} \csc x \tan x\,dx$
7. $\int_0^{\pi} -3\cos x\,dx$
8. $\int_1^3 \left(x^2+x+\sqrt{x}\right)dx$
9. $\int_2^3 e^{3x}\,dx$

Suggested Resources

Section 5.4

Outcomes	Elaboration – Instructional Strategies/Suggestions
SCO: By the end of this course, students will be expected to **C9** construct antiderivatives using the Fundamental Theorem of Calculus. **C10** find antiderivatives of polynomials, e^{kx}, and selected trigonometric functions of *kx*	**C9/C10** In earlier outcomes **B18**, **B19**, and **C8**, students were exposed to many different antiderivatives. These outcomes are continuations of the previous outcomes. Students will use their knowledge of antiderivatives and the Fundamental Theorem of Calculus to find the antiderivatives of exponential and trigonometric functions. The indefinite integral of *f* with respect to *x* is the set of all antiderivatives of the function *f*(*x*) and is denoted $\int f(x)dx$. Remind students that $\int f(x)dx = F(x)+C$, where $F(x)$ is an antiderivative and *C* is the constant of integration. Students need to be careful to include this constant. All antiderivatives look like $F(x)+C$ which reminds us that we can't be certain which antiderivative we have. Students should be reminded to take the derivative of their answers to check that they have the correct antiderivative. Students know now that integration and differentiation are opposite operations and should easily be able to move back and forth. All of the antiderivatives are laid out in Table 6.2 on page 307 of *Calculus: Graphical, Numerical, and Algebraic*.

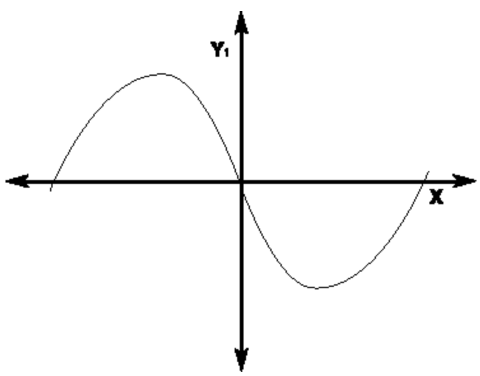
Worthwhile Tasks for Instruction and/or Assessment

Performance (**C9/C10**)

Integrate the following:

1. $\int x^3 + 3x + 2\,dx$
2. $\int e^{-4x} + 3x^2\,dx$
3. $\int \sin(6x)dx$
4. $\int x^3\,dx$
5. $\int \frac{10}{x-7}\,dx$
6. $\int \sec(2x)\tan(2x)dx$

7. Given a graph of *f*(*x*) like this:



Sketch several antiderivatives of *f*(*x*).

8. Find an antiderivative *F*(*x*) of $f(x) = 2x + 1$ that satisfies $F(0) = 2$.

Suggested Resources

Section 6.1

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **B20** compute indefinite and definite integrals by the method of substitution.	**B20** A good introduction to this outcome is trial and error. As a class, or small groups students should try to integrate functions like $\int (x+3)^4\,dx$, $\int (\sqrt{x-2})dx$, $\int (5x+3)^3\,dx$, etc. Once students have integrated these functions, get them to take the derivative to see if they did it correctly. After a discussion, students should see that a simpler method than trial and error would be more effective. Integration through substitution should be introduced at this point. Students should see that it is the reverse of the chain rule. It is an extremely powerful method of dealing with integration. The power rule for integration should be introduced first followed by trigonometric integration. You may wish to review trigonometric identities briefly before continuing on with the unit. It is extremely important that students recognize the most common mistake, which is multiplying by the wrong scalar. Students often use the inverse of the scalar that they need. Once students have firmed up their understanding of integration by substitution on the indefinite integral, the definite integral should come naturally. Some serious time should be spent on this outcome because of its importance to the study of calculus at the university level.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B20**)

Evaluate the following:

1. $\int (x^2+1)^2 2x\,dx$

2. $\int 6\sin 6x\,dx$

3. $\int x\sqrt{2x-1}\,dx$

4. $\int \frac{t+2t^2}{\sqrt{t}}\,dt$

5. $\int \frac{-4x}{(1-2x^2)^2}\,dx$

6. $\int \sqrt{\cot x}\csc^2 x\,dx$

7. $\int_1^2 (x-1)\sqrt{2-x}\,dx$

8. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (x+\cos x)\,dx$

9. $\int_0^1 x\sqrt{1-x^2}\,dx$

10. $\int \tan x\,dx$

Suggested Resources

Section 6.2

Outcomes

SCO: By the end of this course, students will be expected to

B21 apply integration by parts to evaluate indefinite and definite integrals. **(Optional)**

Elaboration – Instructional Strategies/Suggestions

B21 Integration by parts is a powerful tool that can work in situations where other techniques do not. If the class is not extremely strong it may be advisable not to complete this outcome.

A good way to begin this discussion is to derive the formula using the product rule, as outlined on page 323 of the text.

The integration by parts formula is: $\int udv = uv - \int vdu$

Students need to be aware of two main things:

1. Like in example 2, on page 324, what are chosen for *u* and *dv* make a world of difference.
2. That the formula may need to be applied several times to reach the answer.

The most common error is students making the wrong choice for *u* and *dv*. Usually we want u to be something that simplifies when differentiated and *dv* to be something that is manageable when integrated. In order to help students find the correct part of the derivative for u, there are several steps, which followed, usually arrive at the correct choice. Choose *u* in this order:

1. Natural logarithm.
2. Inverse trigonometric function.
3. Polynomial.
4. Exponential.
5. Trigonometric.

It is worthwhile to look through the standard collection of functions to see which functions we easily know how to integrate and which ones we do not. We know, for example, how to differentiate and integrate e^x and e^{cx}, but we do not know how to integrate ln *x*. This understanding can aid in the choosing of *u* and *dv*.

When integrating by repeated use of integration by parts the *u* and *dv* must be the same at each step, or the process will not work.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B21**)

Integrate the following:

1. $\int x^2 \ln x dx$

2. $\int_1^4 \ln \sqrt{x} dx$

3. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \csc^2 x dx$

Enriched (**B21**)

Integrate the following by repeated use of integration by parts:

4. $\int x^2 \sin x dx$

5. $\int e^x \cos 2x dx$

6. $\int \sec^3 x dx$

7. $\int x \tan^{-1} x dx$

Suggested Resources

Section 6.3

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to **B22** solve problems in which a rate is integrated to find the net change over time.	**B22** A good start to this lesson it to review the relationship between position of an object, velocity and acceleration. For example: Find the velocity and acceleration of an object whose position over time is represented by the function $d(t) = t^2 - t$. The velocity would be $v(t) = d'(t) = 2t - 1$. The acceleration would be $a(t) = d''(t) = 2$. Now work the question backward. Find the function that represents the position over time of an object if $a(t) = 2, v(0) = -1, d(0) = 0$. Students should be able to work this backwards and find the equations that were used in the previous example. Students need to understand that the integral of rate gives the net change. Students should do a variety of the problems in this section in order to get used to applying their knowledge to different situations. You may want to put the students in small groups and assign different questions to each group to present to the class. Then give them more to work on independently.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B22**)

1. The function $v(t)$ is the velocity in m/s of a particle that is moving along the x-axis. Find the following:
   a. Determine when the particle is moving to the right.
   b. Determine when the particle is moving to the left.
   c. Determine when the particle is stopped.
   d. Find the particle's displacement for the given time interval.
   e. Find the total distance traveled by the particle.
      i. $v(t) = 5t^2 - 10t - 15,\ 0 \le t \le 10$
      ii. $v(t) = e^{\cos t} \sin t,\ 0 \le t \le 2\pi$
   f. Graph using technology and calculate the area under the curve.

2. A ball is rolled in a straight line along a horizontal surface at a constant speed of 2 m/s. The ball reaches a straight incline that is 3 m long and rolls up the incline. As it rolls up the incline it decelerates at a constant rate of 0.5 m/s^2.
   a. Find the expression for the velocity of the ball while it is rolling up the incline.
   b. Find the expression for the position of the ball while it is rolling up the incline.
   c. When does the ball stop rolling?
   d. How far up the incline will the ball get before it starts to roll back down it?

Suggested Resources

Section 7.1

Outcomes

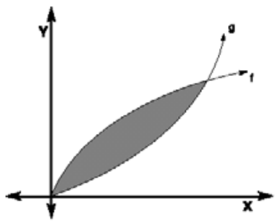
SCO: By the end of this course, students will be expected to

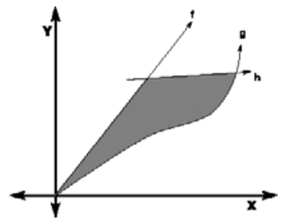
D5 apply integration to calculate areas of regions in a plane.

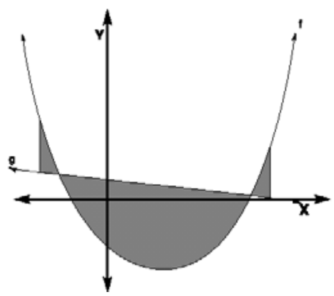
Elaboration – Instructional Strategies/Suggestions

D5 Students have found the area of regions in this course already, and they had to be aware of the zeroes of the function. Students should be reminded of this before they proceed with this outcome.

A class discussion showing a few curves like the three below is a good way to start the lesson. Ask how they believe the area could be found and if they see any points that need to be considered.







In figure 1 one curve is always greater than the other one. Therefore it is relatively easy to find the area between the curves. The points that are needed are the intersection points of the two curves.

In figure 2, the upper curve changes at $x = 1$, therefore there are two areas that need to be found separately and then combined. Again the points of intersection are needed.

Figure 3 has three distinct areas that need to be calculated and then added together. The intersection points of the two curves are needed to find the area in this plane. Look at what happens if it is done as one integral.

With the above discussions students may have been able to predict that the definition for area between curves is:

If *f* and *g* are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then the area between the curves $y = f(x)$ and $y = g(x)$ from *a* to *b* is the integral of $[f - g]$ from *a* to *b*,

$$A = \int_a^b [f(x) - g(x)]dx$$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**D5**)

1. Determine the area bounded by the following graphs:

   a. $y = \cos 2x$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$

   b. $y = x^{\frac{1}{2}}$, $y = 0$, $x = 4$, $x = 9$

   c. $y = 25 - x^2$, $y = 0$, $x = 0$, $x = 8$

   d. $y = x^2$, $y = 18 - x^2$, $x = 0$, $x = 5$

   e. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$

   f. $y = \sec x \sin x$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$

   g. $y = \begin{cases} 1 - x^2 & x \leq 0 \\ 1 + x^2 & x > 0 \end{cases}$, $x = -1$, $x = 1$

2. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at (1,1), and the x-axis.

3. Find the number *a* such that the line $y = a$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal areas.

Suggested Resources

Section 7.2

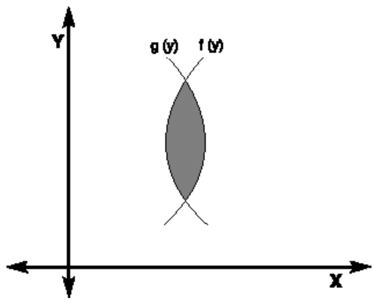
Outcomes

SCO: By the end of this course, students will be expected to

D5 apply integration to calculate areas of regions in a plane.

Elaboration – Instructional Strategies/Suggestions

D5 Once students have found the area under the curve and are comfortable with the technique, integration with respect to y should be introduced. Show a graph like the one below.



In this graph the boundaries of the region are more easily described by functions of y rather than functions of x. We use the same technique but the area between the curves becomes:

$A = \int_a^b [f(y) - g(y)]dy$, where the boundaries of the shape are $y = a$ and $y = b$.

Worthwhile Tasks for Instruction and/or Assessment	Suggested Resources

Section 5 (Optional)

TECHNIQUES OF INTEGRATION

10 Hours

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to	**D6** This outcome, although optional, is highly valuable to students who are going to continue in a science field, and is recommended to be completed, time permitting.
D6 apply integration (by slices or shells) to calculate volumes of solids. **(Optional)**	A possible introduction of this outcome is a discussion of what volume really is. Students have a good understanding from a previous course that volume is usually area • height • a scalar. Of course there are many variations on this, and many of them can be discussed here.

Discuss how solids can be divided into cross sectional areas that the formulas can be discovered for. Both square and circular cross sectional areas should be discussed, as they are the most common.

The volume of a solid of known cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)dx$$

The four-step process to applying this formula is as follows:

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

The washer question is a great way to rap up this unit. Exploration 1 on page 387 is an example of a washer type question.

Like outcome **D5**, sometimes integration with respect to y is the better way to go. Again the formula is replaced with:

$V = \int_a^b A(y)dy$, where the boundaries of the shape are $y = a$ and $y = b$.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**D6**)

1. Find the volume of the solid created by revolving the region formed by $y = x^{\frac{2}{3}}$, $x = 1$, $y = 0$ about the y-axis.

2. Find the volume of the solid created by revolving the region formed by $y = \frac{1}{x}$, $x = 1$, $x = 3$, $y = 0$ about the x-axis.

3. Find the volume of the solid created by revolving the region formed by $y = 4x - x^2$ and the x-axis about the x-axis.

4. Find the volume of the solid created by revolving the region formed by $y = \sqrt{\cos x}$, $y = 0$, $x = 0$, $x = \frac{\pi}{2}$ about the x-axis.

5. Find the volume of the solid created by revolving the region formed by $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$ about the x-axis.

6. A square-based pyramid is bounded by $y = x$, and the x-axis. Find the volume of the solid over the interval $0 \le x \le 4$.

Suggested Resources

Section 7.3

Outcomes

SCO: By the end of this course, students will be expected to

C11 construct slope fields using technology and interpret them as visualizations of differential equations. **(Optional)**

Elaboration – Instructional Strategies/Suggestions

C11 Slope fields should be used to help students visualize solutions to integration.

The usual slope field of the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is a plot of short line segments with slopes $f(x, y)$ for the lattice of points (x, y) in the plane.

Slope fields describe the "flow" of the solutions to a differential equation. A differential equation has lots of solutions. By examining the slope field we are able to gain insight into the differential equations. In particular, if we choose an initial condition, (x_0, y_0), then the solution of the differential equation through this point can be "seen" by following the slope field.

For a function of one or two variables, the slope field is the field (graph) whose value at any point is the slope of the function at that point.

Students should complete Exploration 1 in order to get an understanding of how a slope field works.

Teachers may want to show this using a computer and a projector provided the software is available.

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C11**)

1. Sketch, using technology, the following slope fields:

   a. $y' = x + y$ if $y(0) = 1$

   b. $y' = y \sin 3x$

   c. $y' = xy + y$ at (0, 2)

2. Using $y' = x^2 + y^2 + 1$ do the following:

   a. Sketch the slope field.

   b. Sketch the solution passing through:

      i. (0,0)

      ii. (0,1)

      iii. (1,0)

Suggested Resources

Section 6.1

Outcomes

SCO: By the end of this course, students will be expected to

B23 solve a differential equation of the form *dy/dx = g(x)h(y)*, in which the variables are separable. **(Optional)**

Elaboration – Instructional Strategies/Suggestions

B23 This method allows us to solve some differential equations which can be separated into two sides, the side with the independent variable and the side with the dependent variable. Once the separation has occurred, each side is integrated and then the dependent variable is solved for.

For example: Solve $\frac{dy}{dx} = (y-2)(x+6)$.

The solution would go as follows:

$$\frac{1}{(y-2)}\frac{dy}{dx} = (x+6)\ , \quad y \neq 2$$

$$\int \frac{1}{(y-2)}\frac{dy}{dx}dx = \int (x+6)dx$$

$$\int \frac{1}{(y-2)}dy = \int (x+6)dx$$

$$\ln|y-2| + C_1 = \frac{x^2}{2} + 6x + C_2$$

$$\text{let } C_3 = C_2 - C_1$$

$$y - 2 = e^{\frac{x^2}{2}+6x+C_3}$$

$$\text{let } C = e^{C_3}$$

$$y = Ce^{\frac{x^2}{2}+6x} + 2$$

Make sure that you show students how to deal with questions like these when initial conditions are present.

Worthwhile Tasks for Instruction and/or Assessment	**Suggested Resources**
Performance (**B23**) Solve the following: 1. $\frac{dy}{dx} = (x+5)(y-3)$ 2. $y' = \frac{3x}{y}$ 3. $y' = xy + x$ 4. $xy' + y = y^2$	Section 6.2

Outcomes

SCO: By the end of this course, students will be expected to

B24 solve problems involving exponential growth and decay. **(Optional)**

Elaboration – Instructional Strategies/Suggestions

B24 This outcome deals with differential equations of the form $y' = ky$ that have solutions $y = y_0 e^{kt}$. Remind students that derivatives that have the original function in them are always from functions that involve e^t.

This outcome allows the student to explore different fields where calculus is important. It is important as a teacher to expose students to as many different uses as possible.

Example: The rate of growth of the number of bacteria in a culture is proportional to the number of bacteria present in the culture. If the initial population of the culture is 6000 and one hour later it is 9000, find the equation defining the culture's population at any time *t*, measured in hours.

Let t = time (hours)
y = population

$y' = ky$

$y = y_0 e^{kt}$

Sub in (0, 6000) $y = 6000e^{kt}$

Sub in (1, 9000) $9000 = 6000e^{k(1)}$

$9000 = 6000e^{k}$

$\ln 9000 = \ln 6000e^{k}$

$\ln 9000 = \ln 6000 + \ln e^{k}$

$\ln 9000 - \ln 6000 = k \ln e$

$\ln \dfrac{9000}{6000} = k$ $\quad\quad k = 0.405$

Therefore, $y = 6000e^{0.405t}$

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B24**)

1. A bacteria population is growing at a rate that is proportional to its size at that instant. The population is presently 100000. One hour ago it was 90000.
   a. Find the equation that gives the population at any time *t* from now.
   b. What will the population be one day from now?
   c. When will the population double?

2. The number of bacteria grows from 40 to 100 in 3 hours. The rate of increase is proportional to the number of bacteria present.
   a. Find the equation that will give the number of bacteria at any time *t* after the initial count.
   b. When will the count be tripled?

3. In a chemical reaction one substance A is converted into a second substance B. Assume 5% of substance A is converted into substance B in 10s.
   a. When will 50% of substance A be converted into substance B?
   b. When will only 1% of substance A remain?

4. Ann invests principal P to be invested at 8%. How much must be invested so that 20 years later she has $500 000.

5. How long will it take $1000 to double if it is invested at 7% compounded:
   a. Annually
   b. Monthly
   c. Daily

1. A radioactive substance decreases in mass from 20 g to 18 g in 24 hours.
   a. Find the equation that defines the mass of radioactive substance left after *t* hours.
   b. When will half of the substance be left?

Suggested Resources

Section 6.4

Outcomes	**Elaboration – Instructional Strategies/Suggestions**
SCO: By the end of this course, students will be expected to	**B25** It is advantageous to explore this outcome using technology. The text lists several programs that can help you introduce the topic to your students with technology.
B25 apply Euler's method to find approximate solutions to differential equations with initial values. **(Optional)**	A little time should be spent investigating Euler's Method for approximating solutions to the differential equation $\frac{dy}{dx} = f(x, y)$ and the initial condition $y(x_0) = y_0$ manually. $L(x) = y_0 + f(x_0, y_0)(x - x_0)$ then $y_1 = L(x_1) = y_0 + f(x_0, y_0)dx$ and $y_2 = L(x_2) = y_1 + f(x_1, y_1)dx$ … $y_n = L(x_n) = y_{n-1} + f(x_{n-1}, y_{n-1})dx$

Worthwhile Tasks for Instruction and/or Assessment	**Suggested Resources**
Performance (**B25**) 1. Use Euler's method to compute the approximate *y*-values y_1, y_2, y_3, and y_4 for $y' = y - 3x$, $y(1) = 0$ using a step size of 0.5. 2. Use Euler's method to estimate $y(0.5)$, using a step size of 0.1, if $y' = xy + y$ and $y(0) = 1$.	Section 6.6