Extended Mathematics 11

Guide

NOVA SCOTIA
NOUVELLE-ÉCOSSE

2017

Extended Mathematics 11

Prepared by the Department of Education and Early Childhood Development

This is the most recent version of the current curriculum materials as used by teachers in Nova Scotia.

Extended Mathematics 11

Implementation Draft

June 2018

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Measurement
20–25 hours

GCO: Students will be expected to develop spatial sense and proportional reasoning.

SCO M01 Students will be expected to solve problems that involve the application of rates. [CN, PS, R]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M01.01 Interpret rates in a given context, such as the arts, commerce, the environment, medicine, or recreation.
M01.02 Solve a rate problem that requires the isolation of a variable.
M01.03 Determine and compare rates and unit rates.
M01.04 Make and justify a decision using rates.
M01.05 Represent a given rate pictorially.
M01.06 Draw a graph to represent a rate.
M01.07 Explain, using examples, the relationship between the slope of a graph and a rate.
M01.08 Describe a context for a given rate or unit rate.
M01.09 Identify and explain factors that influence a rate in a given context.
M01.10 Solve a contextual problem that involves rates or unit rates).

Scope and Sequence

Mathematics 10	EXT Mathematics 11	Mathematics 12
M02 Students will be expected to apply proportional reasoning to problems that involve conversions between SI and imperial units of measure. **FM01** Students will be expected to solve problems that involve unit pricing and currency exchange, using proportional reasoning.	**M01** Students will be expected to solve problems that involve the application of rates.	

Background

In Mathematics 8, students explored the difference between a rate and a unit rate (8N05). In Mathematics 9, they used the concept of scale factor to create enlargements and reductions of 2-D shapes (9G03). In Mathematics 10, students explored the concept of slope as a measure of rate of change (RF03). In Mathematics 10 students also solved problems that involve unit pricing and currency exchange by using proportional reasoning (FM01).

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for M01 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

In this unit, students will be expected to represent a rate in different ways. Students will also be expected to use rates to solve problems and make decisions.

Students will explore "rates of change" in a variety of practical situations such as medicine, trajectory, travel, and economic and population growth. They should begin with simple models represented by linear functions in which the average rate of change is constant; students should then progress to models in which the rate of change is variable, which implies some sort of curve.

Students should explore some situations like heart rate, the rate at which runs are scored in a ball game, birth rate, population growth rate, and employment rate and learn how to calculate these rates.

For example, heart rate can be calculated by counting the number of beats in a time interval, divided by the time interval.

$$\text{heart rate} = \frac{\text{number of pulse beats in a time period}}{\text{time}}$$

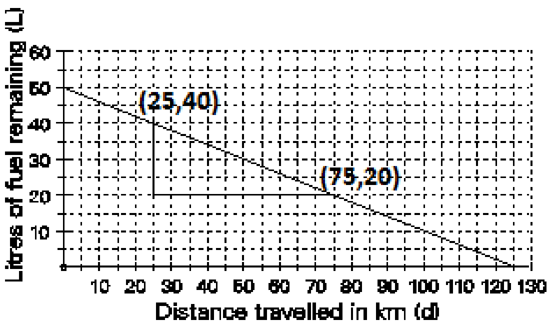
Rates are important because they tell students how one thing is changing in relation to another.

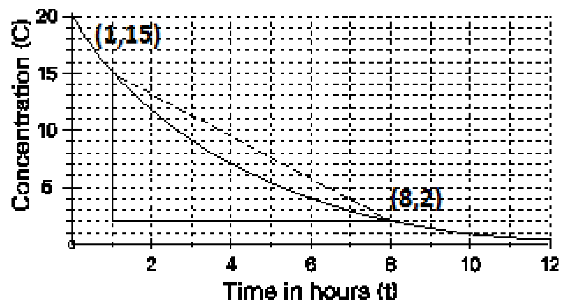
Students might read about an oil spill in the newspaper and how pollution control authorities monitor the spread by estimating the area covered by the oil at different times, then calculating the rate of spread in order to determine how fast the area of the oil slick is growing. Students can undergo a similar process to help them understand rates. Students might consider the rate of change in an oil spill's area of spread to be represented by

$$\frac{\Delta A}{\Delta t} = \frac{\text{change in area}}{\text{change in time}} = \frac{\text{ft.}^2}{\text{hour}}$$

Specifically, if students were given the following information, $\frac{\Delta A}{\Delta t} = \frac{5 \text{ ft.}^2}{2 \text{ hour}} = 2.5\frac{\text{ft.}^2}{\text{hour}}$, it would mean that during the first 24-hour period after the spill, the oil slick's area was increasing at an average rate of 5 ft.^{2} every two hours or that it was increasing at an average rate of 2.5 ft.^{2} every hour.

Students should be able to discern different rates of change visually. Have students look at a graph and estimate how quickly the dependent value is changing with respect to the independent value. Students should also be able to determine if the dependent value is changing at the same rate all the time or if the rate of change varies over time. If a function is given by $y = f(x)$, then students should talk about "the average change in y with respect to x."





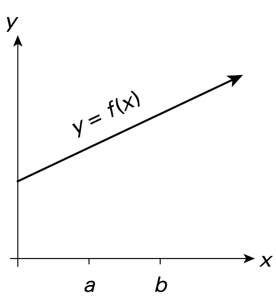
$$\frac{\Delta L}{\Delta d} = \frac{20-40}{75-25} = -\frac{20}{50} = -0.4\frac{L}{km}$$

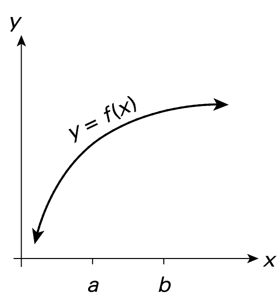
Constant change for all distances in domain.

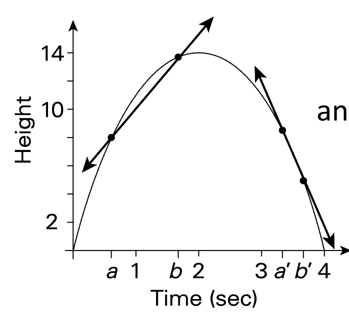
$$\frac{\Delta C}{\Delta t} = \frac{2-15}{8-1} = -\frac{13}{7} = -1.86\frac{\text{concentration}}{\text{hour}}$$

Non-constant change. Average change for time between 1 and 8 hours.

$\dfrac{\Delta y}{\Delta x}$ represents the change in y per unit change in x. If this change is constant, the relationship is linear. For non-linear functions, this average change will not be constant.







For non-linear functions, the rate of change is not the same everywhere. Motion is often represented as a curve. If the graph slopes upward to the right, it means the rate of change is positive and that function is said to be increasing. For example, for each second that passes as a soccer ball is kicked into the air, the soccer ball's height continues to increase until the ball reaches its maximum height. At the maximum height, the rate of change is zero. As the ball begins to fall to the ground, the rate of change is negative. When a function's rate of change is negative, that function is said to be decreasing.

Students should recognize when they have been given a rate of change and what specific variables are being compared. For example, if students were told that air is being pumped into a spherical balloon at a rate of 20 in.^{3}/min., they should understand that the average rate of change is $\frac{\Delta V}{\Delta t}$ = 20 in.^{3}/min. and should be able to describe this change as an increase in volume of 20 in.^{3} every minute. If the function is linear, the rate is constant; otherwise the rate is changing over time and $\frac{\Delta V}{\Delta t}$ represents an average change rather than a constant change.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Serena used proportional reasoning to do the following conversion: 0.78 kg = ___ mg
  She wrote: $\frac{10\,000 \text{ mg}}{1 \text{ kg}} = \frac{? \text{ mg}}{0.78 \text{ kg}}$
- Where did she make an error? Complete the conversion correctly.
- It takes Aisha 8 hours to wash 12 cars. How long will it take her to wash 21 cars?
- Vadim buys a package of 15 pencils for $4.50 at the corner store. Angela buys a box of 50 pencils at the grocery store for $14.00. Which is the better buy?

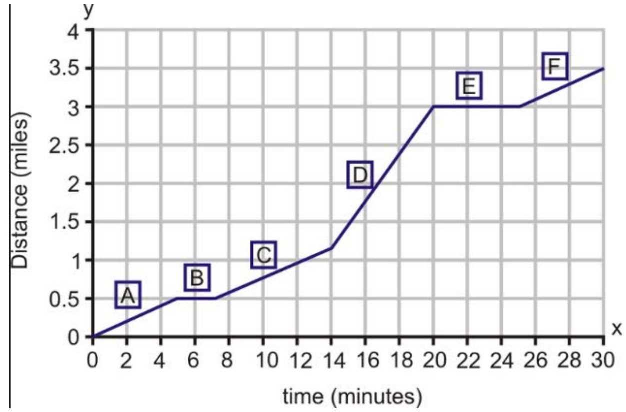
  Paper towels are sold in a 2-roll package for $2.49 and a 12-roll package for $12.99.

  (a) What package has the lower unit price?

  (b) If you need 12 rolls of paper towels, which is the better buy: one 12-roll package or six 2-roll packages?

  (c) When deciding which package size is the better buy for you, what should you consider in addition to unit price?

- The graph shown to the right shows Sam's distance, in metres, from his apartment over a 30 minute period of time. Describe his motion during each interval.



Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Each of your fingernails grows at about 0.05 cm/week. Each of your toenails grows at about 0.65 cm/year. Do your toenails or fingernails grow faster?
- What rates are being implied in the following infographic?

- The following table represents the average cost per litre of regular gasoline in Nova Scotia for the first six months of 2013.

Month	January	February	March	April	May	June
cost/volume ($/L)	121.2	123.6	131.4	139.2	133.1	131.9

Using a graph, determine which two months had the smallest amount of decrease in the price of gas. Give reasons why you think this occurred.

- The average monthly temperature in °C for Truro, Nova Scotia is recorded in the chart below. The sampling period for this data covers 30 years.

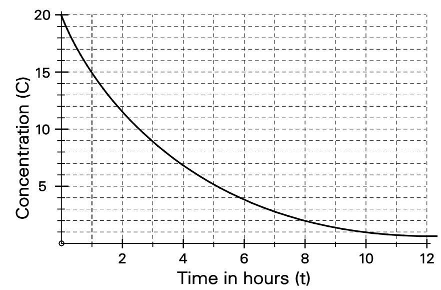
Month	**Average Temperature (°C)**
January	– 6.9
February	– 6.5
March	– 1.8
April	3.9
May	9.8
June	14.7
July	18.4
August	17.8
September	13.4
October	7.7
November	2.8
December	– 3.5

Using a graph, identify the two months with the greatest rate of change in temperature from one to the next. Which two months had the least rate of change?

- Betty earns $463.25 in five weeks. Assuming she does not receive any raises, how much will she earn in two years?
- A 12-bottle case of motor oil costs $41.88. A mechanic needs to order 268 bottles of motor oil. If she can only order by the case, how much money does she spend?
- A jet-ski rental operation charges a fixed insurance premium plus an hourly rate. The total cost for two hours is $50 and for five hours is $110. Determine the hourly rate to rent the jet-ski.
- As a custodian, John makes a cleaning solution by mixing 30 g of concentrated powdered cleanser into 2 L of water. At the same rate, how much powder will he need for 5 L of water?
- An office has decided to track how much paper it uses to reduce waste. At the end of each month, the administrative assistant records the total number of sheets used and their weight. If paper weighs 10.8 lb. for every 500 sheets, how much will 700 sheets weigh?
- Saad jogs at a rate of 10 kmh. When he jogs at this rate for two hours, he burns 760 calories. Claudette jogs more slowly at 8 kmh, burning 150 calories in 30 minutes. If Saad jogs for three hours, how much longer will Claudette have to jog in order to burn the same number of calories?

- The low temperature in Summerside, Prince Edward Island, for a certain day was 9°C, at 4:30 a.m. The temperature then rose steadily at a constant rate until the high temperature of 25.5°C was recorded at 3:30 p.m. A weather forecaster predicted that the temperature would increase at the same rate for the next day, from a low of 11°C at 4:00 a.m. At that rate, what will be the temperature at 1:00 p.m. on the next day?

- When tested for diabetes, Sam was asked to consume a sweetened drink. After she finished drinking, her blood sugar level decreased slowly as time passed. Blood sugar concentration (C) is a function of time (t) in minutes.

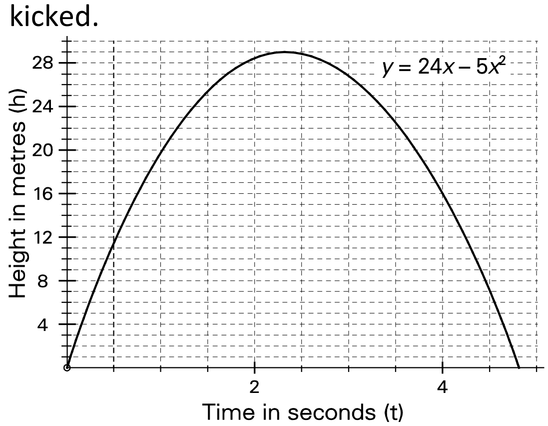


  (a) Is the rate of change positive or negative over the first three hours?

  (b) Referring to the graph shown at the right, describe the rate of change of concentration in terms of time as positive or negative, and as constant or changing.

  (c) Estimate $\dfrac{\Delta C}{\Delta t}$ for the period from one hour to three hours after Sam finished drinking. Explain what it means.

- A football is kicked. The height, $h(t)$, of the football is depicted on the graph below. The height is in metres above the ground and the time is in seconds since the football was kicked.



  (a) What is the maximum height that the ball reaches?

  (b) When is the height of the ball increasing?

  (c) Referring to the graph shown above, describe the rate of change of height compared to time as positive or negative and as constant or changing.

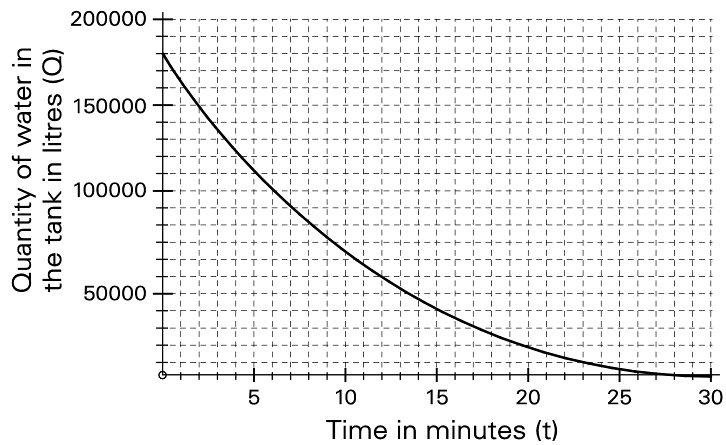
  (d) What is the average rate of change of the ball's height for the interval [1, 4]?

- The number of litres of water in a tank (Q) can be described by the equation $Q(t) = 200(30 - t)^2$, where t is the number of minutes since the tank began to drain.

  (a) What is the average rate at which the water flows out during the first 10 minutes?

  (b) What is the average rate at which the water flows out during the 10-minute period from 10 minutes to 20 minutes?

(c) The graph of the function $Q(t) = 200(30 - t)^2$ is shown below. Explain how you could describe how the average rate at which the water flows out of the tank is changing as time progresses.



- A horse is running a five-furlong race. As the horse passes each furlong marker (F), a steward records the time elapsed (t) since the beginning of the race, as shown in the table below:

F	0	1	2	3	4	5
t (sec)	0	25	38	49	65	79

(a) How long does it take the horse to finish the race?

(b) What is the average speed of the horse during the last three furlongs of the race?

(c) During what part of the race is the horse running the fastest?

- Kim had a job mowing a lawn last Saturday. His employer paid him $50 to cut his lawn. Five hours later Kim was finished and walked home. On his way home, the mower sprung a leak and left a trail of gas. The tank was leaking 0.1 litres per minute, and the cost of gas is $1.339 per litre. If it took Kim one hour to get home, draw a graph that shows the relationship between time and net income for Kim's mowing.

- Kiyomi thinks that when finding the average rate of change, there is no difference whether the function is linear or non-linear. Would you agree with Kiyomi? Explain your thinking.

- The quantity (Q) of a chemical that is responsible for elasticity in human skin is given by $Q(t) = 100 - 10\sqrt{t}$, where t is the age of a person.

(a) Find $Q(0)$, $Q(25)$, and $\dfrac{Q(25) - Q(0)}{25}$.

(b) Describe what each of the values in part (a) represent.

(c) Find $Q(49)$, $Q(81)$, and $\dfrac{Q(81) - Q(49)}{81 - 49}$.

(d) Describe what each of the values in part (c) represent.

(e) Describe how the average rate of change in elasticity in the human skin changes as a person ages.

- A 25-foot ladder is leaning against a vertical wall with its base two feet from the wall. The floor is slightly slippery and the base of the ladder slips farther away from the wall at the constant rate of 0.2 inches per second.
  (a) Draw a graph, with the axes labelled, that illustrates this rate of change.
  (b) Complete the following chart:

Time since ladder begins to slip (seconds)	Distance foot of ladder is from the wall (inches)	Size of angle that the ladder makes with the wall (degrees)
0	2	
1	2.2	
2		
3		
4		

  (c) For the time interval $0 \leq t \leq 3$, determine the average rate at which the angle that the ladder makes with the wall is changing. Include units.
  (d) For the time interval $0 \leq t \leq 4$, determine the average rate at which the angle that the ladder makes with the wall is changing. Include units.
  (e) Is the angle changing at a constant rate? Explain.
- A company's productivity (P) measured by the number of items produced is a function of the number of people (n) working on a job. For the interval [20, 50] the rate of change for P in terms of n is $\frac{\Delta P}{\Delta n} = 200$.
  (a) Draw a graph, with the axes labelled, that illustrates this rate of change.
  (b) Describe the meaning of the rate of change in terms of the number of people and the number of items produced.
- The number of four-litre containers of paint that a company sells (N) depends on the cost (c) of the container of paint. For prices between \$30 and \$45, the rate of change for the number of gallons sold in terms of the price charged per four-litre paint container can be described by $\frac{\Delta N}{\Delta c} = -\frac{500}{2}$.
  (a) Draw a graph, with the axes labelled, that illustrates this rate of change.
  (b) What does this rate of change mean in terms of the number of four-litre containers of paint the company sells and the cost of the container of paint?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Use the Heart Rate activity which is meant to review rates of change and to introduce unit analysis and/or solving questions by proportions. This activity reminds students that they can use rates of change to produce estimates. The activity also reminds students that rates of change can be readily viewed by observing graphs.
- Give the students a graph, such as the one created by Pew Research Center about the use of social networking by age group, 2005–2013. This graph can be found on page 4 of their 2013 report, *72% of Online Adults Are Social Networking Site Users* (Brenner and Smith) at www.pewinternet.org/files/old-media/Files/Reports/2013/PIP_Social_networking_sites_update_PDF.pdf.
- Using the graph, discuss the change in use of social networking sites from 2005 to 2011.

- Pose the following questions to promote student discussion:
  - What does a positive slope represent? A negative slope?
  - What does a horizontal slope represent?
  - What does the steepness of the slope represent?
  - Locate the largest rate of change and explain what it represents.

- Students have previously been introduced to rate as a comparison between two things with different units. Invite students to talk about comparisons they make in their lives, such as fuel consumption, speed, prices in supermarkets, or monthly fees of a fitness centre.

- Providing students the opportunity to work with rates in real-life scenarios should reinforce the students' understanding of rates, both their usefulness and their reasonableness. The focus here is to describe a situation in which a given rate might be used and is most useful. For example, ask students to answer questions such as the following if they were planning a road trip to Las Vegas:
  - Is it reasonable to discuss a road trip to Las Vegas in terms of *m*/*s*?
  - What rate(s) could be used to describe this road trip?
  - What factors might affect your rate of speed on this trip?
  - What factors might affect your fuel consumption?

- Students should be provided with several examples using various units and rates. Teachers should take this opportunity to work with problems involving rates that can be solved using equivalent ratios, proportional reasoning, and/or unit analysis. Examples may include the following:
  (a) During a Terry Fox Run, student volunteers distribute 250 mL cups of water to participants as they cross the finish line. Each volunteer has a cooler that can hold 64 L of water. How many cups of water can each volunteer dispense?

  (b) Loose-leaf paper costs $1.49 for 200 sheets or $3.49 for 500 sheets. What is the least you will pay for 100 sheets? 1600 sheets?

- In groups of two, ask one student to state the rate while the other student states it as a unit rate. Examples might include the following:

Rate	**Unit Rate**
Ten additional hats sell for every additional 120 people who attend the concert.	12 people to one hat
The amount of fuel in the car's tank decreases by 4.2 litres every 100 km travelled.	0.042 litres per km or 23.8 km/litre

- In groups of two, ask one student to state the rate of change in words, while the other student states it using delta notation. Examples might include the following:

Rate (description)	Rate (delta notation)
The water flows into a conical water tank at a rate of 20 ft.^{3}/min.	$\frac{\Delta V}{\Delta t}$ = 20 ft.^{3}/min.
A man walks away from a lamp at a rate of 5 metres/min.	$\frac{\Delta d}{\Delta t}$ = 5 m/min.
A tutor is scheduling her time each week and considers a commitment of two hours per person she tutors.	$\frac{\Delta h}{\Delta p}$ = 2 hr./person
A child lets out the string of a kite at a rate of 2.5 ft./second.	$\frac{\Delta L}{\Delta t}$ = 2.5 ft./min.
Ten additional hats sell for every additional 120 people who attend the concert.	$\frac{\Delta h}{\Delta n} = \frac{1}{12}$ hats/person or $\frac{\Delta n}{\Delta h}$ = 12 persons/hat
The amount of fuel in the car's tank decreases by 4.2 litres for every 100 km travelled.	$\frac{\Delta L}{\Delta km} = \frac{4.2}{100}$ litres/km or $\frac{\Delta km}{\Delta L}$ = 23.8 km/litre

- In groups of two, ask one student to state the rate using delta notation while the other student describes its meaning. Examples might include the following:

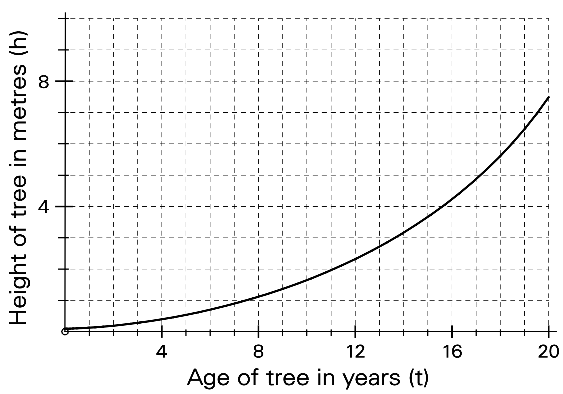
Rate (delta notation)	Rate (description)
$\frac{\Delta V}{\Delta t}$ = 20 ft.^{3}/min.	The volume is increasing by 20 ft.^{3} every minute, or there is an increase in volume of 20 ft.^{3} for every additional minute.
$\frac{\Delta d}{\Delta t}$ = 5 m/min.	The distance is changing by five metres every minute, or there is an increase of five metres in distance for every additional minute.
$\frac{\Delta h}{\Delta p}$ = 2 hr./person	The hours are changing by two for every person, or there is an additional two hours for every additional person.
$\frac{\Delta L}{\Delta t}$ = 2.5 ft./min.	The length is changing by 2.5 feet every minute, or there is an increase of 2.5 feet for every additional minute or the length increases by five feet every two minutes.
$\frac{\Delta h}{\Delta n} = \frac{1}{12}$ hats/person	There is one hat for every 12 people present, or one in 12 people wear a hat.
$\frac{\Delta n}{\Delta h}$ = 12 persons/hat	For every 12 people there is one hat, or 12 people share one hat.
$\frac{\Delta L}{\Delta km} = \frac{4.2}{100}$ litres/km	The car uses 4.2 litres of fuel for every 100 km driven, or the number of litres of fuel used increases by 4.2 every 100 km.
$\frac{\Delta km}{\Delta L}$ = 23.8 km/litre	The car drives 23.8 km using one litre of fuel.

- Present students with graphs such as the ones shown below and ask them to
  - o describe the rate of change as positive or negative, and as constant or changing
  - o calculate the rate of change for a specific interval
  - o use delta notation to describe the average rate of change
  - o state the meaning of the rate of change for a specific interval

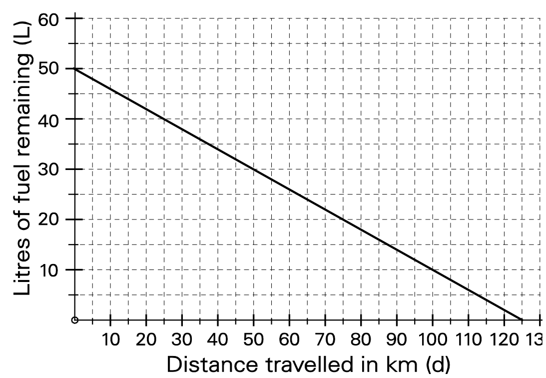
The mass of a polar bear in kg is shown in terms of its girth in cm.

The temperature of food placed in cold storage.

Height of a particular type of tree in metres as a function of its age in years.



Number of litres of fuel remaining as a function of the number of km driven.



Suggested Models and Manipulatives

- graph paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- average rate of
- change
- constant change
- negative rate of change (decreasing)
- positive rate of change (increasing)
- unit rate

Resources/Notes

Internet

- *72% of Online Adults Are Social Networking Site Users* (Brenner and Smith 2013) www.pewinternet.org/files/old-media/Files/Reports/2013/PIP_Social_networking_sites_update_PDF.pdf.

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada) Sections 8.1–8.2, pp. 452–469

Notes

SCO M02 Students will be expected to solve problems that involve scale diagrams, using proportional reasoning. [CN, PS, R, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M02.01 Explain, using examples, how scale diagrams are used to model a 2-D shape or a 3-D object.
M02.02 Determine, using proportional reasoning, the scale factor, given one dimension of a 2-D shape or a 3-D object and its representation.
M02.03 Determine, using proportional reasoning, an unknown dimension of a 2-D shape or a 3-D object, given a scale diagram or a model.
M02.04 Draw, with or without technology, a scale diagram of a given 2-D shape according to a specified scale factor (enlargement or reduction).
M02.05 Solve a contextual problem that involves scale diagrams

Scope and Sequence

Mathematics 10	Mathematics 11	Mathematics 12
M02 Students will be expected to apply proportional reasoning to problems that involve conversions between SI and imperial units of measure.	**M02** Students will be expected to solve problems that involve scale diagrams, using proportional reasoning.	

Background

In Mathematics 9, students were introduced to scale factors and scale diagrams (9SS03). Students explored the concepts of enlargements and reductions. They also determined the scale factor given the scaled diagram of two-dimensional images, and used a scale factor to create an image from its original figure.

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for M02 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

In this unit, students will use scale diagrams involving 2-D shapes before moving on to 3-D objects.

Although students were exposed to scale factors and scale diagrams of 2-D shapes in Mathematics 9, a review of these concepts is important. This is the first time that a variable, such as k, will be used to represent the scale factor, $k = \frac{\text{diagram measurement}}{\text{actual measurement}}$.

In this Measurement unit, using scale factors and measurements, students should be able to determine the dimensions of a reduced or enlarged object.

Students are expected to determine scale factors from a variety of sources, such as models and scale diagrams, by using corresponding lengths. As students work through different examples, they should recognize that the scale factor for an enlargement is greater than one and the scale factor of a reduction is between 0 and 1. When working with problems, students should be exposed to scale factors in a variety of forms including decimals, fractions, and percentages. If available, students should be encouraged to explore scale diagrams with the use of technology such as Geometer's Sketchpad (Key Curriculum Press 2013), Google SketchUp (SketchUp 2013) or GeoGebra (International GeoGebra Institute 2013).

Review of the conversion of different units in both the metric and imperial systems may be necessary. It is also important to emphasize the meaning of units so students better understand the concept of scale and can gain a visual appreciation of the object that is being reduced or enlarged.

Students will extend their work with scale factors and scale diagrams of 2-D shapes to scale factors and scale diagrams of 3-D objects. This will be their first exposure to relating scale factors to a 3-D object.

Students will use a scale factor to determine unknown measurements of similar 3-D objects. This would be a good opportunity to bring in a regular-size cereal box and a similar jumbo-size cereal box. Students can create nets of the boxes and record the measurements. Ask if the boxes are similar and why. Are the dimensions related by a scale factor? Students should be able to determine if corresponding measurements of the boxes (length, width, and height) are proportional.

Students should also be able to use a given scale factor to determine the unknown dimensions of a 3-D object. If the dimensions of a scale drawing of a patio chair are 2 cm × 1.5 cm × 4 cm, for example, and a scale factor of 1:30 is applied, students are expected to determine the actual dimensions of the patio chair.

It is important to use various everyday objects and apply a given scale factor when asking students to determine the dimensions of the actual object or its image.

Note: While scale factors may sometimes be written as 1 cm = 2 km for example, this does not mean that these are equivalent, and it could be more accurately written as 1 cm represents 2 km or 1 cm: 2 km.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

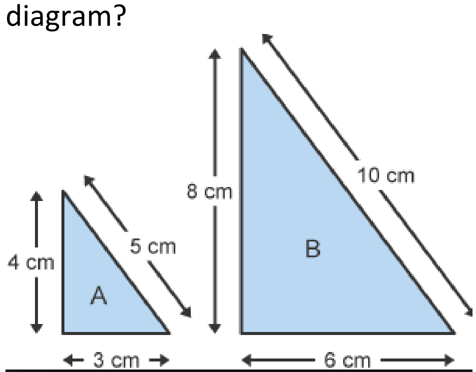
Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

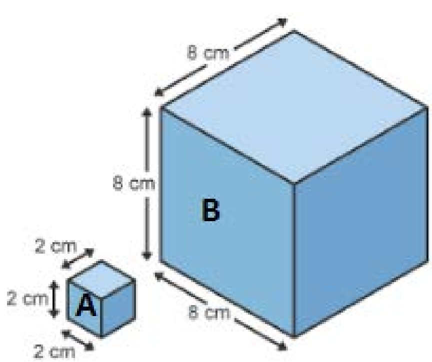
Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Filipo bought a second-hand treadmill online. It would only register in miles. Describe a conversion factor that could be used to estimate a conversion from miles to kilometres or vice versa.
- What is the scale factor for Triangle A to transform into Triangle B in the following diagram?



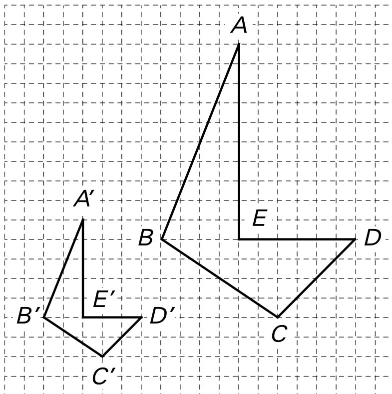
- What is the scale factor for Cube A to transform into Cube B in the following diagram?



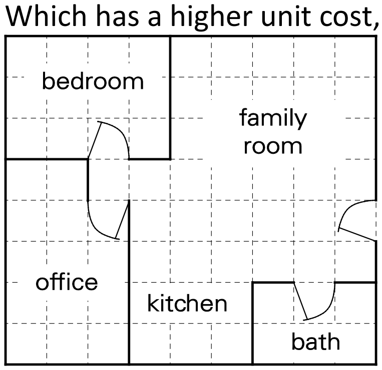
Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

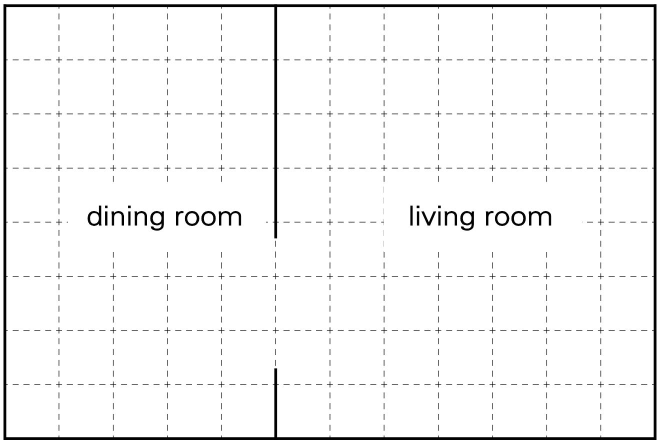
- A circle has been transformed so that its image radius is 14 cm. If the scale factor is 0.4, what is the radius of the original circle?
- Determine the scale factor from the diagram below.



- ΔABC is similar to ΔSTU. If the sides of ΔABC measure 5.2, 3.8, and 6.4 respectively, and the scale factor is 2:3, what is the perimeter of ΔSTU?
- In the following blueprint, each grid mark represents 4 feet.
  - (a) Ceramic tile costs $5 per square foot. How much would it cost, before tax, to purchase tiles for the bathroom?
  - (b) Carpet costs $18 per square yard. The bedroom and living room are to be carpeted. How much would the carpet cost, before tax?
  - (c) Which has a higher unit cost, the tile or the carpet? Explain.



- Louis intends to have the floor of the dining room in his rental property redone with ceramic tiles that come in one-foot wide squares. He has a scale diagram of the dining room and living room, shown below, where one unit represents two feet. A 12-pack of squares sells for $38.40 and single squares can be purchased for $3.70 each. Determine the cost, in dollars, for Louis to buy just enough squares to cover his dining room floor?



- During an art class, students are projecting the image of a can of evaporated milk on the wall. The projector applies a scale factor of 250%. If the can has a diameter of 10 cm and a height of 12.5 cm, what are the dimensions of the image on the wall?
- Tony drew a scale diagram of his new skateboard to show a friend. He used a scale factor of 0.4. The scaled diagram has dimensions 3.2 in. × 1.8 in. × 10.8 in. What are the dimensions of the skateboard?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Consider using a model dinosaur (or some other scale model)
- A dinosaur model has a scale of 1:12. If the head of the dinosaur model is 8 cm in length, how long was the head of the real dinosaur?

  For this example use questions such as the following to promote student discussion:
  - What is the scale factor?
  - What does this value mean?
  - Will the value of the scale factor result in an enlargement or reduction? Explain.
  - How did you determine the length of the head of the dinosaur?

  Students should recognize that for each 1 cm the model measures, the corresponding part of the dinosaur measures 12 cm. They may set up and solve a proportion equation or they may simply recognize the dinosaur is 12 times the length of the model dinosaur.

- Examples of real-world applications, such as maps, sewing patterns, car models, doll house furniture, and construction blueprints, would be an effective way to introduce this concept and capture students' interest. Students could be given a road map, the price of gasoline per litre, and the miles-per-gallon value for a vehicle and asked to determine an estimate of the cost for a trip between two towns.
- Refer to the emergency exit plan that is posted in each classroom in your building.
  - Ask students how they could determine if the schematic is drawn to scale. You could ask students to make this determination and justify their answer.
  - If the schematic is drawn to scale, ask them to explain how they could determine what scale was used and then to find the scale used.
- An appropriate task would involve students designing a floor plan of their dream bedroom, kitchen, or a room of their choice. They will have to consider what to include in their drawing, the scale they will use, and the measurements needed. Students must also realize that it is important for the measurements to be realistic. If a blueprint is

using a scale of 1 in. representing 1 ft., for example, the doorway should be at least 3 in. wide on the drawing so that in real-life it would be 3 ft. wide.

Consider asking the following questions to guide students with their design:

- How are the dimensions recorded on the diagram?
- How are the doors, windows, closets, and walls represented?
- Is the scale indicated?
- Should a key be included to identify the symbols used in the drawing?
- Will furniture be included?

Completed floor plans can be posted around the classroom for students to see other examples.

- Have students bring in some models, such as small versions of cars, trucks, airplanes, dolls, figurines, or action figures. Ask them to do some research to determine the scale factor that such models represent.

Suggested Models and Manipulatives

- cereal boxes and other objects of various sizes
- toy cars, trucks, airplanes, dolls, doll furniture, or action figures
- grid paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- scale factor

Resources/Notes

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada) Sections 8.3–8.6, pp. 474–510

Notes

SCO M03 Students will be expected to demonstrate an understanding of the relationships among scale factors, areas, surface areas, and volumes of similar 2-D shapes and 3-D objects. [C, CN, PS, R, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M03.01 Determine the area of a 2-D shape, given the scale diagram, and justify the reasonableness of the result.
M03.02 Determine the surface area and volume of a 3-D object, given the scale diagram, and justify the reasonableness of the result.
M03.03 Explain, using examples, the effect of a change in the scale factor on the area of a 2-D shape.
M03.04 Explain, using examples, the effect of a change in the scale factor on the surface area of a 3-D object.
M03.05 Explain, using examples, the effect of a change in the scale factor on the volume of a 3-D object.
M03.06 Explain, using examples, the relationships among scale factor, area of a 2-D shape, surface area of a 3-D object and volume of a 3-D object.
M03.07 Solve a spatial problem that requires the manipulation of formulas.
M03.08 Solve a contextual problem that involves the relationships among scale factors, areas, and volumes.

Scope and Sequence

Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
M03 Students will be expected to solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including right cones, right cylinders, right prisms, right pyramids, and spheres.	**M03** Students will be expected to demonstrate an understanding of the relationships among scale factors, areas, surface areas, and volumes of similar 2-D shapes and 3-D objects.	

Background

In Mathematics 10, students solved problems related to surface area and volume of 3-D objects (M03). It would be beneficial to review the surface area and volume formulas of 3-D objects such as a rectangular prism, right cylinder, right cone, right pyramid, and sphere.

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for M03 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

In this Measurement unit, students will focus on the relationship between scale factor and area of similar 2-D shapes. They will solve problems that involve scale factor, surface area, and volume of 3-D objects.

Students will analyze how area is affected when the lengths of shapes are enlarged or reduced by a particular scale factor. This may be a good opportunity to review area formulas for shapes such as a parallelogram, triangle, rectangle, and circle.

It is important for students to recognize that the scale factor is applied to each dimension of the 2-D shape. As a result, the area will change by a factor of k^2. (The new area will be k^2 times the original area.)

Therefore, $k^2 = \dfrac{\text{area of similar 2D shape}}{\text{area of original shape}}$.

Using manipulatives such as linking cubes, students will investigate the relationship between the scale factor and the surface area of two similar 3-D objects, in addition to investigating the relationship between the scale factor and the volume of two similar 3-D objects.

It is important for students to recognize that the scale factor is applied to each dimension of the 3-D shape. As a result, the surface area will change by a factor of k^2 and, the volume will change by a factor of k^3. (The new surface area will be k^2 times the original surface area. The new volume will be k^3 times the original volume.)

Therefore, $k^2 = \dfrac{\text{surface area of similar 3D shape}}{\text{surface area of original 3D shape}}$ and $k^3 = \dfrac{volume\ of\ similar\ 3D\ shape}{volume\ of\ original\ 3D\ shape}$.

Students are expected to use the dimensions of a scale diagram of a 3-D object as well as the scale factor to determine the surface area and volume of the enlarged/reduced object.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

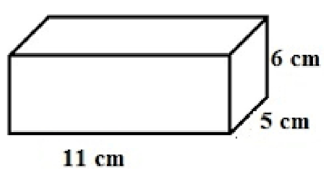
Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Calculate the volume of a cylinder with a height of 10 cm and a diameter of 20 cm.
- How does the volume of a sphere with a radius of four inches compare to a sphere with a radius of eight inches?
- Draw a square that has a side length of 2 units. Calculate the perimeter and area of this square.
- Calculate the surface area and volume of the rectangular prism shown below.

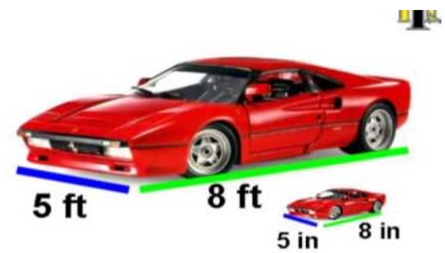


Whole-Class/Group/Individual Assessment Tasks

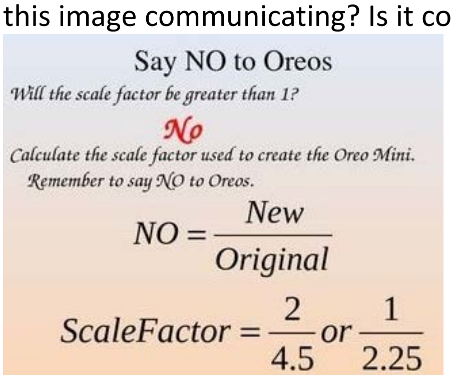
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- A 4 in. × 6 in. picture has dimensions that have been tripled. What is the area of the new picture?
- Chad and Sheina painted a mural on the wall, measuring 12 ft. × 8 ft. using an overhead projector. If the original sketch had an area of 216 $in.^2$, what is the scale factor?

- For the diagram shown below how does the surface area of the actual car compare to that of the model?



- Find at least three examples of enlargements or reductions in real-world objects. Estimate their scale factors.
- Dashiell claims that when you enlarge every side of a cube *n* times, its volume also increases *n* times. Jane says that the volume of a cube increases 3*n* times, and Élaine is convinced that the volume increases n^3 times. Who do you agree with and why?
- The surface area of a cone is 36 ft.^{2}. What is the surface area of its image if a scale factor of 1:4 is applied?
- Find the volume of a cylinder if its image has a volume of 450 cm^3 and a scale factor of 2:3. Round your answer to the nearest cubic centimetre.
- What is the scale factor of the following pairs of similar spheres?
  - Volume of the original is 450 mm^3 and its image is 1518.75 mm^3.
  - Surface area of the original is 248 in.^{2} and its image is 126.5306 in.^{2}.
- Samantha found the following image when doing a Google Search. What information is this image communicating? Is it correct?



Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

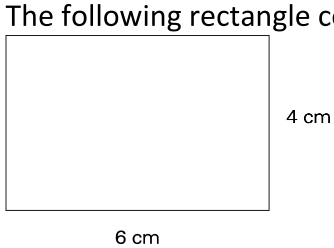
Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Begin by giving examples of careers that use scale models. Professionals such as architects, engineers, filmmakers, and sales associates use scale-size models to do their jobs. Show pictures of these examples or bring in actual examples for the students to observe.
- The following rectangle could be used as an illustration.

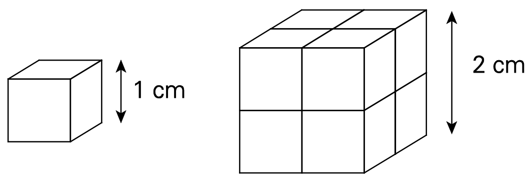


  Students will explore the relationship between scale factor and area. They should use different scale factors such as 2, 3 or 0.5. Consider the following questions for discussion:

  - What is the area of the original rectangle?
  - What are the dimensions and area of the resulting similar rectangles?
  - What do you notice?

  This simple example should help students observe that the resulting areas are not directly proportional to the lengths. When students double the sides of a rectangle, for example, the area does not just double, it quadruples.

- Videos can be used to stimulate student interest (such as the DVD extras showing the making of films where scale models were used—*Star Wars*, *Lord of the Rings*, etc.). Scale factor is normally represented in science in terms of a fraction equal to a number less than one, reduced so that the numerator is a one (i.e., $\frac{1}{20}$ scale). However, it can also be expressed as an integer multiplier (i.e., a scale factor of 20). It may be simpler to work with the latter idea, but students and teachers should be comfortable working with either conceptualization.
- The use of manipulatives is very important as students discover the impact on the surface area and volume when the dimensions of an object are multiplied by a scale factor.
    - Ask students to measure the dimensions of one linking cube. The volume of such a cube is 1 cm^3 while its surface area is 6 cm^2.

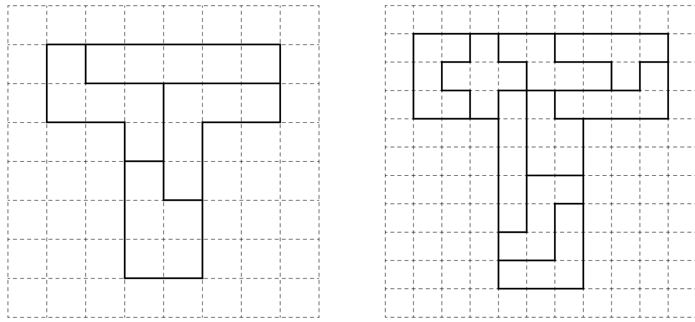


    - Encourage students to work in groups, build various cubes using pre-determined scale factors, find the surface area and volume, and make the connection to its scale factor. Information could be organized in a chart.

Scale Factor	Length	Width	Height	Surface Area	Volume
1	1	1	1	6	1
2	2	2	2	24 (changes by a factor of 4)	8 (changes by a factor of 8)
3	3	3	3	54 (changes by a factor of 9)	27 (changes by a factor of 27)
4	4	4	4	96	64

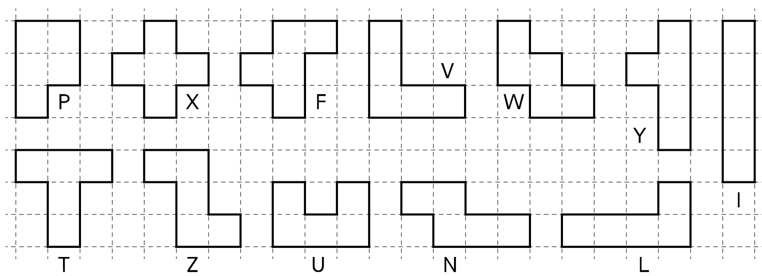
    - Ask students what will happen if the length of each edge of the cube is doubled. [They should observe that the volume of the cube changes by a factor of 8 or 2^3, while the surface area only changes by a factor of 4 or 2^2.]
    - Ask students what will happen to the surface area and the volume of the original cube if the length of every edge of the original cube triples.
    - Students should recognize that when the dimensions of similar 3-D objects are related by a scale factor k, their surface areas are related by k^2 and their volumes are related by k^3.

- Ask students to build the scale models of the T pentomino pieces, shown below.



Ask them to describe the scale factor for each and then compare their areas.

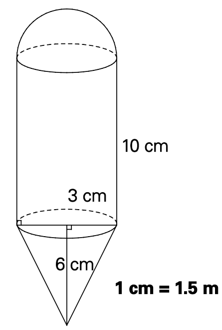
Note: The 12 Pentominoes are shown below.



Use some of the following challenges to integrate LR02 (Analyze puzzles and games that involve spatial reasoning, using problem-solving strategies.) into this outcome.

- Using the four pentominoes named L, N, P, and V, create a shape that is double the dimensions of the Z pentomino.
- Using the four pentominoes named L, N, P, and V, create a shape that is double the dimensions of the U pentomino.
- Using the four pentominoes named L, N, P, and V, create a shape that is double the dimensions of the L pentomino in two different ways.
- Using the four pentominoes named L, N, P, and Y, create a shape that is double the dimensions of the N pentomino.
- Using the four pentominoes named I, L, N, and P, create a shape that is double the dimensions of the L pentomino.
- Not every pentomino can be doubled. For example, the pentominoes named V and X cannot be doubled, although the rest can. Find solutions for doubling as many of the remaining pentominoes as you can?
- Using all the pentominoes except those named U, X, and Y, create a shape that is triple the dimensions of the L pentomino.
- Using all the pentominoes except those named T, W, and Z, create a shape that is triple the dimensions of the W pentomino.
- Using all the pentominoes except those named L, W, and Y, create a shape that is triple the dimensions of the Z pentomino.

   - Using all the pentominoes except those named F, I, and X, create a shape that is triple the dimensions of the X pentomino.
   - The dimensions of all the pentominoes can be tripled. Find solutions for tripling as many of the remaining pentominoes as you can.
- Once students have discovered the impact of scale factor on surface area and volume, post a question such as the following:
   - Consider the scaled-down diagram of the storage tank to the right.
   - Guide students through the process using the following instructions and questions.
      (a) Find the total surface area of the original tank.
      (b) Find the total volume of the original tank.
      (c) Was it necessary to determine the dimensions of the original drawing to answer the above questions?
      (d) How would you determine the surface area and volume of the original if the scale diagram was not given?



Suggested Models and Manipulatives

- linking cubes
- pentominoes

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- scale
- scale diagram
- scale factor

Resources/Notes

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  Section 8.4, pp. 483–490
  Section 8.6, pp. 502–510

Software

- Geometer's Sketchpad (Key Curriculum Press 2013) (NSSBB #: 50474, 50475, 51453)
- SketchUp 2013 (Google.com 2013)
  www.sketchup.com
- GeoGebra (International GeoGebra Institute 2013)
  www.geogebra.org/cms/en

Geometry

48–52 hours

GCO: Students will be expected to develop spatial sense.

SCO G01 Students will be expected to derive proofs that involve the properties of angles and triangles. [CN, R, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(*It is intended that deductive reasoning be limited to direct proof.*)

G01.01 Generalize, using inductive reasoning, the relationships between pairs of angles formed by transversals and parallel lines, with or without technology.
G01.02 Prove, using deductive reasoning, properties of angles formed by transversals and parallel lines, including the sum of the angles in a triangle.
G01.03 Generalize, using inductive reasoning, a rule for the relationship between the sum of the interior angles and the number of sides (*n*) in a polygon, with or without technology.
G01.04 Identify and correct errors in a given proof of a property involving angles.
G01.05 Verify, with examples, that if lines are not parallel the angle properties do not apply.
G01.06 Verify, through investigation, the minimum conditions that make a triangle unique.

Scope and Sequence

Mathematics 10	Mathematics 11	Mathematics 12
RF08 Students will be expected to solve problems that involve the distance between two points and the midpoint of a line segment.	**G01** Students will be expected to derive proofs that involve the properties of angles and triangles.	**LR03** Students will be expected to solve problems that involve conditional statements.

Background

In M10 students have been used their knowledge of distance and midpoint formulas and slope to if a triangle is right, isosceles, equilateral or scalene or if a quadrilateral is a square, rectangle, rhombus or parallelogram. (RF08)

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for G01 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

Students have had no formal instruction with proof in previous mathematics courses. If students have completed the Logical Reasoning (LR01) outcome prior to beginning this unit, they will have had experience with making, testing, and proving conjectures.

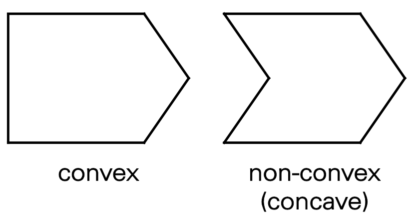
In the Geometry and Logical Reasoning Units students are introduced to the notion of formal **proof** including the **two-column proof** format and other formats.

When constructing proofs, students must use proper mathematical terminology.

In Grade 6, students learned that the sum of the angles in a triangle totals 180° and that the sum of the angles in a quadrilateral is 360°. They will now use deductive reasoning to prove that this is true for all triangles and for all quadrilaterals. They will also use deductive reasoning to determine a formula for finding the sum of the angles in a polygon with *n* sides.

Students should distinguish between convex and non-convex (concave) polygons. The focus, however, will be on convex polygons.

Convex polygon: A polygon in which each interior angle measures less than 180°.

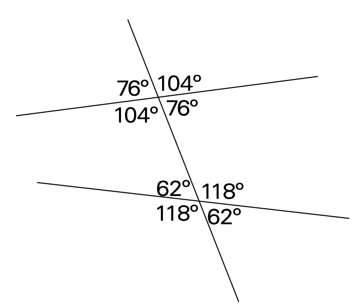


In Grade 7, students identified parallel and perpendicular lines. They also used various strategies to draw a line segment that was perpendicular (or parallel) to a given line segment.

Students are expected to discover relationships both with and without technology. Software programs such as FX Draw (Efofex Software 2013) and Geometer's Sketchpad (Key Curriculum 2013) or an interactive whiteboard could also be used to develop an understanding of the angle relationships. The construction of a geometric situation, either electronically or by hand, is as powerful for learning as the analysis of the situation.

Students will be introduced to the term transversal before developing the various angle properties. They will discover that vertically opposite angles are congruent and that when two parallel lines are intersected by a transversal, the following are true: corresponding angles are congruent; alternate interior angles are congruent; alternate exterior angles are congruent; and interior angles on the same side of the transversal are supplementary.

Students should recognize that when a transversal intersects a pair of non-parallel lines, the angle properties do not apply except for vertically opposite angles.

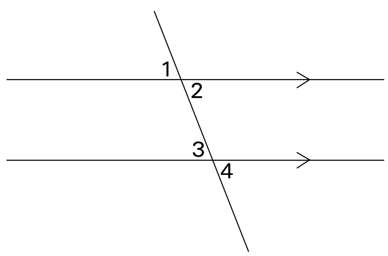


Once students have explored the angle relationships when two parallel lines are cut by a transversal, they will use their knowledge of corresponding angles, vertically opposite angles, and supplementary angles to formally prove the other relationships, such as alternate interior angles.

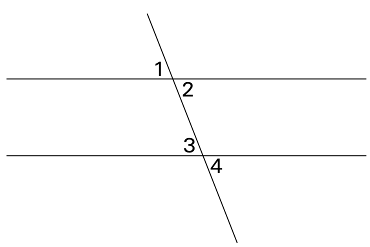
A **two-column proof** and a **paragraph proof** are two of the most common strategies used to construct proofs involving properties of angles formed by transversals and parallel lines.

Teachers could ask students to make a conjecture that involves alternate exterior angles formed by parallel lines and a transversal. Students may conjecture that alternate exterior angles are equal and use the following to prove their conjecture.

Statement	Justification
$\angle 1 = \angle 2$	vertically opposite angles
$\angle 2 = \angle 4$	corresponding angles
$\angle 1 = \angle 4$	transitive property



In both the Geometry and Logical Reasoning units, students are expected to identify errors in reasoning and proofs (both formal and informal) such as the one shown below.



Statement	Reason
$\angle 1 = \angle 2$	Vertically opposite angles are congruent.
$\angle 2 = \angle 3$	Alternate interior angles are congruent.
$\angle 1 = \angle 3$	Both equal $\angle 2$.
$\angle 3 = \angle 4$	Vertically opposite angles are congruent.
$\angle 1 = \angle 4$	Both equal $\angle 3$.

In Grades 8 and 9, students were exposed to the properties of congruent and similar polygons.

The intent of this Geometry unit is for students to investigate what makes a triangle unique. It is not the aim of the unit to have students prove triangles congruent.

Using the investigative approach, students should recognize that there are four sets of conditions that guarantee the uniqueness of a triangle:

- side-side-side (SSS)
- side-angle-side (SAS)
- angle-side-angle (ASA)
- side-angle-angle (SAA)

Teachers should not present these as theorems without students having discovered the requirements for a triangle to be unique first. If the lengths of three sides of a triangle are given, for example, only one unique triangle can be constructed. As a result, any triangles constructed with these side lengths will be a replica of the given triangle and will therefore be congruent to the given triangle.

Students should also be given time to investigate other relationships. Students may determine the side-side-angle (SSA) only works for a right triangle, known as the hypotenuse-leg (HL) theorem. Although this theorem is not part of this outcome, discussion may be warranted here as to why it works for a right triangle. They will also determine that the angle-angle-angle (AAA) property only indicates similarity not congruency.

The SSA non-unique case is also called the ambiguous case, and this will be considered in detail when the sine law is studied (G03).

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

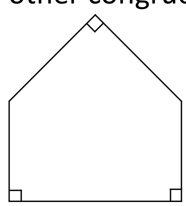
Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- How can you determine if two lines are parallel?
- Draw a quadrilateral with two parallel sides and use a protractor to measure the angles in the quadrilateral.
- In what real-life situations do you see parallel lines?

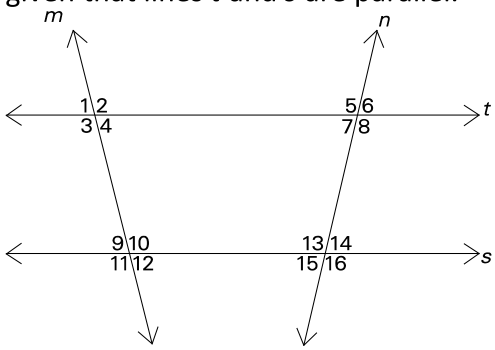
Whole-Class/Group/Individual Assessment Tasks

- Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.
- Find an object that models lines cut by a transversal. Sketch the object and highlight the various pairs of angles that are formed. State which angles are congruent and why you can be certain that they are without measuring them.
- In baseball, the home plate is shaped like the one shown. It has 3 right angles and 2 other congruent angles (A and B). Find the measures of $\angle A$ and $\angle B$.

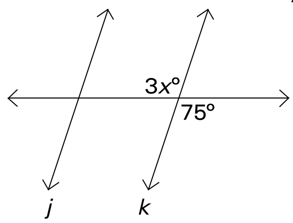


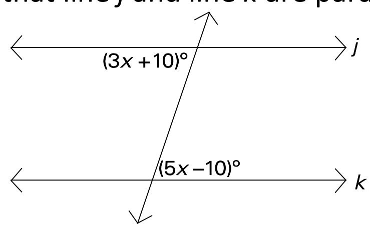
- Using at least three examples, verify inductively that the conjectures regarding the sum of interior angles of a polygon are valid for convex polygons.
- Draw ΔDEF with DF = 8 cm, $\angle D$ = 35° and $\angle F$ = 95°. Is your triangle unique? Explain. Compare your triangle with several classmates to check.
- When asked to construct ΔPQR with PQ = 12", QR = 15", and $\angle D$ = 27°, Marie and Jackie drew congruent triangles. Are there any other possible triangles with these measurements that are not congruent to the ones that were drawn? Explain.
- A daycare is building a triangular sandbox. In the plan, two of the sides of this sandbox are 20 feet and the third side is 16 feet. Is there more than one sandbox that will meet these specifications?
- Two neighbors have built triangular fire pits with the following specifications. Two of their three sides are 64 inches and 30 inches in length. The angle opposite the 30 inch side is 60°. Are the fire pits guaranteed to be the same shape and size?

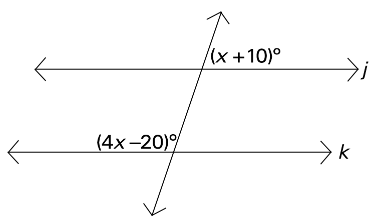
- For the following diagram $\angle 2 = 95°$ and $\angle 6 = 80°$, determine the other angle measures given that lines *t* and *s* are parallel.

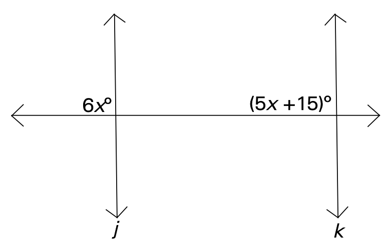


- Find the value of the variable, given that line *j* and line *k* are parallel.

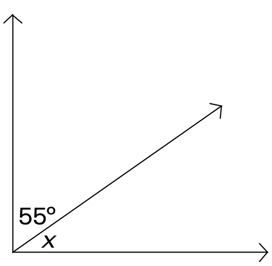


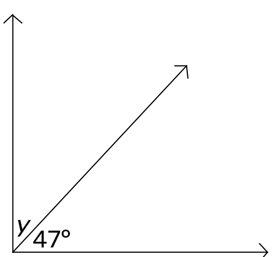




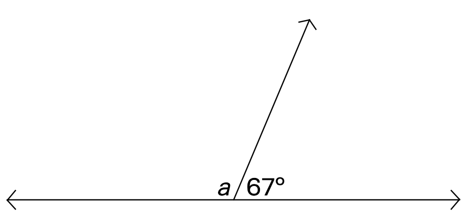


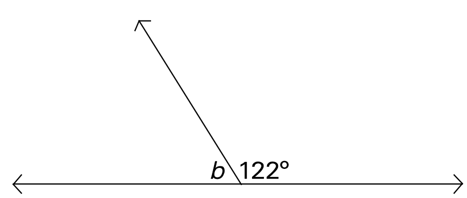
- Given that the following are right angles, find the value of the variables.





- Given that the following are straight angles, find the value of the variables.





Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

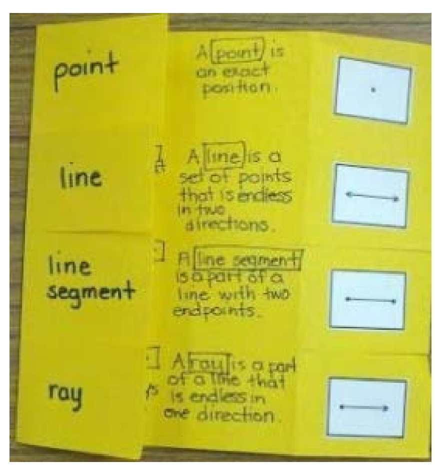
Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

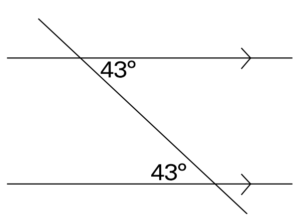
- Students struggle with language in this unit, this activity will provide students with an opportunity to articulate properties in their own language first. Once common mathematical language is introduced, students will describe these properties in their own words with the hopes connections between what they are trying to explain and the math language involved will become easier and more memorable.



Have students explore relationships between parallel lines and a transversals. Students will use their own language to explain patterns, through discussions teachers talk about the importance of having a common language to use when discussion properties and introduce correct terminology. Students will then create a line “foldable” explaining terms in their own words.

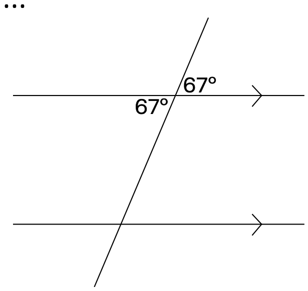
- Students need to be introduced to the term **transversal** before developing the various angle properties. It is important that students realize that a transversal does not have to cut parallel lines.
- Ask students, in groups of three or four, to draw five sets of lines cut by a transversal.
    - Draw the first four sets with non-parallel lines crossed by a transversal, with each set getting closer to being parallel. The fifth set should be parallel lines crossed by a transversal.
    - Number the sets from 1 to 5.
    - Measure the same pair of alternate interior angles. Also have them find the sum of the same-side exterior angles in each set of lines and to record their data in an organized chart.
    - After reviewing and discussing the data in their chart, students should draw conclusions about the measures of alternate interior angles and the sum of same-side exterior angles for lines that are not parallel and lines that are parallel.
    - Ask students to measure all of the angles and record the patterns they observe. Which angles are equal? Which angles are supplementary? Is this an example of inductive reasoning?
- Patty paper (tracing paper) or geo-strips can be used as an effective tool to investigate which angles are congruent when two lines, parallel and non-parallel, are intersected by a transversal.
- Students can work in groups of six to play the game “I Have ... , Who Has ... ”. Each student is given a card such as the following:

I have ...



Who has vertically opposite angles?

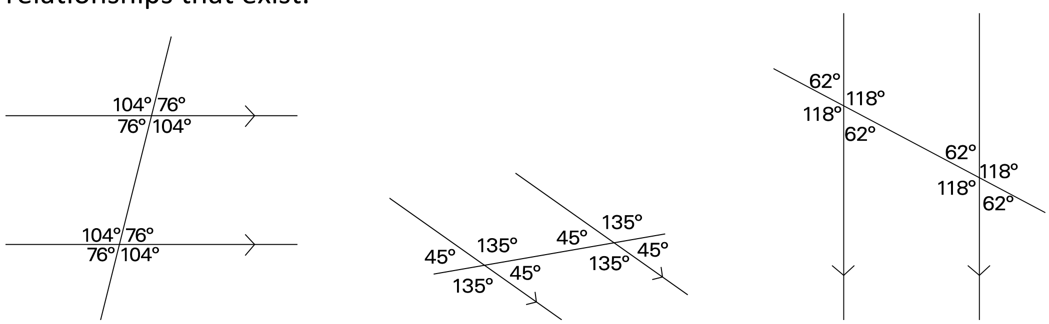
I have ...



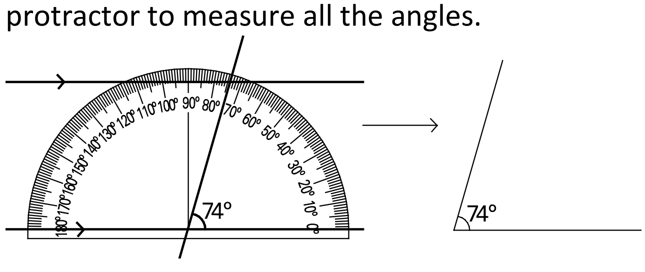
Who has corresponding angles?

Student #1 begins by identifying the pair of angles illustrated in their diagram and reading the “Who has ...” statement at the bottom of their card. All other students must then look at their pictures to see who has the card illustrating that particular pair of angles. Play continues until the game comes back to the original card.

- Use tape to create two parallel lines with a transversal on the floor. Students should work in pairs. Student A stands on the angle. Student B picks a card (from a deck with “vertically opposite,” “alternate interior,” “alternate exterior,” “corresponding,” “same side interior,” and “congruent”) to direct Student A to move to the described angle. Students switch roles after six moves.
- Ask students to create a graphic organizer highlighting the relationships between
  - vertically opposite angles
  - alternate interior angles
  - alternate exterior angles
  - corresponding angles
  - same side interior angles
- After exploring multiple examples, students make their generalizations about the angle relationships that exist.

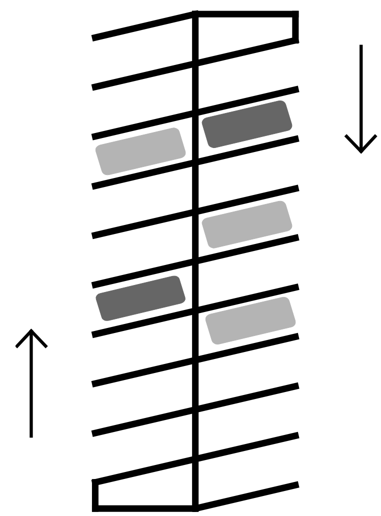


- Software programs such as FX Draw (Efofex Software 2013) and Geometer’s Sketchpad (Key Curriculum 2013) could also be used to develop the angle relationships. This alternative to paper and pencil creates a dynamic learning environment where students can interactively investigate and generalize the relationship between pairs of angles formed by transversals and parallel lines. An interactive whiteboard could also be used to develop these relationships. Students could be invited to use the interactive protractor to measure all the angles.



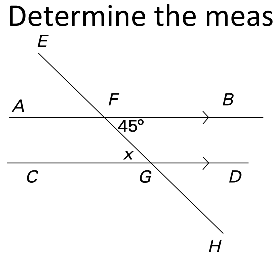
- Ask students to look for examples of parallel lines cut by a transversal around their school. Some examples might include court lines on the gymnasium floor or floor tile patterns in the classroom. Students could trace the lines onto a piece of paper and then measure the angles using a protractor to determine relationships amongst the angles. Have students look to see if they can find parallel lines with a non-right transversal.

- As an alternative task, students could investigate these angles using a parking lot. It would be beneficial if students were exposed to a parking lot where all the angles are not right angles. When workers paint lines for a parking lot, they aim to paint lines that are parallel to each other. The lines in a parking lot, therefore, provide an ideal illustration of the relationship between angles created by parallel lines and a transversal. For example, students can discuss and mark the different types of angles in the school's parking lot with chalk. This task could also be set up in the gymnasium using tape to represent parking spaces. Students can then measure the angles to determine which angles are equal and which are supplementary.



- This would be a good opportunity to promote discussion by asking students the following questions:
    - Why would a parking lot have parallel lines that intersect at non-right angles?
    - Why would a parking lot have one-way traffic?
- It is beneficial to have students analyze solutions that contain errors, explain why errors might have occurred, and explain how errors can be corrected. This reinforces the angle relationships that have been developed throughout this unit. Have students identify and correct errors such as those present in the following example:

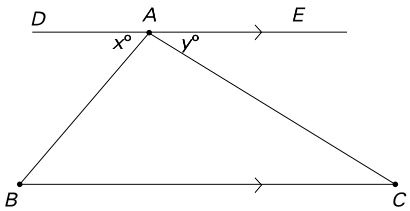
Determine the measure of *x*.



Statement	**Justification**
$\angle BFG = 45°$	given
$\angle BFG = \angle FGD$	interior angles on the same side of the transversal are equal
$\angle FGD + \angle FGC = 180°$ $\angle FGC = 180° - \angle FGD$ $\angle FGC = 180° - 45°$ $\angle FGC = 135°$ $x = 135°$	supplementary angles (angles forming a straight line)

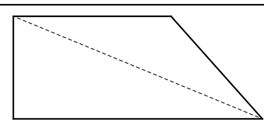
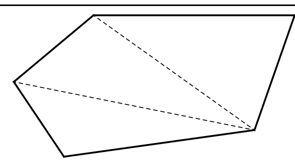
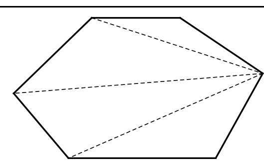
Students should recognize that interior angles on the same side of the transversal are not equal. They are supplementary angles.

- Students will prove, using deductive reasoning, that the sum of the interior angles of any triangle is 180°. Give students direction by asking them to draw a triangle and to draw a line that is parallel to one of the sides of the triangle and tangent at one of the vertices.



Students should be able to complete the proof using the properties of angles formed by transversals and parallel lines.

- Encourage students to discover the relationship between the sum of the interior angles and the number of sides in a convex polygon using the angle sum property. They are aware that the sum of the angles in a triangle is 180°. Students can separate each polygon into triangles by drawing diagonals. They can then use the following table to help them with their investigation. Each vertex of a triangle must be a vertex of the original polygon.

Number of Sides	**Diagram**	**Number of Triangles Formed**	**Sum of Angles**
4		2	360°
5		3	540°
6		4	720°

The objective of this investigation is for students to recognize that the sum of the interior angles increases by 180° as the number of sides increase by one. They should also observe that the number of triangles formed is always two less than the number of sides in the polygon. Using this information, encourage students to develop a formula for the sum of the measures of the interior angles of a polygon, $S = 180°(n - 2)$ where S represents the sum of the interior angles and n is the number of sides of the polygon.

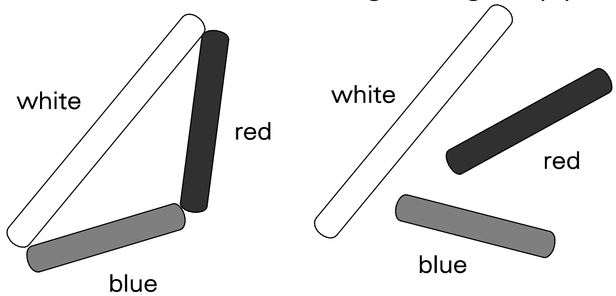
If the polygon is regular, students can use their knowledge of the sum of the measures of the angles in a polygon to determine the measure of each interior angle.

- The following task, which uses either pipe cleaners or geo-strips, will help students conceptualize the idea of the necessary conditions for unique triangles.

  Note: After giving students three different lengths of pipe cleaners or geo-strips, have them build a reference triangle that will be used for all of the following investigations.

  Side-Side-Side (SSS)

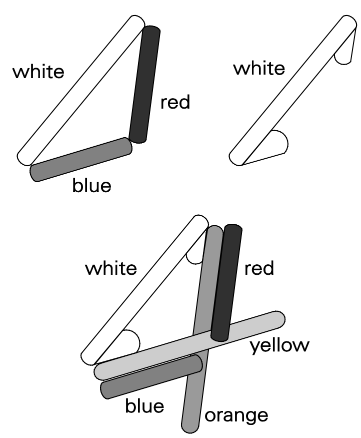
  - Provide students with a set of pipe cleaners or geo-strips that are identical to those they used to form their reference triangle.
  - Ask students to build a triangle using the pipe cleaners or geo-strips.



  - Ask students what they noticed about both triangles.
  - Students should notice that no matter where they put the sides, they will build a replica of the reference triangle. Therefore knowing the three sides of a triangle is one condition that will guarantee a unique triangle because any other triangle built with the same conditions will be a replica of the given triangle. This will lead into a discussion around the side-side-side relationship (SSS) of congruent triangles because what makes a triangle unique will also make triangles congruent.

Angle-Side-Angle (ASA)

  - Provide students with additional pipe cleaners or geo-strips, a protractor, scissors, and construction paper.
  - Ask students to use one pipe cleaner or geo-strip identical to one side in the reference triangle. They should then measure the angles formed by that pipe cleaner or geo-strip and the other sides of the reference triangle and cut an angle out of construction paper to represent each of these measured angles.



  - Ask students to attach an angle to each end of the selected pipe cleaner or a geo-strip and then use the extra pipe cleaners to construct the other two sides of the triangle.
  - Ask students to compare the side lengths of their construction to the side lengths of the base or reference triangle.

  - o Encourage discussion using questions such as the following:
    - Are these triangles unique? Why
    - What did you notice about the side in relation to the two angles?
    - What are the conditions that will create a unique triangle?
    - Will these conditions produce congruent triangles?

Side-Angle-Side (SAS)

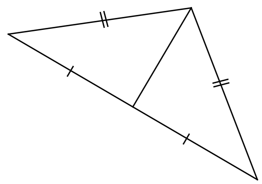
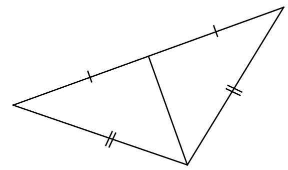
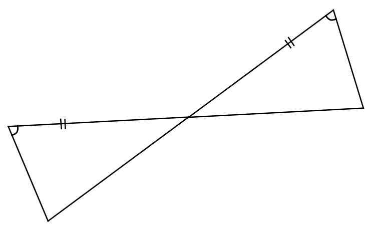
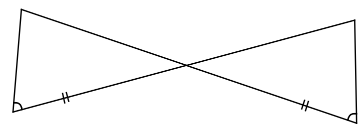
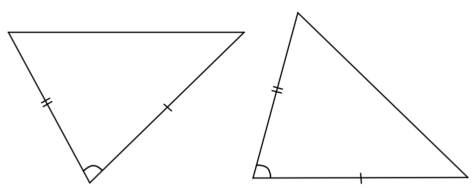
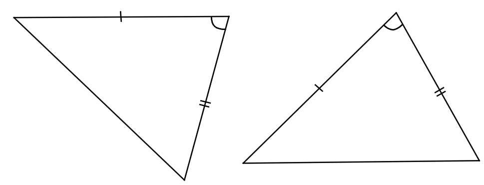
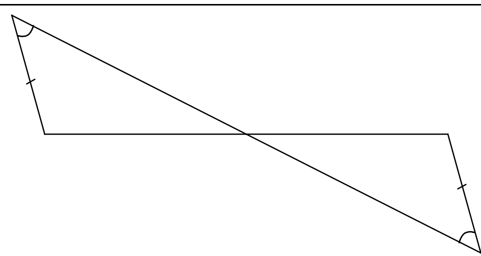
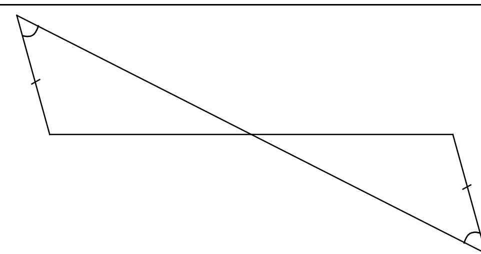
  - o Provide students with additional pipe cleaners or geo-strips, a protractor, scissors, and construction paper.
  - o Ask students to use two pipe cleaners or geo-strips identical to those in the base triangle. They should then measure the angle formed between the two selected pipe cleaners or geo-strips and cut an angle out of construction paper to represent this measured angle.
  - o Ask students to attach the two pipe cleaners so the construction paper angle fits at their joining point. Then use the extra pipe cleaner to construct the other side of the triangle.
  - o Ask students to compare the side lengths of their construction to the side lengths of the base or reference triangle. Ask the same questions that you asked for the angle-side-angle relationship.

Side-Angle-Angle (SAA)

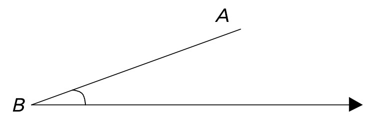
  - o Provide students with additional pipe cleaners or geo-strips, a protractor, scissors, and construction paper.
  - o Ask students to use one pipe cleaner or geo-strip identical to that in the reference triangle. They should then measure two of the angles formed in the base triangles. One of these angles must be across from the selected pipe cleaner or geo-strip. Students should then cut angles out of construction paper to represent each of these measured angles.
  - o Ask students to use their pipe cleaner or geo-strip and the constructed angles along with additional pipe cleaners or geo-strips to form a triangle.
  - o Ask students to compare the side lengths of their construction to the side lengths of the base or reference triangle.
  - o Use questions such as the following to help students work through the task.
    - Can congruent triangles be constructed using these properties?
    - What three pieces of information are required to prove congruent triangles?

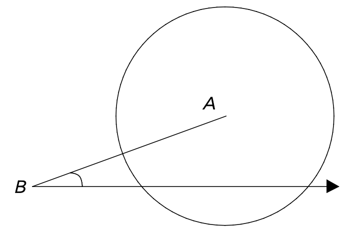
- To reinforce the conditions that will ensure that you have congruent triangles, have students play the following memory game.
  - o Students have to find three matching cards. One card will state a condition for congruent triangles (SSS, SAS, ASA) and the other two cards show sample triangles that satisfy the postulate. In groups of two, students will take turns

trying to find the three matching cards for each postulate, and the student at the end of the game with the most cards wins.

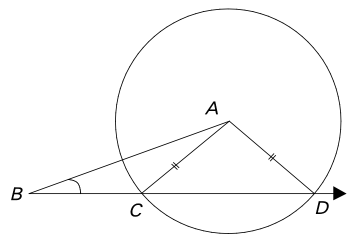
SSS		
ASA		
SAS		
AAS		

- Using a compass, you may wish to have students explore why SSA does not work for non-right triangles
    - Draw a diagram with a fixed angle *B* and fixed side *AB*. Create a circle with centre at *A*.

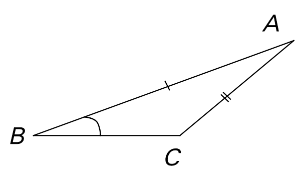


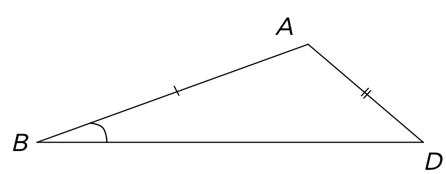


    - Make a triangle by drawing two lines that are radii of the circle.



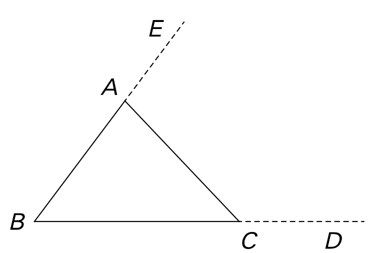
    - Two triangles are constructed. However, ΔABC is not congruent to ΔABD.



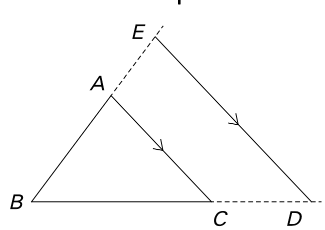


    - Using this construction, students should recognize that knowing only side-side-angle (SSA) does not work because the unknown side could be located in two different places.

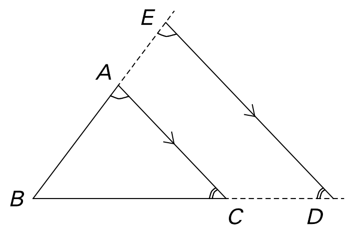
- A construction can also clarify the angle-angle-angle (AAA) case.
    - Draw a triangle *ABC*. Extend the lines on two sides of the triangle.



    - Draw a line parallel to *AC*.



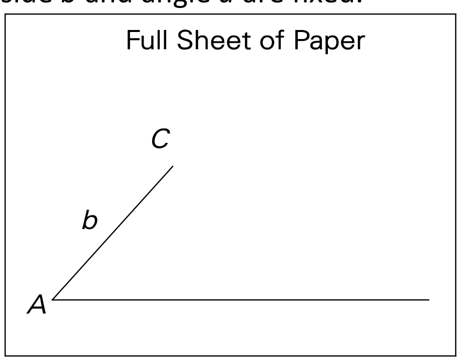
- o Use the fact that the lines are parallel to mark the corresponding angles.



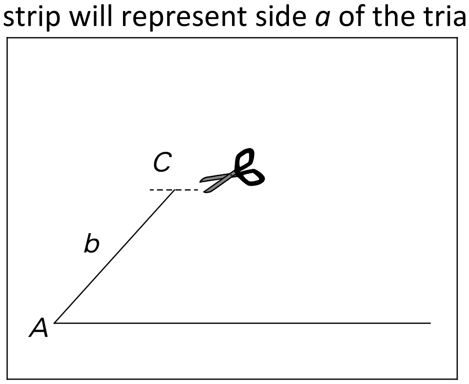
- o Two triangles are constructed. However, ΔABC is clearly not congruent to ΔEBD .
- o From this, students should recognize that knowing only angle-angle-angle can produce similar triangles but cannot guarantee congruent triangles.

- The following task can be used with students to discover that, under certain conditions, there may be no possible triangles, exactly one unique triangle, or two different triangles. This sets the background that will be explored further as part of the ambiguous case for the law of sines.

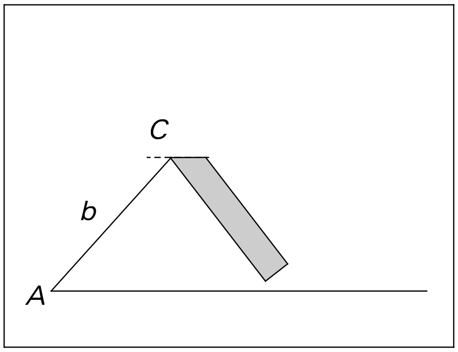
  Step 1: Provide or have students construct a diagram similar to the one below, where side *b* and angle *a* are fixed.



  Step 2: Cut a slit in the paper at point *C* large enough to fit a small strip of paper. This strip will represent side *a* of the triangle.



Step 3: Insert the strip into the slot at *C* as shown.



Step 4: Have students explore various lengths for side *a* by pulling the strip out from *C* and rotating it left and right.

As teachers observe, consider the following questions.

- How many triangles can be formed when the strip is too short? (no triangle)
- How many triangles can be formed when the strip is perpendicular to side *C*?
- (one right triangle)
- Create a triangle using a longer strip. Can another triangle be created using the same strip length? (two unique triangles)

Suggested Models and Manipulatives

- Compasses
- Card stock
- dynamic geometry software
- foam or cardboard angles
- patty paper or tracing paper
- pipe cleaners or geo-strips
- protractors
- rulers
- scissors

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- adjacent interior angles
- alternate exterior angles
- alternate interior angles
- converse
- convex polygon
- corresponding angles

- exterior angles
- interior angles
- non-adjacent interior angles
- regular polygon
- transversal

Resources/Notes

Internet

- *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2013)
  http://illuminations.nctm.org
  This interactive game allows students to explore the conditions to create congruent triangles. (Keyword: congruence theorems)

- Math Warehouse (Morris 2013)
  www.mathwarehouse.com
  An interactive game where students can explore and discover the rules for angles of parallel lines cut by a transversal. (Keyword: parallel line and angle)

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada) Sections 2.1–2.4, pp. 70–99

SCO G02 Students will be expected to solve problems that involve the properties of angles and triangles. [CN, PS, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

G02.01 Determine the measures of angles in a diagram that involves parallel lines, angles, and triangles and justify the reasoning.
G02.02 Identify and correct errors in a given solution to a problem that involves the measures of angles.
G02.03 Solve a contextual problem that involves angles or triangles.
G02.04 Construct parallel lines, using only a compass and straight edge or a protractor and straight edge, and explain the strategy used.
G02.05 Determine if lines are parallel, given the measure of an angle at each intersection formed by the lines and a transversal.

Scope and Sequence

Mathematics 10	Mathematics 11	Mathematics 12
RF10 Students will be expected to solve problems that involve systems of linear equations in two variables, graphically and algebraically.	**G02** Students will be expected to solve problems that involve the properties of angles and triangles.	

Background

Students have worked with parallel lines in previous grades. In G01 they determined various angle relationships when two parallel lines were cut by a transversal. They are now asked to verify that the lines are parallel by measuring the angles they have created (corresponding, vertically opposite, alternate interior, alternate exterior, interior angles on the same side of the transversal). They should understand how many angle relationships are necessary to measure in order to prove the lines are parallel. A common student error occurs when students identify lines as being parallel based only on vertically opposite angles that are equal.

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for G02 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

Students will be expected to construct two parallel lines using both a compass-straight edge combination and a protractor-straight edge combination.

Students will use angle properties to calculate identified angles in diagrams involving parallel lines, transversals, and triangles, as well as using these angle relationships to solve contextual problems.

Students are expected to analyze solutions that contain errors, explain why errors might have occurred, and explain how they can be corrected. This process reinforces the angle relationships that have been developed throughout this unit. Students should be able to identify and correct errors.

Students could be exposed to examples where variables represent the angles requiring them to solve a linear equation. In ΔABC, for example, $\angle A = 8x - 15$, $\angle B = -x + 42$, $\angle C = 2x$. Ask students to determine the measure of $\angle A$.

Students should be exposed to problems where they can make a connection between mathematics and their environment.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

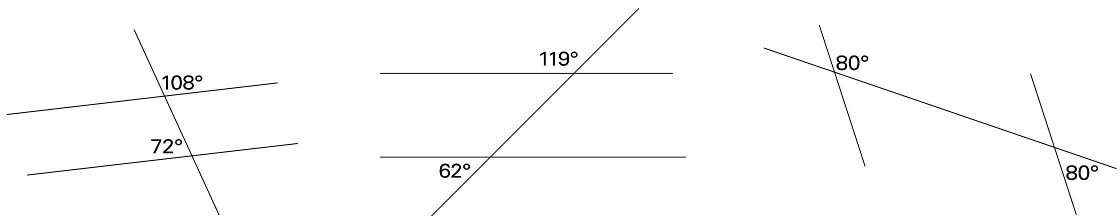
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve for x given that $2x + 15 = 5x + 3$
- If one of the acute angles of a right triangle is 42°, what is the measure of the other angle?
- Provide an example of a situation where it is important for lines to be parallel. What would happen if they were not parallel?

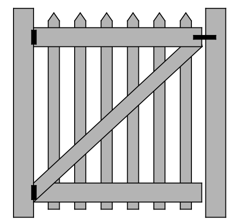
Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

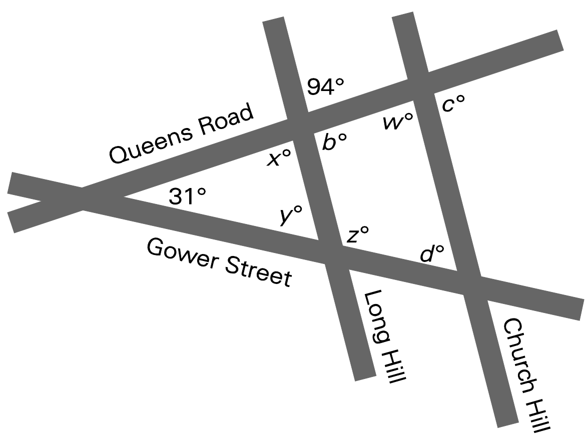
- Determine if the lines shown below are parallel. Explain your reasoning.



- What measurements would be necessary to determine if the top and bottom rails of the gate are parallel?

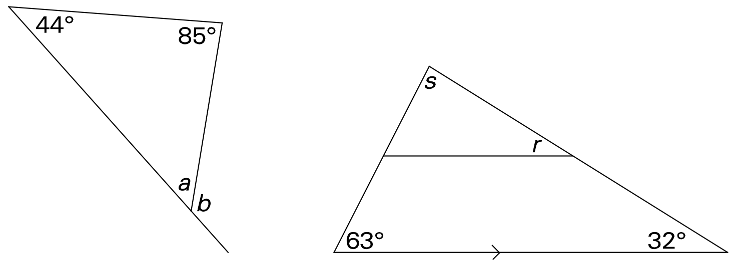


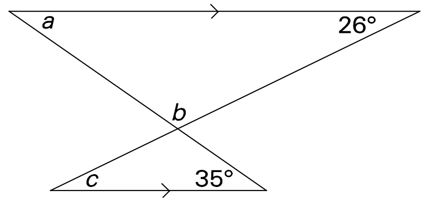
- Long Hill and Church Hill are parallel to each other. Determine the missing measures on the map shown below:



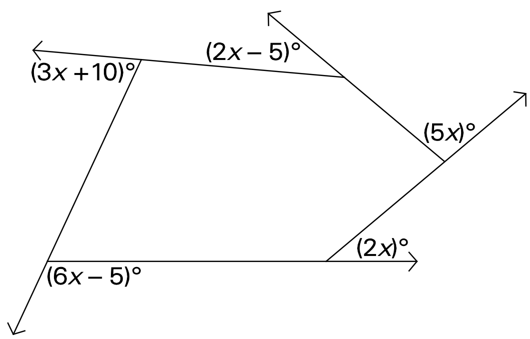
- Design a map involving parallel lines cut by a transversal. Provide at least one angle measurement, and create a question to identify unknown angles.

- Determine the measures of unknown angles indicated by the variables in the diagrams.

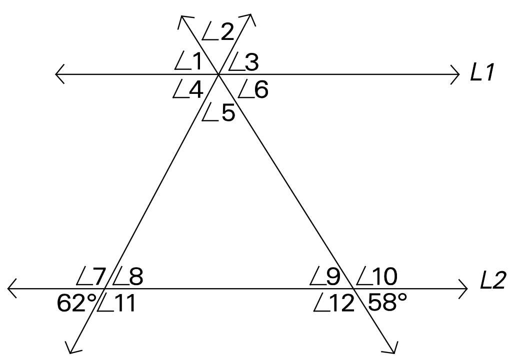




- Determine the value of *x* in the diagram.

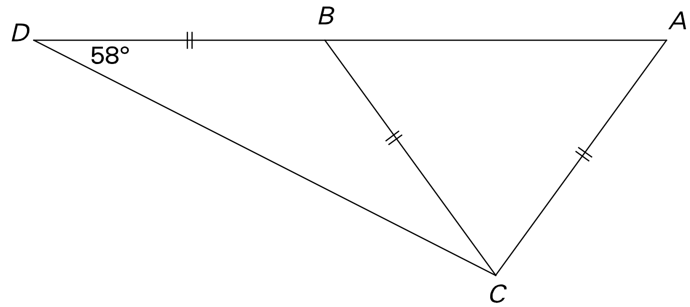


- The measure of which of the following angles can be determined to ensure that the lines L1 and L2 are parallel?



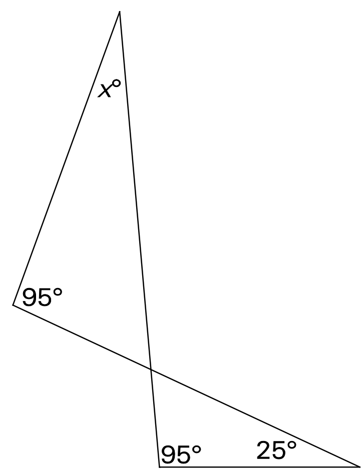
- In quadrilateral *KLMN*, $\angle K = 6x - 30$, $\angle L = -2x + 50$, $\angle M = 5x + 10$, and $\angle N = 3\angle M$. Determine if the quadrilateral has a right angle.

- Determine the measure of $\angle A$ and explain your reasoning.

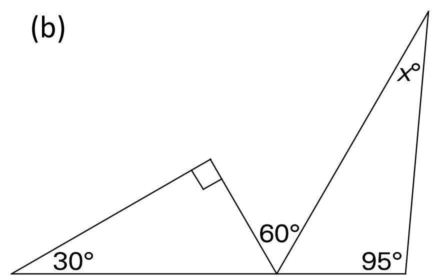


- Determine the value of x.

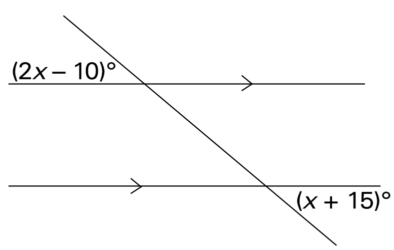
(a)



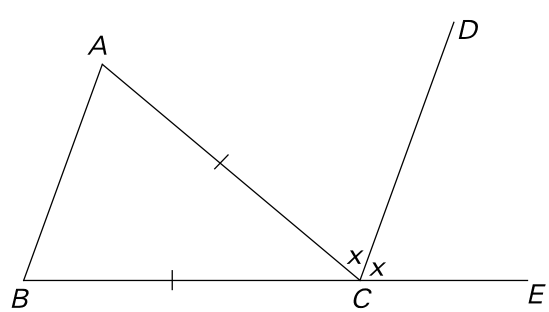
(b)



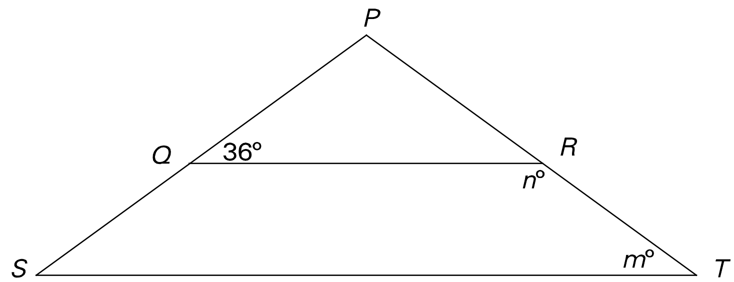
(c)



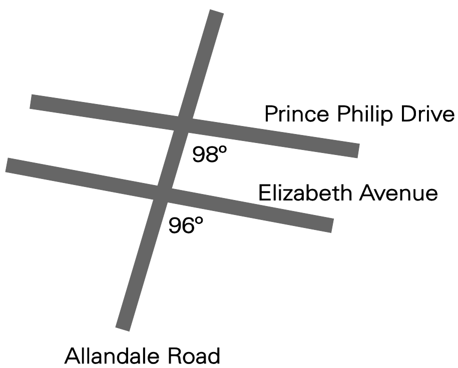
- Given the information in the diagram below, prove that $AB \parallel DC$.



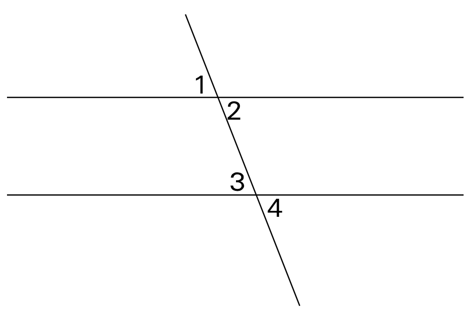
- Calculate the values of m and n, if $PQ = PR$ and $QR \parallel ST$.



- Suppose Prince Philip Drive and Elizabeth Avenue each follow a straight line path and intersect at Allandale Road at angles of 98° and 96°, as shown in the map below. If the streets were to continue in a straight line, would their paths ever cross? Explain your reasoning.



- McKenzie said that if a triangle is obtuse, two of the angles of the triangle are acute. Ask students if they agree with McKenzie. Explain your reasoning.
- Identify and correct errors found in the following example:



Statement	Reason
$\angle 1 = \angle 2$	Vertically opposite angles are congruent.
$\angle 2 = \angle 3$	Alternate interior angles are congruent.
$\angle 1 = \angle 3$	Both are congruent $\angle 2$.
$\angle 3 = \angle 4$	Vertically opposite angles are congruent.
$\angle 1 = \angle 4$	Both are congruent $\angle 3$.
Lines are parallel	Alternate exterior angles are congruent.

- Prove that the line formed by joining the midpoints of two sides of a triangle is parallel to the third side of the triangle.
- Look at a map of the major streets in a city or town. Use your protractor to determine which, if any, streets are parallel.

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

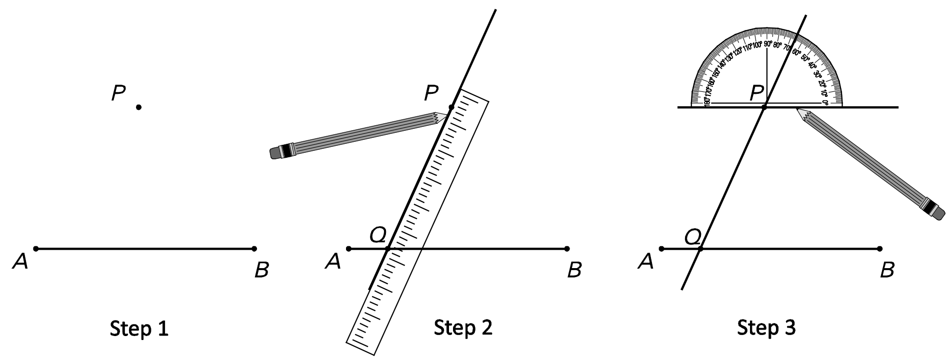
- Consider the following sample instructional strategies when planning lessons.
- Have the students construct parallel lines and verify the angle relationships that were developed earlier. Use a variety of methods—a compass and straight edge, a protractor and ruler, or paper folding and protractor—to ensure that the lines are parallel, and then consider the following:

Using a protractor and a straight edge

Step 1: Draw a segment *AB*. Place a point *P* above the line.

Step 2: Draw a line through *P* and intersecting *AB* at *Q*.

Step 3: Using a protractor, measure $\angle QPB$. Use your protractor to then form an angle having the same measure at *P*. One arm of the angle is parallel to *AB*.



Using paper folding and a protractor

Step 1: Take a blank sheet of paper and fold it in half.

Step 2: Fold it in half again.

Step 3: Unfold the paper and trace any two of the fold lines, using a ruler and protractor.

Step 4: Construct a transversal.

Using a compass and straight edge

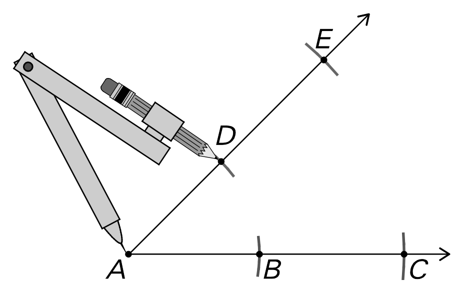
Step 1: Draw a straight line.

Step 2: Use your compass to mark off segments of equal length along this line. Label these points *A*, *B*, and *C*.

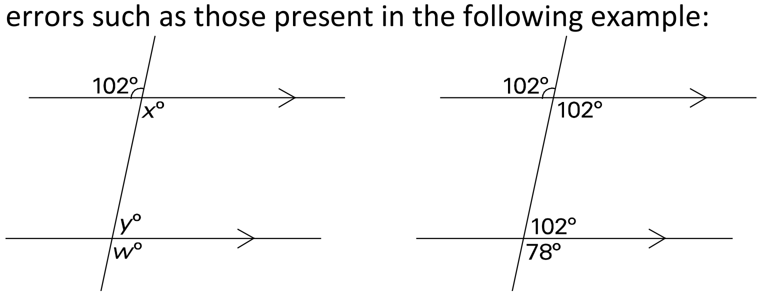
Step 3: Draw a second straight line that intersects the first one at point *A*.

Step 4: Use your compass to mark off segments, beginning at point *A*, of equal length along this second line. Label these points *D* and *E*.

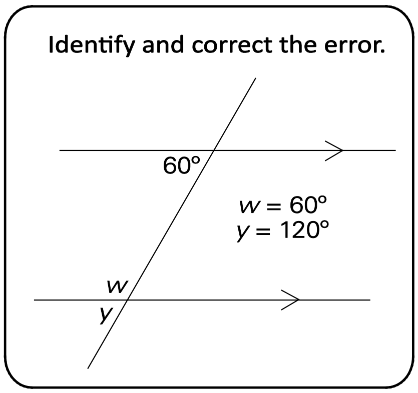
Step 5: Join *B* and *D*; join *C* and *E*.

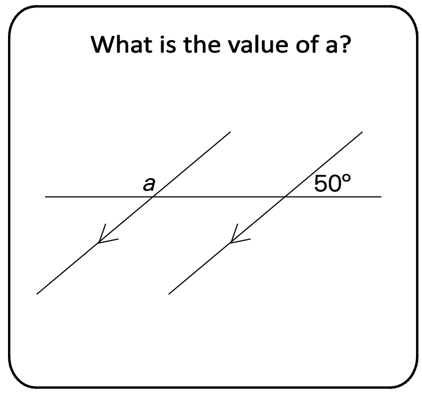


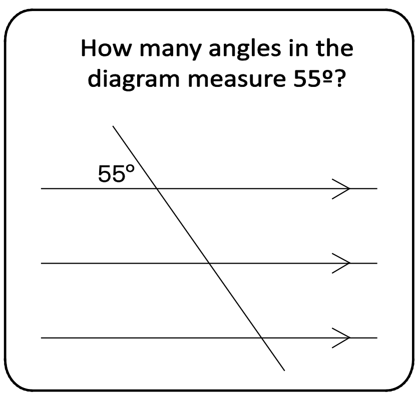
- A common error occurs when students incorrectly identify pairs of angles, leading to an incorrect measurement. For example, students identify same side interior angles as congruent rather than supplementary. Students should be able to identify and correct errors such as those present in the following example:



- Quiz-Quiz-Trade Game:
  - Each student is given a card with a problem. The answer is written on the back of the card.
  - In groups of two, partner *A* asks the question and partner *B* answers.
  - They switch roles and repeat.
  - Partners trade the cards and then find a new partner.
  - Sample cards are shown below.







- Create centers in the classroom containing solutions to problems that involve the measurement of angles. Students will participate in a carousel activity in which they will be asked to move throughout the centers to identify and correct errors.

Suggested Models and Manipulatives

- compasses
- patty paper or tracing paper
- protractor
- straight edges

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- acute angle
- adjacent interior angles
- alternate exterior angles
- alternate interior angles
- converse
- convex polygon
- corresponding angles
- exterior angles
- interior angles
- non-adjacent interior angles
- obtuse angle
- regular polygon
- right angle
- transversal

Resources/Notes

Internet

- Math Is Fun, Parallel Lines, and Pairs of Angles (MathIsFun.com 2011)
  www.mathsisfun.com/geometry/parallel-lines.html
  Basic interactive site that can be used to review the terminology.

Print

- Foundations of Mathematics 11 (Canavan-McGrath et al. 2011, Nelson Canada)
  Sections 2.2–2.4, pp. 73–99

SCO G03 Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case. [CN, PS, R]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

G03.01 Draw a diagram to represent a problem that involves the cosine law and/or sine law.
G03.02 Explain the steps in a given proof of the sine law and of the cosine law.
G03.03 Solve a problem involving the cosine law that requires the manipulation of a formula.
G03.04 Explain, concretely, pictorially or symbolically, whether zero, one or two triangles exist, given two sides and a non-included angle.
G03.05 Solve a problem involving the sine law that requires the manipulation of a formula.
G03.06 Solve a contextual problem that involves the cosine law and/or the sine law.

Scope and Sequence

Mathematics 10	Mathematics 11	Pre-calculus 12
M04 Students will be expected to develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles.	**G03** Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case.	**T06** Students will be expected o prove trigonometric identities, using ▪ reciprocal identities ▪ quotient identities ▪ Pythagorean identities ▪ sum or difference identities (restricted to sine, cosine, and tangent) ▪ double-angle identities (restricted to sine, cosine, and tangent)

Background

In Mathematics 10 students used the primary trigonometric ratios to solve problems involving right triangles. They will now use the sine law and cosine law to solve problems that can be modelled with acute triangles, including the ambiguous case.

In Mathematics 10 students used the terminology angle of elevation, angle of depression, and angle of inclination.

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for G03 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

Students are expected to explain the steps in the derivation of both the sine law (also commonly referred to as the law of sines), and the cosine law (also commonly referred to as the law of cosines), and then use the relationships to calculate unknown sides and angles in triangles. Students apply the laws to calculate unknown side lengths and angle measures, explaining their reasoning.

Students need to be able to explain their thinking about whether they will use the law of sines, the law of cosines, the trigonometric ratios, or the Pythagorean Theorem.

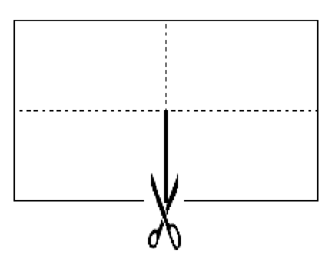
Students examine the conditions that lead to the ambiguous case of the sine law—two given sides and the non-included angle, SSA. Based on given information, students decide whether zero, one, or two triangles exist in non-contextual situations and explain their reasoning in a variety of ways. For contextual problems in which the given information leads to two possible triangles, students decide whether the acute or obtuse situation applies and justify their decision.

Students will solve problems represented by more than one triangle. They will use a combination of strategies such as the primary trigonometric ratios, the Pythagorean Theorem, the sum of the angles in a triangle, the sine law, and the cosine law to solve problems.

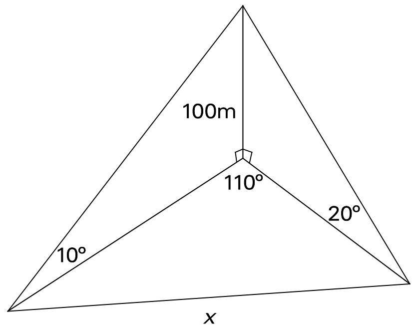
In Mathematics 10 students solved 3-D problems in which the planes were at right angles to each other. In this unit, students will extend these 3-D situations to include instances in which the planes are not perpendicular. Models will be essential for many students when visualizing these 3-D problems.

This can be done simply by using a piece of paper, card stock, or a cue card.

- Fold along the dotted lines, as shown. Next, cut along one of the dotted lines from the edge to the centre.



- Fold to create the corner of a box. The lengths and angles can then be placed in the appropriate locations on the card as shown below.



Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

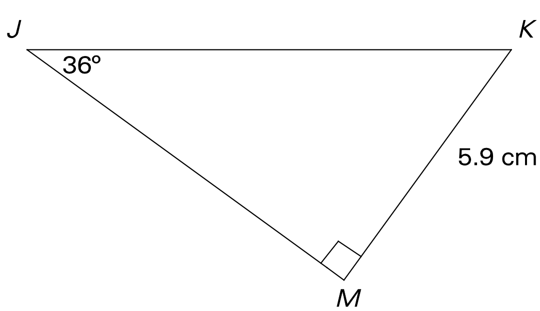
Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

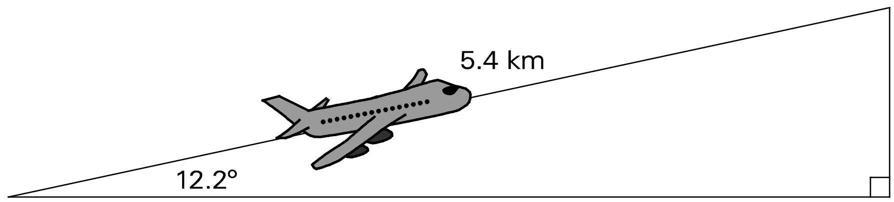
Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Determine the length of the hypotenuse.



- A pilot starts her takeoff and climbs steadily at an angle of 12.2°. Determine the horizontal distance the plane has travelled when it has climbed 5.4 km along its flight path. Express your answer to the nearest tenth of a kilometre.



- The angle between the shorter side of a rectangle and its diagonal is 56°. The shorter side of the rectangle is 2.3 cm. How long is the diagonal?
- Solve the following equations for the unknown:

a) $\frac{3}{x} = \frac{8}{12}$

b) $\frac{2}{5} = \frac{sin30°}{x}$

c) $x^2 = 5^2 + 6^2 - 2(5)(6)(0.5)$

d) $100 = 49 + 4 - 2(7)(2)(x)$

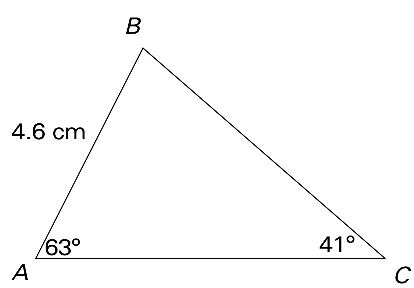
e) $x^2 + 3x - 8 = 0$

f) $100 = 36 + x^2 - 6(x)(0.5)$

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the length of *AC* in the following diagram using two different methods.



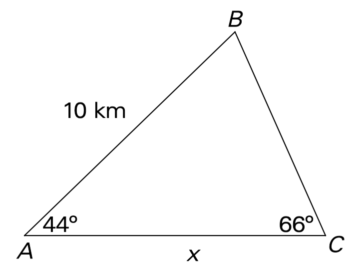
- Answer the following questions:
  (a) Why does the sine law have three ratios in its equation?

  (b) Do you use all three ratios at once? How can you tell which ratios to use?

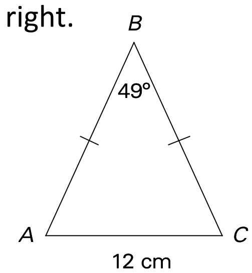
  (c) How many pieces of information about a triangle’s side lengths and angles are needed in order to solve it using the sine law? What possible combinations will work?

(d) Would you use the sine law to solve right triangles? Explain.

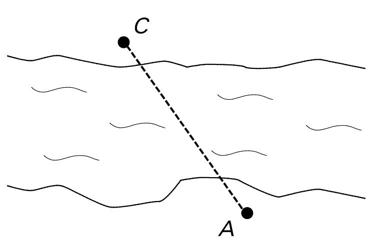
- Two observers sight a plane at angles of elevation of 44° and 66° respectively. If one observer is 10 km away from the plane, how far apart are the two observers from each another?



- Make up a problem in which a side length can be determined using law of sines. Show the solution.
- Make up a problem in which an angle measure can be determined using law of sines. Show the solution.
- Make up a problem in which a side length can be determined using law of cosines. Show the solution.
- Make up a problem in which an angle measure can be determined using law of cosines. Show the solution.
- Find the missing side lengths in the isosceles triangle to the right.

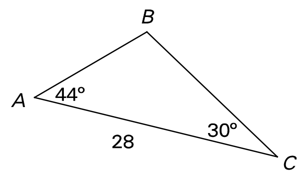


- A surveyor is located on one side of a river that is impossible to cross and only has a 100 m measuring tape and a sextant (used to measure angles) in his possession. Explain how the surveyor could use only these two tools and the law of sines to find the distance from point *A* to point *C*.



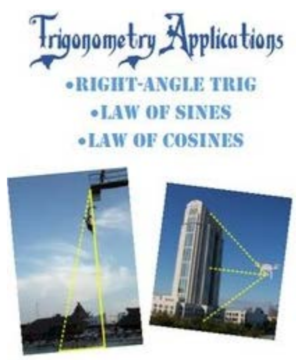
- Explain how the law of cosines validates the Pythagorean theorem if the included angle is 90°.

- Obtain a set of cards/questions from your teacher. Each card or question will contain a triangle with information regarding the lengths of the sides and angles. Sort these cards/questions according to a specific strategy you would use to solve the question. [Pythagorean theorem, sine law, cosine law, primary trigonometric ratios].

Pythagorean theorem	Sine law 
Cosine law	Primary trigonometric ratios

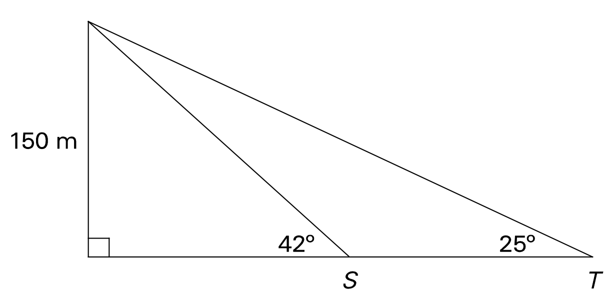
- Research a real-world problem that makes use of the sine law, the cosine law, or both the sine and the cosine laws. Share the problem with the class and discuss the method for solving.

- Consider the infographic shown below. What changes could you make so that it would be more informative?

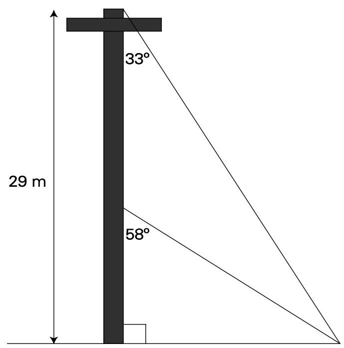


- What quantities must be known in a triangle before the sine law can be used? The cosine law?

- A tower is 150 metres high. Two wires located at positions *S* and *T*, are fastened at the top of the tower and make angles of elevation of 42° and 25° with the ground. How far apart are the two wires?



- Yujin enjoys swimming in the ocean. One day, Yujin decides to swim 9.2 km from Island *A* to Island *B*; then, after resting a few moments, she swims 8.6 km to Island *C*. If Island *C* to Island *A* to Island *B* forms a 52° angle, determine how much further Yujin has to swim by swimming to Island *B* first rather than simply swimming straight from Island *A* to Island *C*.
- A 29-foot-high pole on a farm is supported by two guy wires making angles of $33° \ and \ 58°$ with the vertical pole, as shown below. Find the length of the two guy wires.



- An engineer is asked to build a triangular support frame for an airplane wing with $\angle A$ = 42° and with sides *b* = 13.2 cm and *a* = 10.1 cm respectively. Can the engineer determine the length of the third side? Explain.
- For ΔABC , *a* = 8, *b* = 9, and *c* = 7. What is the measure of $\angle C$?
- A forest ranger is standing at the top of a 100-foot-high tower. She observes a fire at an angle of depression of 25°, turns 120°, and sees, at a 40° angle of depression, a group of campers. How far are the campers from the fire?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

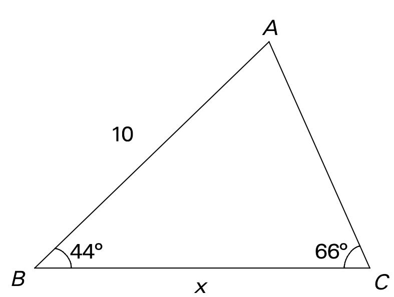
Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

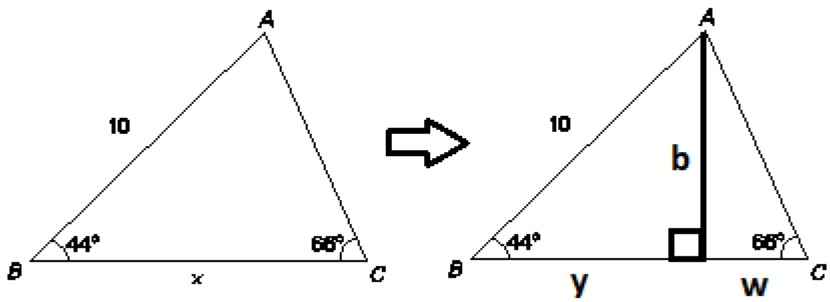
Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Students have been exposed to right-triangle trigonometry to solve problems involving right triangles. Before introducing the sine law, it may be beneficial for students to solve, using the primary trigonometric ratios, a triangle that is not a right triangle. Encourage students to draw a diagram to represent the problem to help them gain a visual understanding of the problem. Consider the following example:



- Ask students if this triangle can be divided into two right triangles and what strategies can be applied to find the indicated length. They should recognize that this requires a multi-step solution. A strategy must be developed before a solution is attempted. Drawing an altitude from vertex *A*, students can use the primary trigonometric ratios and the Pythagorean theorem to solve for the unknown value.



$\cos 44° = \dfrac{y}{10}$

$(10)\cos 44° = y$

$7.19 = y$

$\sin 44° = \dfrac{b}{10}$

$(10)\sin 44° = b$

$6.95 = b$

$\tan 66° = \dfrac{6.95}{w}$

$(w)\tan 66° = 6.95$

$w = \dfrac{6.95}{\tan 66°}$

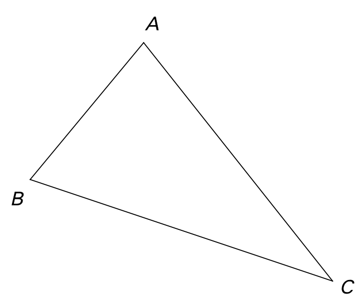
$w = 3.09$

$x = y + w$

$x = 7.19 + 3.09$

$x = 10.28$

- Provide students with a triangle and have them measure the side lengths and angles using a ruler and protractor. Students can then use inductive reasoning to make a conjecture about the law of sines prior to proving it deductively.



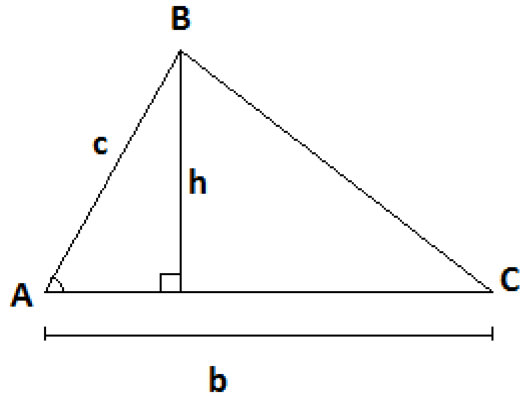
Ask students to answer the following questions:

(a) What conjecture can you make regarding the ratios calculated below?

(b) Would your conjecture be valid if you were to use the reciprocal of the ratios?

	Measure		Measure		Calculate
$\angle A$		Side *a*		$\dfrac{\sin A}{a}$	
$\angle B$		Side *b*		$\dfrac{\sin B}{b}$	
$\angle C$		Side *c*		$\dfrac{\sin C}{c}$	

- The law of sines can be derived using the area formula of a right triangle. Consider the following diagram:



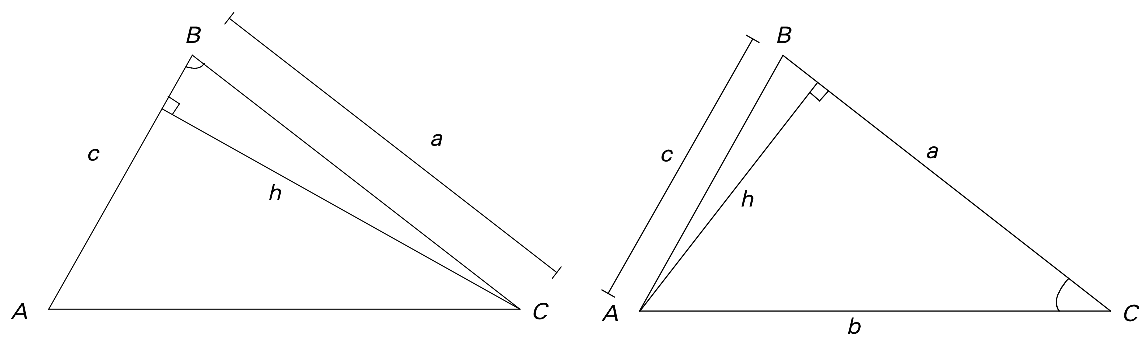
  - Triangle *ABC* is not a right triangle. Therefore, students will draw an altitude from vertex *B*.
  - The area formula for a right triangle is Area = $\frac{1}{2}$ (base)(height).
  - Ask students to write an expression for the height (h) using the sine ratio.

    Since, $\sin A = \frac{h}{c}$

    then (c) sin $A = h$

    Therefore, Area = $\frac{1}{2}(b)(h) = \frac{1}{2}(b)(c \sin A)$

- Ask students to repeat this procedure by drawing an altitude from the other vertices and writing an equation for the area of triangle *ABC*.



They should conclude that Area = $\frac{1}{2}(b)(c \sin A) = \frac{1}{2}(a)(c \sin B) = \frac{1}{2}(a)(b \sin C)$

- Promote student discussion as to why they can set the three area expressions equal to each other.

  $\frac{1}{2}(b)(c \sin A) = \frac{1}{2}(a)(c \sin B) = \frac{1}{2}(a)(b \sin C)$

  $(b)(c \sin A) = (a)(c \sin B) = (a)(b \sin C)$

  Dividing each expression by *abc*:

  $$\frac{(b)(c \sin A)}{abc} = \frac{(a)(c \sin B)}{abc} = \frac{(a)(b \sin C)}{abc}$$

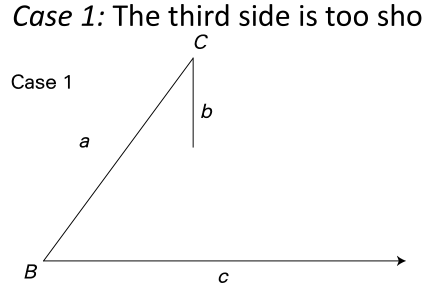
  Simplifying this yields the law of sines:

  $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

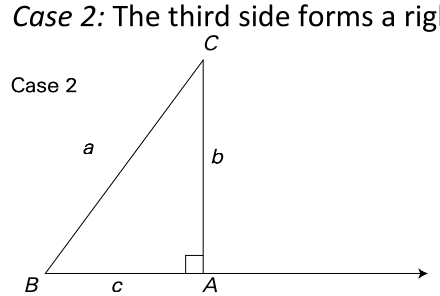
  In other words, the law of sines is a proportion that compares the ratio of each side of a triangle to the sine of its opposite angle.
- Encourage students to draw diagrams with both the given and unknown information marked. Students should recognize that when they have ASA there is a unique triangle that can be found using the law of sines.
- Students should also recognize that when they have SSA, there *may* be more than one solution. When this happens it is called the ambiguous case. There are eight cases to consider (five when the angle is acute and three when the angle is obtuse).

  If the angle is acute, then there are five cases possible, only one of these situations yields the ambiguous case.

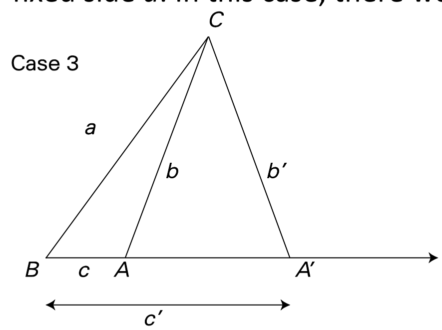
  Case 1: The third side is too short, and there is no solution.



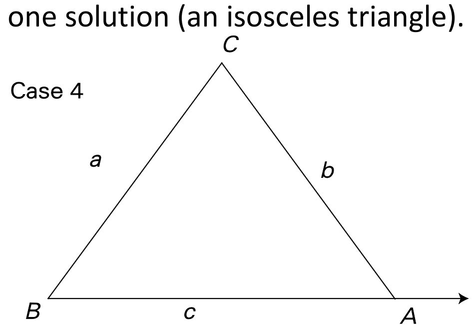
  Case 2: The third side forms a right angle triangle, and there is a single solution.



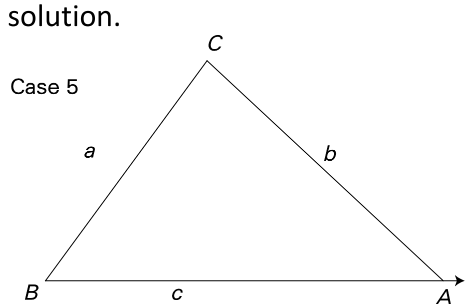
Case 3: The third side is longer than the perpendicular would be, but shorter than the fixed side *a*. In this case, there would be two solutions.



Case 4: The third side is the same length as the fixed side *a*. In this case there is only one solution (an isosceles triangle).

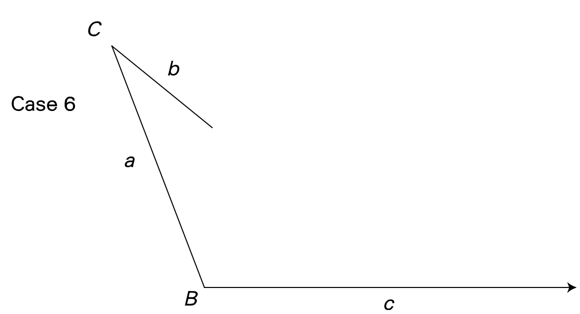


Case 5: The third side is longer than the fixed side *a*. In this case there is only one solution.

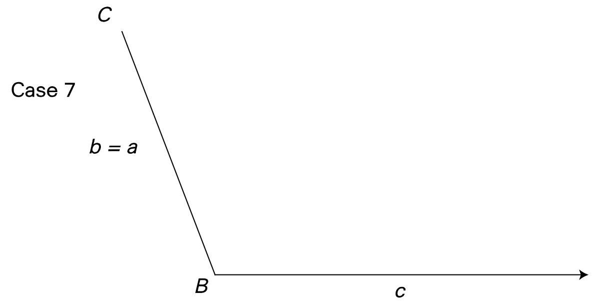


If the angle is obtuse, there are three situations possible; none of these situations yields the ambiguous case.

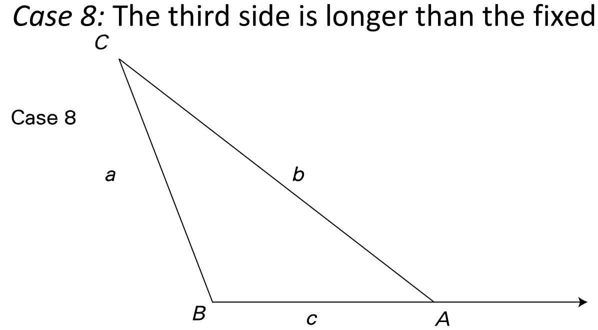
Case 6: The third side is shorter than the fixed side. There will be no possible solution.



Case 7: The third side is the same length as the fixed side. There will be no possible solution.

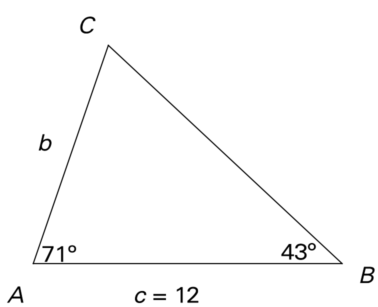


Case 8: The third side is longer than the fixed side. There will be one solution.



Students should recognize when they have information that would create the ambiguous case.

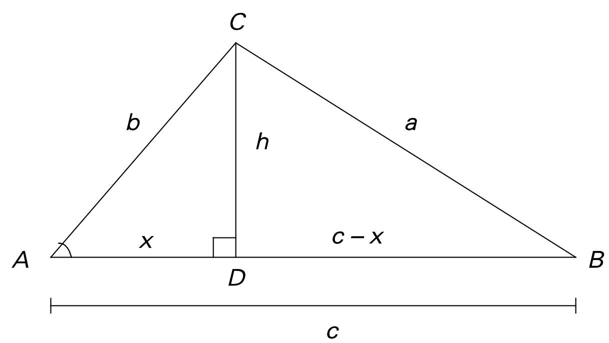
- When working with the sine law, students sometimes incorrectly identify side and opposite angle pairs. To avoid this error, encourage them to use arrows on the diagram when identifying the angle and its opposite side. They could also encounter problems when they multiply to solve the ratios. When solving $\frac{\sin A}{12} = \frac{\sin 30^\circ}{4}$, for example, students may incorrectly write 4 sin *A* = sin 360° instead of 4 sin *A* = 12 sin 30°. The use of brackets may clarify the situation for many students and thus teachers should encourage students to write 4(sin *A*) = 12(sin 30°). Another common student error occurs when students try to solve a triangle given two angles and an included side, mistakenly thinking that there is not enough information to use the sine law. Consider an example such as the following:



Students can use the property that the sum of the angles in a triangle is 180°. The measure of $\angle C$, therefore, is 66°. They can then proceed to use the sine law to find the length of side *AC*.

- Encourage students to check the reasonableness of their answer. For example, since $\angle C$ is a little smaller than $\angle A$, we expect the length of side *AB* to be a little shorter than the length of side *CB*. Students should also consider asking questions such as the following: Is the shortest side opposite the smallest angle? Is the longest side opposite the largest angle?

- Students will prove and use the cosine law to solve triangles. Provide students with a triangle *ABC* with side lengths *a*, *b* and *c*. Ask them to draw an altitude, *h*, from vertex *C* and let *D* be the intersection of *AB* and the altitude, as shown in the figure below. If *x* is the length of *AD*, they should recognize $BD = c - x$. Guide students through the following process:



  - Use the Pythagorean theorem in ΔBCD: $a^2 = h^2 + (c - x)^2$
  - Expand the binomial: $a^2 = h^2 + (c^2 - 2cx + x^2)$
  - Rearrange to obtain: $a^2 = x^2 + h^2 + c^2 - 2cx$
  - For ΔACD: $x^2 + h^2 = b^2$
  - substitute to obtain: $a^2 = b^2 + c^2 - 2cx$
  - In ΔACD: $\cos A = \dfrac{x}{b}$
  - resulting in: $b \cos A = x$
  - Substitute for *x* into $a^2 = b^2 + c^2 - 2cx$: $a^2 = b^2 + c^2 - 2c(b \cos A)$ or $c^2 = a^2 + b^2 - 2a(b \cos C)$

- Students can then express the formula in different forms to find the lengths of the other sides of the triangles. It is important that students be able to apply the cosine formula flexibly.
  (as $a^2 = b^2 + c^2 - 2bc \cos A$ or $b^2 = a^2 + c^2 - 2ac \cos B$ or $c^2 = a^2 + b^2 - 2ab \cos C$)

- The cosine law can be used to determine an unknown side or angle measure in a triangle. Continue to encourage students to draw diagrams with both the given and unknown information marked when solving problems. Students should consider why the cosine law is used to find the unknown angle if three sides are known (SSS) or if two sides and the included angle are known (SAS).

Have students consider the following possibilities:

Included angle is acute	**Included angle is right**	**Included angle is obtuse**
$c^2 = a^2 + b^2 - 2ab \cos 60°$ $c^2 = a^2 + b^2 - 2ab(0.5)$ $c^2 = a^2 + b^2 - ab$ (Side opposite an acute angle is smaller than if it had been opposite a right angle.)	$c^2 = a^2 + b^2 - 2ab \cos 90°$ $c^2 = a^2 + b^2 - 2ab(0)$ $c^2 = a^2 + b^2$ (Pythagorean theorem)	$c^2 = a^2 + b^2 - 2ab \cos 150°$ $c^2 = a^2 + b^2 - 2ab(-0.5)$ $c^2 = a^2 + b^2 + ab$ (Side opposite an obtuse angle is larger than if it had been opposite a right angle.)

- When three sides of a triangle are known, students will use the cosine law to find one of the angles. Some students may rearrange the equation to solve for a particular angle. Others may substitute the unknown values into the cosine law and then rearrange the equation to find the angle. It is important for students to recognize that they have a choice when trying to find the second angle—they can either use the cosine law or the sine law. Students should notice the third angle can then be determined using the sum of the angles in a triangle.
- When solving triangles, encourage students to consider the following questions:
  - What is the given information?
  - What am I trying to solve for?
  - With the given information, should I use the sine law or the cosine law? Is there a choice?
- If students know two sides and a non-included angle (SSA) of a triangle, they can use the cosine law in conjunction with the sine law to find the other side and avoid the need to determine the second possible angle by applying the sine law twice.

  As an alternative, they could apply the sine law twice if they are aware of the possible ambiguous case.

  Students need to be exposed to numerous examples in order to find the method that works best for them.

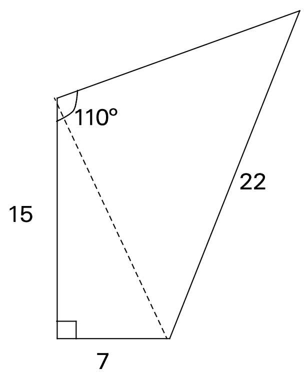
Example

Two sides of a triangle are 5 " and 3 " in length. The angle opposite the 5 " side is 70°. Determine the length of the other side.

Law of cosines and law of sines	Law of sines done twice
$\frac{\sin 70^\circ}{5} = \frac{\sin A}{3}$ $\frac{3\sin 70^\circ}{5} = \sin A$ $0.5638 = \sin A$ $A = 34.32^\circ$ $\angle X = 180^\circ - (34.32^\circ + 70^\circ) = 75.68^\circ$ $X^2 = 3^2 + 5^2 - 2(3)(5)\cos 75.68^\circ$ $X^2 = 9 + 25 - 7.42$ $X^2 = 26.58$ $X = 5.16$	$\frac{\sin 70^\circ}{5} = \frac{\sin A}{3}$ $\frac{3\sin 70^\circ}{5} = \sin A$ $0.5638 = \sin A$ $A = 34.32^\circ$ $\angle X = 180^\circ - (34.32^\circ + 70^\circ) = 75.68^\circ$ $\frac{x}{\sin 75.68^\circ} = \frac{5}{\sin 70^\circ}$ $x = \frac{5\sin 75.68^\circ}{\sin 70^\circ}$ $x = 5.16$
Law of cosines (using quadratics)	
$5^2 = 3^2 + x^2 - 2(3)(x)\cos 70^\circ$ $25 = 9 + x^2 - 2.052(x)$ $0 = x^2 - 2.052(x) - 16$ Using the quadratic formula $x = \frac{-(-2.052) \pm \sqrt{(-2.052)^2 - 4(1)(-16)}}{2(1)}$	$x = \frac{2.052 \pm \sqrt{(68.211)}}{2}$ $x = \frac{2.052 \pm 8.259}{2}$ $x = 5.16$ *or* -3.10 Only the solution of 5.16 is possible for a side length.

- When working with the cosine law, students sometimes incorrectly apply the order of operations. When asked to simplify $a^2 = 365 - 360\cos 70^\circ$, for example, students often write $a^2 = 5\cos 70^\circ$. To avoid this error, teachers should emphasize that multiplication is to be completed before subtraction.
- Use observation to assess student understanding by providing students with several practice problems that use the sine law and/or the law of cosines. As teachers observe students working through the problems, ask them the following questions:
  - What is the unknown? Is it an angle or a side?
  - How can you isolate the unknown?
  - How can you complete the calculations?
  - Does your conclusion answer the question asked?

- Students may need to use a strategy or a combination of strategies to solve problems represented by one or more than one triangle. Provide students with an example in which a playground, in the shape of a quadrilateral, is to be fenced. Ask them to determine the total length of fencing required.

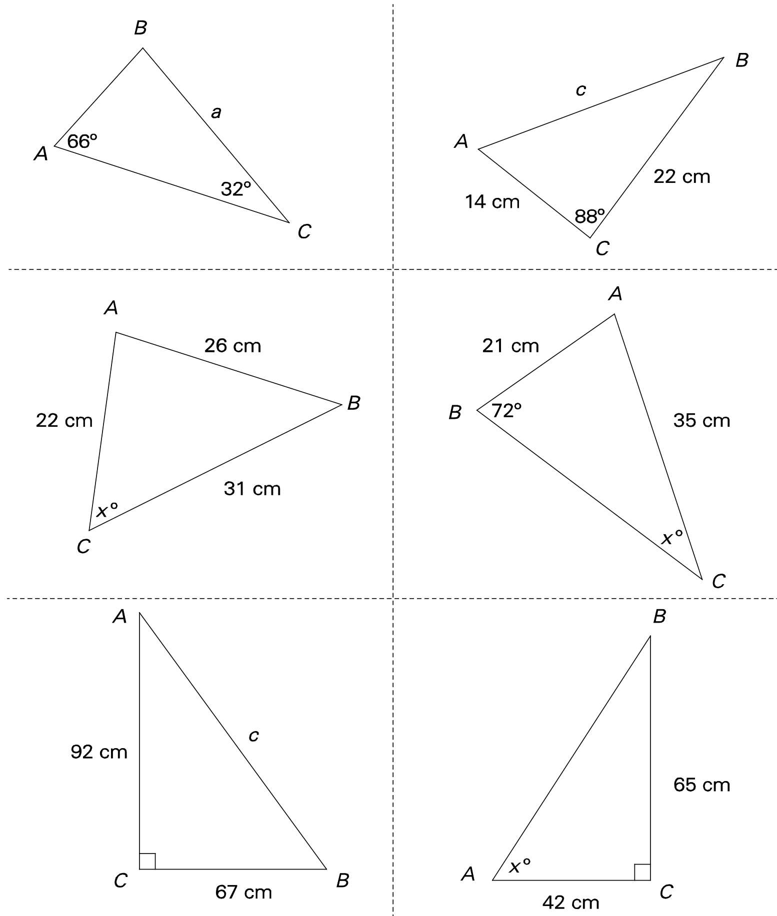


- Discussion around various strategies one might use, as well as which method is the most efficient, is encouraged. Ask students how changing the right angle in the figure to an acute angle, such as 50°, would affect their strategy.
- Ask students to create a graphic organizer. When solving triangles, the organizer can guide students as they decide upon the most efficient method to use when solving for an unknown angle and/or side.

SSS	**ASA**	**SAS**	**SSA**	**Right Triangle**
Law of cosines	Law of sines	Law of cosines	Law of sines (ambiguous case) and/or law of cosines (using quadratic formula)	Pythagorean theorem and/or trigonometric ratios

- In the task Four Corners, students have to think about which method they would use to solve a triangle. Post four signs, one in each corner of the room labelled sine law, cosine law, Pythagorean theorem, and trigonometric ratios. Provide each student with one triangle. Instruct the students to make a decision as to which method they would use to find the missing angle or side and to stand in the corner where it is labelled. Once students are all placed, ask them to discuss why their triangle(s) would be best solved using that particular method.

Sample triangles are given below:



Suggested Models and Manipulatives

- protractors
- rulers

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- ambiguous case
- law of cosines
- law of sines

Resources/Notes

Print

Foundations of Mathematics 11 (Canavan-McGrath et al. 2011, Nelson Canada)
Sections 3.1–3.4, pp. 116–148
Sections 4.1–4.4, pp. 162–196

Logical Reasoning
18-22 hours

GCO: Students will be expected to develop logical reasoning.

SCO LR01 Students will be expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems.
[C, CN, PS, R]

[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation
[T] Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

LR01.01 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
LR01.02 Explain why inductive reasoning may lead to a false conjecture.
LR01.03 Compare, using examples, inductive and deductive reasoning.
LR01.04 Provide and explain a counterexample to disprove a given conjecture.
LR01.05 Prove algebraic and number relationships, such as divisibility rules, number properties, mental mathematics strategies, or algebraic number puzzles.
LR01.06 Prove a conjecture, using deductive reasoning (not limited to two column proofs).
LR01.07 Determine if an argument is valid and justify the reasoning.
LR01.08 Identify errors in a given proof.
LR01.09 Solve a contextual problem involving inductive or deductive reasoning.

Scope and Sequence

Mathematics 10	Mathematics 11	Mathematics 12
	LR01 Students will be expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems.	**LR01** Students will be expected to analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies. (M12)* **T06** Students will be expected to prove trigonometric identities using • reciprocal identities • quotient identities • Pythagorean identities • sum or difference identifies (restricted to sine, cosine, and tangent) • double-angle identities (restricted to sine, cosine, and tangent) (PC12)**

* M12—Mathematics 12
** PC12—Pre-calculus Mathematics 12

Background

Students have had no formal instruction with this topic in previous mathematics courses. However, as part of their life experiences, students are expected to have drawn conclusions (conjectures) based on observations.

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for LR01 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

To develop logical reasoning skills, students will examine situations, information, problems, puzzles, and games. They will form **conjectures** through the use of **inductive reasoning** and prove their conjectures through the use of **deductive reasoning**.

A **conjecture** is a testable expression that is based on available evidence but is not yet proven.

Inductive reasoning is a form of reasoning in which a conclusion is reached based on a pattern present in numerous observations. The premise makes the conclusion likely, but does not guarantee it to be true.

Deductive reasoning is the process of coming up with a conclusion based on information that is already known to be true. The facts that can be used to prove a conclusion deductively may come from accepted definitions, properties, laws, or rules. The truth of the premises guarantees the truth of the conclusion.

Students are introduced to inductive reasoning, through the investigation of geometric situations and are introduced to conjectures through the observation of patterns. They will be required to justify the reasoning used for all conjectures.

Students will be presented with situations to explore and will be encouraged to develop conjectures based on these situations. They will also explore the role that *counterexamples* play in disproving conjectures.

Inductive Reasoning	Deductive Reasoning
Begins with experiences or a number of observations.	Begins with statements, laws, or rules that are considered true.
An assumption is made that the pattern or trend will continue. The result is a conjecture.	The result is a conclusion reached from previously known facts.
Conjectures may or may not be true. One counterexample proves the conjecture false.	Conclusion must be true if all previous statements are true.
Used to make educated guesses based on observations and patterns.	Used to draw conclusions that logically flow from the hypothesis.

In this Logical Reasoning unit students are introduced to the notion of formal *proof* including, but not restricted to, the *two-column proof* format.

When constructing proofs, students must use proper mathematical terminology. Students have been exposed to the Pythagorean Theorem, the number system, and divisibility rules from previous grades. There are some concepts, however, that will need to be introduced, such as **supplementary angles**, **complementary angles**, and the **transitive property**.

Angles are *supplementary* when their measures add up to 180°.

Note: Students have been exposed to the term **straight angle** and the concept that if angles *A*, *B*, and *C* make a straight angle then $\angle A + \angle B + \angle C = 180°$. However, they are not familiar with the term **supplementary**. Angles are **complementary** when their measures add to 90°.

The **transitive property** in mathematics and logic, is a statement that if *A* bears some relation to *B* and *B* bears the same relation to *C*, then *A* bears that relation to *C*.

Students will examine arguments and proofs and judge whether or not the reasoning presented is valid. They will determine if there is an error in the reasoning used and, if so, will identify the error.

Contextual problems will be solved using inductive or deductive reasoning.

Deductive reasoning involves drawing a specific conclusion through logical reasoning by starting with general statements that are known to be valid. As a part of the logical reasoning outcome, students will be introduced to geometric proofs using deductive reasoning. However, there will be a much greater emphasis placed on proofs as a part of the geometry outcomes.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Given the visual below, develop an explanation for the possible events that have occurred.

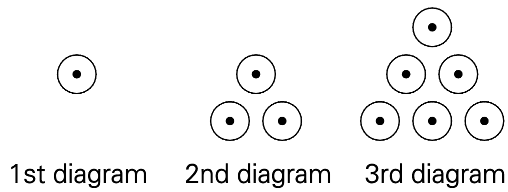
- Determine the next number in the pattern 2, 6, 12, 20, 30, 42, __.
- What is the maximum number of squares that can be created using 20 segments of equal length?
- Using a copy of Pascal's triangle, identify a pattern. (See appendix.)
- How does the way in which optical illusions "trick" your eyes relate to the ideas of valid versus invalid conjectures?

Whole-Class/Group/Individual Assessment Tasks

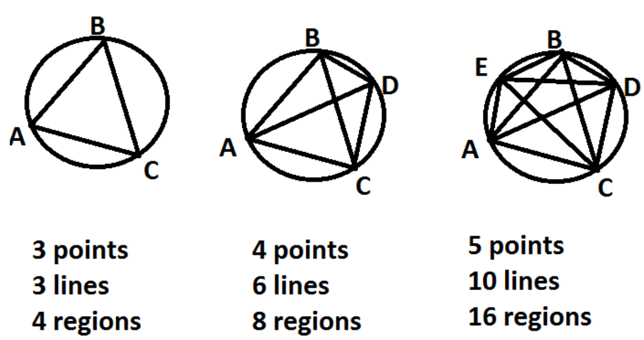
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Draw any quadrilateral. Find the midpoint of each side and connect them. What type of quadrilateral results? Compare your findings with other students and make a conjecture based on your observations.

- Jamal says that if a quadrilateral has four sides of equal length then it must be a square. Is Joe correct? Explain.
- Make a conjecture based on the following statement:
  - Shannon took her umbrella with her when she left for work each of the last four times that rain was in the forecast. Today rain is in the forecast and Shannon is getting ready to leave for work. Are you certain that your conjecture is correct? Explain.
- Find a pattern in a magazine or newspaper and describe it in words. Write a conjecture about the pattern and present your example to the class.
- Describe several situations in your life in which you make conjectures.
- What would the sixth diagram look like in the following sequence?



- Make a conjecture about
  - the sum of the angles in a quadrilateral
  - the product of two consecutive integers
- Pierre found the following pattern on REDDIT. Make a conjecture and test your conjecture.

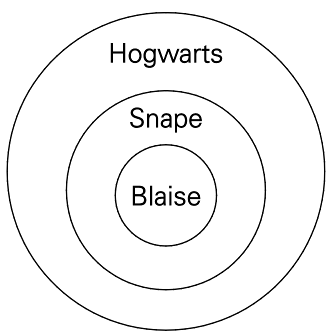


- Complete the following conjecture that holds for all the equations: The sum of two odd numbers is always ...
  3 + 7 = 10
  11 + 5 = 16
  9 + 13 = 22
  7 + 11 = 18

- Write a conjecture about combining two numbers using a mathematical operation. Show a number of examples supporting your conjecture.
- Provide an example in which a limited number of observations might lead to a false conjecture.
- Mike had breakfast at Tarek's Cafe, a local restaurant, for three Saturdays in a row. He saw Sophia there each time. Mike told his friends that Sophia always eats breakfast at Tarek's on Saturdays.
  (a) What type of reasoning did Mike use?
  (b) Is his conclusion valid? Explain.
- Provide a counterexample, if possible, for each of the following conjectures:
  (a) If a number is divisible by 2, then it is divisible by 4.
  (b) If $x + 4 > 0$, then x is positive.
  (c) For all positive numbers n, $\frac{1}{n} \leq n$
- Give a counterexample, if possible, to the following conjecture: The difference between two positive numbers is always a positive number.
- For all real numbers x, the expression x2 is greater than or equal to x. Do you agree? Justify your answer.
- Construct different sized polygons (triangle, quadrilateral, pentagon, hexagon, and octagon). Measure the interior angles. Next, find the sum of the interior angles for each polygon. Make a conjecture to determine the sum of the interior angles for a polygon with n sides.
- If $x < 0$, then $x < -x$. Do you agree? Justify your answer.
- Prove that the sum of a two-digit number and the number formed by reversing its digits will always be divisible by 11.
- Prove that the difference between an odd integer and an even integer is an odd integer.
- Do the following number puzzles and prove them deductively:
  (a) Fabulous Five:
    - Choose a number from 1 to 10.
    - Double the number.
    - Add 10 to your new number.
    - Divide the total by 2.
    - Subtract the original number that you started with.
    - Your answer will always be 5.
  (b) Is This Your Number?
    - Think of any number and write it on a small sheet of paper. Fold it up and place it in your pocket. Remember this number.

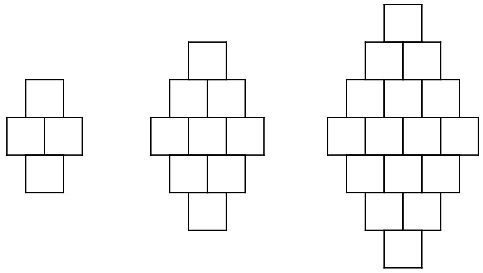
- Find a partner and ask them to think of a number. Make sure they do not tell you.
- Tell them to double the number they chose.
- This is where the number on your paper comes in. In your head, double the number you wrote.
- Tell your partner to add the number you doubled in your head.
- Then tell them to divide this new number by 2.
- Next, tell them to subtract the number they currently have from the first one they started with, or vice versa.
- Pull out the piece of paper in your pocket and give it to your partner. They will be amazed because the number on the paper will be their number!

- Create a number puzzle that always results in a final answer of 4. Prove the conjecture deductively.
- Find an example of a number puzzle and prove it deductively.
- Identify situations in your everyday life in which you use inductive reasoning and in which you use deductive reasoning. Provide examples of each.
- Dean was given the following situation:
  Snape is a teacher at Hogwarts. Blaise is a student at Hogwarts. Dean concluded that Blaise is a student of Snape's and represented his conclusion in the Venn Diagram shown below. Identify his error and construct a new diagram to represent this situation.



- Ten women meet for a bowling tournament, and each shakes the hand of every other woman. Determine the number of handshakes that occurred. Explain the strategy used to arrive at the answer.
- Look at a monthly calendar and pick any three numbers in a row, column, or diagonal. Using inductive reasoning, make a conjecture about the middle number. Use deductive reasoning to prove your conjecture.
- You have a 5-L bottle and a 3-L bottle, neither with any markings on the sides. How can you measure 4 L of water?
- Predict the number of squares that would be in the next (4th) diagram. Explain how you could predict the number of squares that would be found in the 10th diagram and

generalize your explanation to predict the number of squares that could be found in the *n*th diagram.



- Explain how a specific game uses inductive and/or deductive reasoning.
- Margaux creates a series of rectangles, each having different dimensions, and calculated the area and perimeter. She recorded her results as shown.

Length (cm)	**Width (cm)**	**Area (cm^2)**	**Perimeter (cm)**
5	4	20	18
10	3	30	26
6	6	36	24
8	3	24	22

Margaux makes the conjecture that the area of a rectangle is always greater than its perimeter. Do you agree with this conjecture? Justify your reasoning.

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

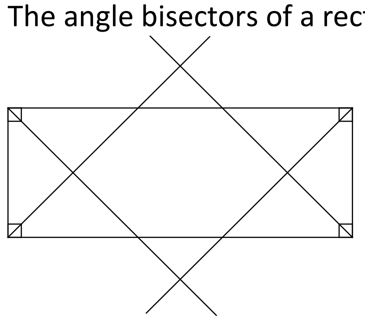
Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

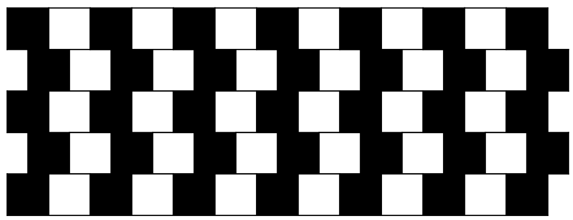
- Students could be introduced to this unit through the analysis of real-life situations.
- Show students a video clip of Young Sherlock Holmes to illustrate inductive reasoning. Ask students to note as many examples of inductive reasoning that they hear and identify which ones may lead to a false conjecture.
- Use a stations activity to have students investigate investigating patterns and making conjectures. For example: Students could investigate the angle relationships in polygons, various picture patters and various number patterns.
- Give students an opportunity to make conjectures using a variety of strategies. Encourage them to draw pictures, construct tables, use measuring instruments such as rulers and protractors and/or use geometry software such as FX Draw (Efofex Software 2013) or Geometer's Sketchpad (Key Curriculum 2013). Dynamic geometry programs, can be used to construct accurate diagrams of geometrical objects. Once students have been exposed to these programs, more complicated conjectures can be investigated which could, in turn, lead to valuable classroom discussions.
- Provide students with specific criteria and encourage them to generate their own conjectures. For example, give students an opportunity to choose a class of quadrilaterals and ask them to identify any properties of the angle bisectors or perpendicular bisectors of the sides of those figures. A sample of answers may include the following:
    - The angle bisectors of a rectangle make a square.



- In a kite, the point where all the angle bisectors meet is the centre.
- In an isosceles trapezoid, the two base angle bisectors form an isosceles triangle.

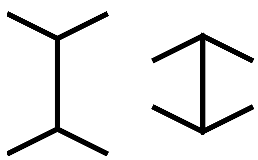
Students should be encouraged to share their answers with other classmates, to discuss what they observe, and to explain how they came up with their conjectures.

- Students should be presented with situations where they are expected to make a conjecture and then verify or discredit it. They will be expected to gather additional evidence, such as taking measurements, performing calculations, or extending patterns, to determine if a given argument is valid. Consider the following examples:
  - Make a conjecture about the horizontal lines in this diagram.



Students may initially believe that the lines are slanted. Using a ruler to investigate their conjecture, they will discover that the lines are, in fact, straight.

  - Make a conjecture about the length of the vertical line segments.



In this example, students may make a conjecture that the vertical line segment on the left is longer than the vertical line on the right. If they use a ruler to test their conjecture, they will discover that both lines are actually the same length. Students should then revise their conjecture.

- Students should develop an understanding that inductive reasoning sometimes leads to false conjectures. They can explore situations in which the collection of additional data, for example, refutes an initial conjecture. Consider an example such as the following:
  - A small bakery is building its business. The owner notices that sales are increasing rapidly in the first three weeks of operation. The following table gives a record of the owner's observations:

Week	1	2	3
Number of Cakes Sold	10	15	20

Jaden conjectures that this bakery is selling 5 additional cakes each week. He further conjectures that by the end of the year (52 weeks) this business will be selling more than 250 cakes per week. Are these conjectures valid? Explain.

- Students may initially believe that these conjectures are valid, but upon further exploration, they need to recognize that, even if this trend could continue through increased consumer demand, at some point the bakery would reach its maximum capacity. Furthermore, it is unlikely that consumer demand would continue to increase in such a tidy and constant way. While a conjecture might be reasonable for a small domain, and perhaps Jaden's conjectures may be correct for a limited number of weeks, using his conjecture to make projections over a larger domain is likely flawed reasoning.
- When students make a conjecture, they will attempt to either prove their conjecture by constructing a valid argument or they will attempt to disprove their conjecture by finding a counterexample. One counterexample is sufficient to disprove a conjecture. To introduce this topic, students could be asked to come up with counterexamples for statements such as the following:
    - Any piece of furniture having four legs is a table.
    - If a tree has no leaves then it is winter.
    - If the grass is wet, it is raining.
- Students will continue to use strategies, such as drawing pictures or using measuring instruments, when looking for a counterexample. After a counterexample is found, students should revise their conjecture to include the new evidence.

  Example
  A student may conjecture that all Pythagorean triples can be created by procedure described below.
    - Start with an odd number (> 1). This is the first of the three numbers in the triple.
    - Square that number, then divide by 2.
    - Round the result down and up, giving the remaining two numbers in the triple.

  Example
    - Start with 5.
    - Square to get 25, and divide by 2 to get 12.5.
    - Round down to get 12, and up to get 13.
    - Triple is 5-12-13.

  This pattern works for the Pythagorean triples 3-4-5; 5-12-13; 7-24-25; 9-40-41, and many others. But there are many Pythagorean triples that begin with an even number (such as 8-15-17) and some Pythagorean triples that begin with an odd number and cannot be created this way (such as 33-56-65). These counterexamples might have the students revise their conjecture.

- To reinforce the idea of counterexamples:
    - Provide students with cards containing a conjecture that may be disproved using counterexamples, and have them complete a Quiz-Quiz-Trade task. Student 1 must read the conjecture on the card, and student 2 must provide a

counterexample to disprove the statement. When both students have disproved the conjectures, they exchange cards and find a new partner. Examples of conjectures include the following:

- If a number is divisible by 2, then it is divisible by 4.
- No triangles have two sides of the same length.
- All basketball players are more than 6 feet tall.
- If it is a cell phone, then it has a touch screen.

o Ask students to participate in the task, Find Your Partner. Half of the students should be given a card with a conjecture on it and the other half should be given a card with a counterexample on it. Students need to move around the classroom to match the conjecture with the correct counterexample. They should then present their findings to the class.

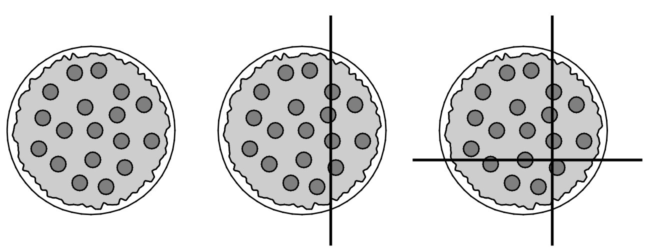
- Have students research either a scientific or mathematical theory that was disproven using counterexamples. Ask them to explain the theory and how it was disproven.

- When testing conjectures involving numbers, students may be unable to find a counterexample because they often use the same types of number. For example, students may only consider natural numbers. They should be encouraged to try various types of numbers, such as whole numbers, fractions, positive numbers, negative numbers, and zero.

- It is important for students to realize that being unable to find a counterexample does not prove a conjecture. The mathematician Christian Goldbach is famous for discovering a conjecture that has not yet been proved to be true or false. Goldbach conjectured that every even number greater than 2 can be written as the sum of two prime numbers. For example, 4 = 2 + 2, 6 = 3 + 3, and 8 = 3 + 5. No one has ever proved this conjecture to be true or found a counterexample to show that it is false, so the conjecture remains just that. As an extension, students can make a list of prime numbers at least up to 100, and use them to test the truth of the conjecture for even numbers up to 100.

- As teachers observe students work, they should question students about the considerations that have to be taken into account. For example, for the following question:

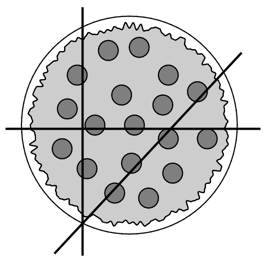
o Melissa made a conjecture about slicing pizza. She noticed a pattern between the number of slices of pizza and the number of cuts made in the pizza.



Number of cuts	0	1	2
Number of slices of pizza	1	2	4

Her conjecture was that the number of pizza slices doubled with each cut. Do you agree or disagree? Justify your decision.

- Prompt discussion using the following questions for the conjecture:
  - Is it important to take into account how the pizza is cut?
  - Do all the slices of pizza have to be the same area?
- A possible counterexample for this situation is shown below:



- Ask students to work in groups for various activities. For example:
  - Each group will be given a deck of cards with a variety of conjectures on them. Students will work together to determine which cards use inductive reasoning and which cards use deductive reasoning.
- Create centres in the classroom containing proofs that contain errors. Ask students to participate in a carousel activity in which they move through the centres while identifying and correcting the errors.
- Video clips can be used effectively to discuss logical reasoning. For example, a video clip of a court scene, such as one from *A Few Good Men*, *CSI*, *Lincoln Lawyer*, or another similar show/movie, could be shown in order to prompt a discussion on how inductive and/or deductive reasoning is used in the clip.
- Students could begin their exploration of proof by looking at examples involving the transitive property. Consider the following example: All natural numbers are whole numbers. All whole numbers are integers. Three is a natural number. What can be deduced about the number 3?
- Students should be exposed to the different strategies they can use to prove conjectures.
  - Students can use a visual representation, such as a Venn diagram, to help them understand the information in an example.
  - They can express conjectures as general statements. This involves choosing a variable to algebraically represent the situation. For example, two consecutive integers can be represented as x and $x + 1$.
  - Students can use a two-column proof that contains both statements and reasons. As an alternative, the steps used to prove the conjecture can be communicated in paragraph form.
- It is important for students to recognize that depending on the problem, certain strategies are more efficient than others. This is an opportunity for students to work in

groups to investigate a number of proofs given a specific method. After students have completed the task, pose the following questions to promote class discussion:

- What strategy was used to prove your conjecture?
- Could you use another strategy to prove the conjecture?
- Is one strategy more efficient than another? Why?
- Students should recognize that proving conjectures for varying situations may require different formats. When proving a conjecture involving a geometric figure, for example, a two-column proof or paragraph form is most appropriate. To prove a conjecture involving number properties, divisibility rules, or algebraic number tricks, expressing conjectures as general statements is often the most appropriate method.
- When constructing proofs, including algebraic and number relationships, a common error occurs when students use a numerical value as their example. Students need to understand that a conjecture is only proved when all cases are shown to be true. Consider the following example: Choose a number. Add 5. Double the result. Subtract 4. Divide the result by 2. Subtract the number you started with. The result is 3.
- Students can show inductively, using three or four numbers, that the result is always 3. To prove deductively, students must show this is the case for all numbers by letting the first number be x.
- It is important for students to compare inductive and deductive reasoning through the use of examples. The example below could be used to highlight their differences.
- The sum of four consecutive integers is equal to the sum of the first and last integers multiplied by two.

Inductive Reasoning	**Deductive Reasoning**
1, 2, 3, 4 $1 + 2 + 3 + 4 = 10$ $2(1 + 4) = 10$	Let the numbers be x, $x + 1$, $x + 2$, $x + 3$. $x + (x + 1) + (x + 2) + (x + 3) = 4x + 6$ $2[x + (x + 3)] = 2(2x + 3) = 4x + 6$

- Exploring the differences between inductive and deductive reasoning through examples will strengthen student understanding of these concepts. The table below could be used to summarize inductive reasoning versus deductive reasoning.

Inductive Reasoning	**Deductive Reasoning**
Begins with experiences or a number of observations.	Begins with statements, laws, or rules that are considered true.
An assumption is made that the pattern or trend will continue. The result is a conjecture.	The result is a conclusion reached from previously known facts.
Conjectures may or may not be true. One counterexample proves the conjecture false.	Conclusion must be true if all previous statements are true.
Used to make educated guesses based on observations and patterns.	Used to draw conclusions that logically flow from the hypothesis.

- It is beneficial to have students analyze proofs that contain errors. To reinforce their understanding of inductive and deductive reasoning, students should identify errors in a given proof and explain why those errors might have occurred and how they can be corrected. Some typical errors include the following:
  - Proofs that begin with a false statement:
    - All high school students like Facebook. Shoshannah is a high school student; therefore, Shoshannah likes Facebook.
    - Gathering a large quantity of data will strengthen the validity of the conjecture, but to prove a conjecture all cases must be considered. Students should consider if it is possible to ask all high school students if they like Facebook.
  - Algebraic errors

Shelby was trying to prove this number trick:	Shelby wrote the following:
	Let n be your number.
– Pick a number.	$2\ n$
– Double your number.	$2n + 20$
– Add 20.	$n + 20$
– Divide by 2.	$n + 20 - n$
– Subtract the original number.	20
– The result is 10.	

Students should identify and correct the error in Shelby's work.

  - Division by zero

Pedro claims he can prove that 2 = 5. His work is shown below.

Suppose $a = b$.

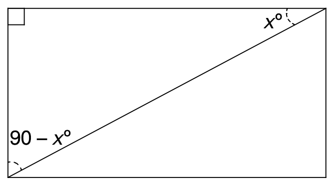
$-3a = -3b$	Step 1: Multiplying by –3.
$-3a + 5a = -3b + 5a$	Step 2: Add $5a$ to both sides.
$2a = -3b + 5a$	Step 3: Simplify
$2a - 2b = -3b + 5a - 2b$	Step 4: Subtract $2b$ from each side.
$2a - 2b = 5a - 5b$	Step 5: Simplify
$2(a - b) = 5(a - b)$	Step 6: Factor
$\frac{2(a-b)}{(a-b)} = \frac{5(a-b)}{(a-b)}$	Step 7: Divide by $(a - b)$
$2 = 5$	

This type of question may be more challenging for students because, algebraically, there does not appear to be any mistakes in Pedro's work. Students should analyze Step 6 and Step 7 carefully.

In Step 6, students may use substitution instead of division. Since $a - b = 0$, students may write $2(a - b) = 5(a - b)$ as $2(0) = 5(0)$ resulting in $0 = 0$. In Step 7, students may write $2(a - b) = 5(a - b)$, as $2 = 5$ without realizing they have even divided. Ask students to substitute $a = b$ back into the equation. When working with proofs that involve division, students should check to see if the divisor is zero.

- Circular reasoning

  An argument is circular if its conclusion is among its premises. Darren claims he can prove that the sum of the interior angles in a triangle is 180°. He draws the following rectangle.

  Here is his proof:

  I constructed a rectangle. Next, I drew a diagonal. I knew that all of the angles in a rectangle are 90°. I labelled one of the other angles in the triangle x. Therefore, the other angle must be
  $180° – 90° – x = 90° – x$. Then, $90° + x + (90° – x) = 180°$.

  Students must realize that they cannot assume a result that follows from what they are trying to prove.

- Students should be exposed to problem-solving situations that require the use of inductive and/or deductive reasoning. They will explore some situations in which they are asked to first show inductively that a pattern exists and then prove it deductively. It is important for students to recognize that inductive and deductive reasoning are not separate entities—they work together. Consider the following example:
- Hani was investigating patterns on the hundreds chart. He was asked to choose any four numbers that form a 2 × 2 square on the chart. He chose the following:

4	5
14	15

- Hani should be able to use inductive reasoning to make a conjecture about the sum of each diagonal and then use deductive reasoning to prove his conjecture is always true.
- When observing groups of students as they are working on a task you may wish to ask questions such as the following:
- What strategy did your group use? Why did you choose this strategy?
- Could you use another strategy? If so, which one? Why?
- Is one strategy more efficient than another?

Suggested Models and Manipulatives

- protractors
- rulers

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- complimentary angles
- conjectures
- counterexample
- deductive reasoning
- inductive reasoning
- proof
- supplementary angles
- transitive property
- two-column proof

Resources/Notes

Internet

- *Basic-mathematics.com*, "Examples of Inductive Reasoning" (Basic-mathematics.com 2008)
  www.basic-mathematics.com/examples-of-inductive-reasoning.html
  Basic examples
- of inductive reasoning.
- Movieclips Beta, "Young Sherlock Holmes: Watson Meets Holmes" (Movieclips, Inc. 2014)
  http://movieclips.com/WijX-young-sherlock-holmes-movie-watson-meets-holmes
  The following site provides a clip from *Young Sherlock Holmes*.
- The iPhone/iPod/iPad application called *Crack the Code* is an application involving deductive and/or inductive reasoning.
- Making and testing conjectures by noticing patterns. NCTM Illumations - Do You Notice Sum-Thing?
  https://illuminations.nctm.org/Lesson.aspx?id=4067

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  Sections 1.1–1.7, pp. 6–57

SCO LR02 Students will be expected to analyze puzzles and games that involve spatial reasoning, using problem-solving strategies. [CN, PS, R, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(It is intended that this outcome be integrated throughout the course by using translations, rotations, construction, deconstruction and similar puzzles and games.)

LR02.01 Determine, explain and verify a strategy to solve a puzzle or to win a game; for example,
- guess and check
- look for a pattern
- make a systematic list
- draw or model
- eliminate possibilities
- simplify the original problem
- work backward
- develop alternative approaches

LR02.02 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.

LR02.03 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

Scope and Sequence

Mathematics 10	Mathematics 11	Mathematics 12
	LR02 Students will be expected to analyze puzzles and games that involve spatial reasoning, using problem-solving strategies.	**LR01** Students will be expected to analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.

Background

This outcome is intended to be integrated throughout the course by using puzzles and games focused on translations, rotations, construction, and deconstruction. These puzzles are intended to help enhance spatial reasoning and deepen problem-solving strategies. Numerical reasoning and logical reasoning using puzzles and games will be addressed in Mathematics 12.

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for LR02 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

Students need time to play and enjoy each game before analysis begins. They can then discuss the game, determine the winning strategies, and explain these strategies by demonstration, orally, or in writing.

A variety of puzzles and games, such as board games, online puzzles and games, and pencil-and-paper games should be used.

Problem-solving strategies will vary depending on the puzzle or game. Some students will explain their strategy by working backwards, looking for a pattern, using guess and check, or by eliminating possibilities, while others will plot their moves by trying to anticipate their opponents' moves. As students play games and analyze strategies, they explore mathematical ideas and compare different strategies for efficiency.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

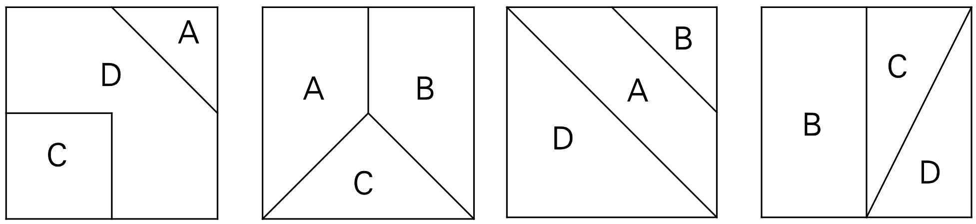
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Name a game or puzzle that you have played.
- Play a game of tic-tac-toe. What strategy do you use?

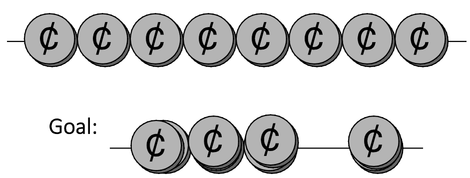
Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

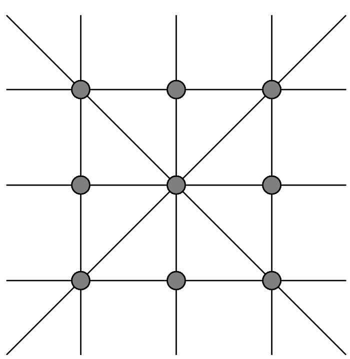
- The squares, shown below, are cut up, and each member of a group of four is given three pieces marked with the same letter. The task for the group is to make four complete squares.



- Take eight coins and arrange them in a row. Your goal is to end up with four piles of two coins each using four moves. Each move consists of picking up a coin, jumping in either direction over two "piles" and landing on the third. Each pile may consist of a single coin or of two coins.
  (a) What strategies did you use?
  (b) What challenges occurred while trying to solve the puzzle?

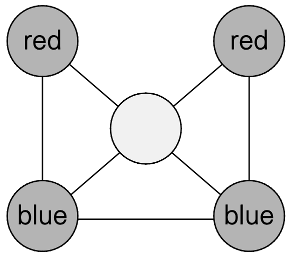


- Take nine coins and arrange them on a 3 × 3 grid as shown to the right. In this configuration, there are eight straight lines that each pass through three coins: three horizontal, three vertical and two diagonal. Now move exactly two coins so that there are ten straight lines, each with three coins. **Note:** In the solution, there are no straight lines with more than three coins, and no coin is placed on top of another.



  (a) What strategies did you use?
  (b) What challenges occurred while trying to solve the puzzle?

- An example of a board game that resembles tic-tac-toe is Pong Hau K'i, which is also played by two players. The game board is made up of five circles joined by seven line segments. Draw a game board.



Rules

- o Each player will have two stones (counters or buttons) of different colours. Place two red stones at the top, for example, and two blue at the bottom as shown.
- o Players take turns sliding one stone along a line to an adjacent empty circle.
- o To win, you have to block the other player so that he or she cannot move.

Play the game several times, taking turns making the first move. Ask the following questions as you play the game:

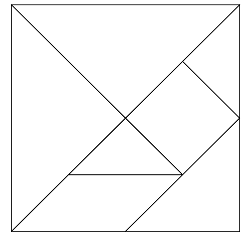
- o Where will the first move always be?
- o What does the board look like when one player is blocked? Why will this help the player?
- o Is it better to have the red stones or blue stones?
- o Is it better to make the first move or second move?
- o Is anticipating the possible moves ahead an effective game plan strategy? Explain why or why not.

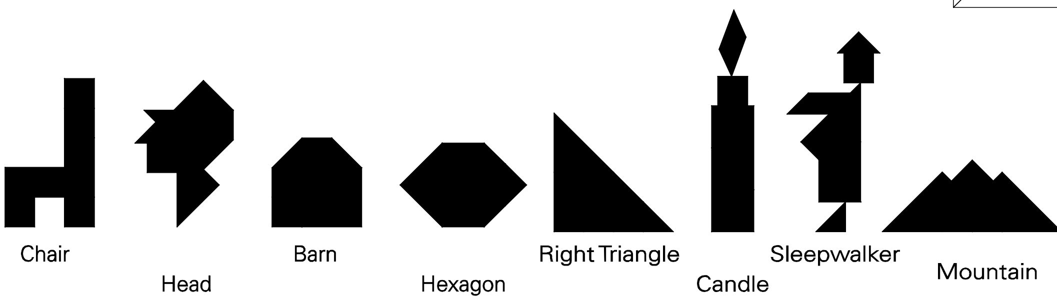
A variation of this game is to start from different positions.

- o Decide which colour each player will use and who will place the first stone.
- o After the first stone is placed on any circle, the other player places a stone on any of the four remaining circles and so on until all four stones are on the board.
- o The game is continued by moving the pieces as previously described.

Play this modified game several times. Is it more challenging than the original game?

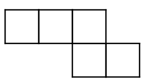
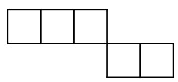
- The tangram puzzle was invented in China thousands of years ago. The object is to arrange all seven pieces of the tangram (cut from a square as shown to the right), into various shapes just by looking at the outline of the solution. Try making the shapes shown below, using all seven pieces.





Design other puzzles and challenge your classmates to solve them.

- Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.
- Pentominoe Challenges:
    - *Challenge #1*: A pentomino is made from five squares. The squares must be properly joined edge to edge so that they meet at the corner. For example,

       is a pentomino, but  is not a pentomino.

      There are only 12 in the set (because shapes that are identical by rotation or reflection are not counted). Create the set of 12 pentomoinoes.

    - *Challenge #2:* The 12 pentomino pieces fit together to form a rectangle. In fact, there is more than one rectangle that can be created with the 12 pentominoes. Form as many different rectangles as possible with all 12 pieces.
    - *Challenge #3—A Game:* Working with a partner and using pentomino playing grids of 6 × 10,
      5 × 12, 4 × 15 and 3 × 20 units, take turns drawing pentominoes in pencil. Co-operative thinking is allowed! When you are satisfied that you have drawn a pentomino that is not already on the grid, colour it (use a different colour for each pentomino). Remember ... no two pentominoes can share a square. Remember ... each of the 12 pentominoes may be used only once.
      Caution: Watch out for translations, reflections, and rotations.
        - The game ends when there is no room available for another pentomino.
        - Record the number of pentominoes created by your team.
        - Play another game using any of the blank pentomino playing grids. The goal is to beat your team score.
        - Continue playing additional games, adjusting the strategy so that the goal is to maximize your score.

- *Challenge #4:* Take the T-pentomino. Use 9 of the remaining 11 pentominoes to make an enlarged scale model of the T-pentomino.
  - What is the area of the enlarged model?
  - How does this area compare with the area of the T-pentomino?
  - What are the lengths of the each of the sides of the enlarged model?
  - How do these lengths compare with the corresponding sides of the T-pentomino?
  - Repeat this process for the W-pentomino and the X-pentomino.
  - What other pentominoes can you make larger versions of?
  - What strategy can you use to determine which pentominoes can have larger versions created?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

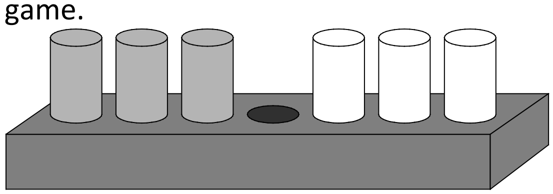
Consider the following sample instructional strategies when planning lessons.

- The following are some tips for using games in the mathematics class:
    - Use games for specific purposes.
    - Keep the number of players in each group to two to four, so that turns come around quickly.
    - Communicate to students the purpose of the game.
    - Engage students in post-game discussions.
    - When students feel that they have obtained a strategy for the solution of a puzzle or game, ask them to share that strategy with the class.

- As students play games or solve puzzles, ask probing questions and listen to their responses. Record the different strategies and use these strategies to begin a class discussion. The following are some possible discussion starters.
    - Thumbs up if you liked the game, thumbs sideways if it was okay, and thumbs down if you did not like it. What did you like about it? Why?
    - What did you notice while playing the game?
    - Did you make any choices while playing?
    - Did anyone figure out a way to quickly find a solution?

    Note: If you are pressed for time, you could use exit cards and then debrief the following day.

- As students play a game, it is important to pose questions and engage students in discussions about the strategies they are using. For example, consider the following peg game.



The goal of the puzzle is to switch the pegs on the left with the pegs on the right by moving one peg at a time. A peg may only be moved to an open slot directly in front of it or by jumping over a peg to an open slot on the other side of it. You may not move backwards. The game ends when you win or get stuck.

As you observe students playing the game, ask them questions such as the following:

- What did you like about the game? Why?
- What did you notice while playing?
- What was your first move?
- What were your first three moves?
- What problems arose when solving the puzzle?
- What are the minimum number of moves to win this game?
- Did you anticipate the next move?
- Did you notice any patterns?

- Students should be encouraged to explore a simpler version of the game where appropriate. Solving the peg puzzle using two or four pegs, for example, may allow students to complete the game more efficiently.
- Try games in advance as instructions to games are not always clear.
- Have students look for patterns and then develop strategies to fit these patterns. Sometimes these patterns are numerical, such in magic squares and Sudoku.
- Have students develop a game for classmates to play and/or use a known game. Students can then change a rule or parameter and then explain how it affects the outcome of the game.
- Find a game online and critique the quality of the game.
- It is important that the teachers at each school discuss which games and puzzles will be used in which courses. Many games and puzzles can be used with more than one type of logical reasoning. One possible division of games and puzzles is illustrated below.

Math at Work 10 / Math 11/ Extended Math 11	Math at Work 11 / Mathematics 12	Math at Work 12 / Mathematics 12
Students will be expected to analyze puzzles and games that involve ***spatial reasoning***, using problem-solving strategies.	Students will be expected to analyze puzzles and games that involve ***numerical reasoning***, using problem-solving strategies.	Students will be expected to analyze puzzles and games that involve ***logical reasoning***, using problem-solving strategies.
• Inversé • Qwirkle Cubes • Rectangles (Shikaku) • Skyscrapers • Blokus • Connect Four • Swish	• Kenken • Kakuro • 24 Game • Cribbage • Yatzee • Farkle	• Sudoku • Set • Bridges • Mastermind • Chess • Nim • Towers of Hanoi

Suggested Models and Manipulatives

- board games such as Blokus and Tetris with an emphasis on spatial reasoning
- coins or counters
- grid paper
- pentominoes (may be made with card stock using the diagrams from the appendix)
- tangrams (may be made with card stock using the diagrams from the appendix)

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- reflection
- rotation
- translation

Resources/Notes

Internet

There are numerous games and puzzles available on the Internet. What follows are just a few suggestions of spatial games and puzzles, available for free online. Many can also be acted out, done on paper, or done with models.

Note: It is expected that the teacher confirm the validity of the site prior to directing students to it.

- Math Is Fun site
  www.mathsisfun.com
  - Towers of Hanoi (2011)
    www.mathsisfun.com/games/towerofhanoi.html
  - Tic-Tac-Toe (2012)
    www.mathsisfun.com/games/tic-tac-toe.html
  - Dots and Boxes Game (2012)
    www.mathsisfun.com/games/dots-and-boxes.html
  - Four in a Line (2012)
    www.mathsisfun.com/games/connect4.html
- Illuminations: Resources for Teaching Math, "Calculation Nation" (National Council of Teachers of Mathematics 2014)
  http://calculationnation.nctm.org/Games
  - Nextu—In this game, players alternate claiming shapes on a tessellation with shapes of greater value than the adjoining shape.
  - Flip-n-Slide—This is a game in which triangles are translated and reflected to capture ladybugs. It is spatially challenging to envision the final position of the triangle.
- *Nrich: Enriching Mathematics* (University of Cambridge 2014)
  http://nrich.maths.org
  - Sprouts is a game for two players, and can be played with paper and a pencil. The rules are simple, but the strategy can be complex. The site discusses the game, its history, and strategies for solving the puzzle.
  - Frogs and Toads is a well-known puzzle in which frogs and toads must change places in as few turns as possible.
- Jill Britton (personal site)
  http://britton.disted.camosun.bc.ca/nim.htm
  - Nim is an ancient game that can be played online, on paper, or using sticks. The winner is the player to not pick up the last stick (Common Nim) or to pick up the last stick (Straight Nim).

- Cool Math
  www.coolmath-games.com/0-b-cubed/index.html
    - Cubed is a game where students must pass over each block prior to reaching the final red block. Other spatial games on this site include Pig Stacks, Aristetris, and Bloxorz.
- Some Board Games
    - Blockers
    - Blokus
    - Chess
    - Connect Four
    - Free Flow
    - Inverse
    - Qwirlke Cubes
    - Rushhour
    - Shikaku
    - Swish
    - Toers of Hanoi

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada) Section 1.7, pp. 52–57

Statistics
Time: 30 – 35 hours

GCO: Students will be expected to develop statistical reasoning.

SCO S01 Students will be expected to analyze, interpret, and draw conclusions from one-variable data using numerical and graphical summaries. [C, CN, R, T, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome:

S01.01 Recognize that the analysis of one-variable data involves the frequencies associated with one attribute.
S01.02 Determine, using technology, the relevant numerical summaries.
S01.03 Generate, using technology, the relevant graphical summaries of one-variable data based on the type of data provided.
S01.04 Interpret statistical summaries to describe the characteristics of a one-variable data set and to compare two or more related one-variable data sets.
S01.05 Make inferences, and make and justify conclusions, from statistical summaries of one-variable data orally and in writing, using convincing arguments.

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
	S01 Students will be expected to analyze, interpret, and draw conclusions from one-variable data using numerical and graphical summaries.	

Background

In Mathematics 7, students explored the measures of central tendency, range, and the effect of outliers on a given data set (SP01). They learned that **measures of central tendency** (mean, median and mode) attempt to summarize an entire set of data with a single value. These measures are sometimes referred to as averages. **Outliers** are values that are not typical or noticeably different than the rest of the values in the set of data. Students in Mathematics 7 explored how outliers can affect each of the measures of central tendency differently. In Mathematics 8, students examined and created various types of graphs that are used in data management: circle graphs, bar graphs, pictographs, line graphs, and double bar graphs. They determined which graph best represents a data set and justified their decision (SP01). In Mathematics 9, students considered sources of bias in data collection (SP01), explored the difference between a population and a sample (SP02), and completed a project to collect and display data (SP03).

"*Data sets can be analyzed in various ways to provide a sense of the shape of the data, including how spread out they are (range, variance) and how they are centered (mean, median, mode). Measures that describe data with numbers are called statistics. Data can be organized in various graphical forms to visually convey information. The use of a particular graph or statistic can mediate what the data tell about the population*" - *Elementary and Middle School Mathematics: Teaching Developmentally* by Van de Walle, 5th Ed. p386

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

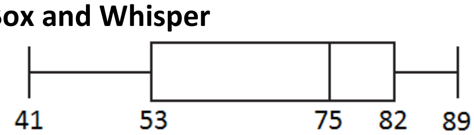
Constructivism is a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for Statistics is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

Students need to understand the difference between mean, median, and mode, and when it is appropriate to use each measure. For example, the mean might be a better summary for normally distributed data such as the height of students but the median might be a better summary for data sets with outliers like house prices. The mode might be the best average to use when describing discrete categorical data such as favorite color. Students should be comfortable using technology to determine mean, median and mode. As part of this statistics outcome, students should be exposed to situations to reinforce the fact that measures of central tendency are never sufficient to represent or compare sets of data.

Students will analyze and interpret one variable data and select appropriate graphical displays for this data. They will analyze data sets using numerical summtries (e.g. measures of central tendency, range, and interquartile range). They will determine these numerical summaries with and without technology (e.g. TI-83/84, spreadsheets, etc). Note that when you have a normal distribution it is appropriate to describe the data with mean and standard deviation ($\bar{x}, S_x$ when working with a sample ; μ, σ when working with a population). Non normal data should be described using a five point summary.

Students will explore and understand the 5 point summary (median, upper and lower quartiles, and extreme values) for a data set and use these values to create box plots and modified box plots. A modified box plot is one where outliers (data points that are more than 1.5 times the width of the box, or inter quartile range (IQR), below Q1 or above Q3 are indicated by a star or circle).

Box and Whisper



Students should be able to construct box plots and modified box plots not only by hand, but also with the use of technology. They should be comfortable interpreting box plots and should learn to interpret the distributions according to the width of the box (which contains at least 50 percent of the data), the position of the median, and the length of the whiskers. For example, the box plot below shows the lengths of lake trout caught in Sherbrooke Lake last year (in cm), what could be concluded about the fish in this lake?

Stem and Leaf

Stem	Leaf
1	0
2	
3	08
4	
5	17
6	1255689
7	34799
8	0815

This will be student's first exposure to stem-and-leaf plots. Stem-and-leaf plots are a way to both organize and display numerical data particularly when the size of the data set is fairly small. A stem-and-leaf plot is a way to quickly visualize the distribution of data within a set. These plots display the actual data values in order and are aligned vertically. This makes it easy to see individual data points and interpret data. In a stem-and-leaf plot, numbers are grouped together by place value. The "stems" are in the left hand column and include the digit or digits of greatest place value. The "leaves" are in the right hand column and include the digit or digits of lesser place value. Stem and leaf plots have the advantage of creating a rough histogram when turned sideways. In this example the stems are the tens digits and the leaves are the ones digits. Students should be encouraged to interpret the stem and leaf plots for clusters of data, gaps in the data and the amount of spread in the data.

In previous grades, students have constructed and interpreted bar graphs, line graphs, and circle graphs. The focus in Extended Mathematics 11 is on constructing box plots and stem-and-leaf plots as well as the ability to interpret or justify the use of a type of graph and its validity in a certain context.

Students should compare two or more related one-variable data sets using graphical summaries such as stacked or parallel box plots and back to back stem-and-leaf plots.

Back to Back Stem and Leaf

Pulse Rate

Before		After
9 8 8 8	6	
6 6 4 1 1 0	7	
8 8 6 2	8	6 7 8 8
6 0	9	0 2 2 4 5 8 9
4	10	0 4 4
0	11	8
	12	4 4
	13	
	14	6

In a back to back stem-and-leaf plot, two distributions are placed back to back along a common column of stems as shown in the example to the right.

Some types of graphs are more useful for comparing data sets than others. Students should discuss the benefits of and limitations of various data displays such as side by side circle graphs, stacked box plots or double bar graphs to compare data sets.

Students should be given the opportunity to have a summative data gathering and analyzing activity. During this activity students should gather and organize data and construct graphical summaries. They should then analyze the data to make and justify conclusions.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

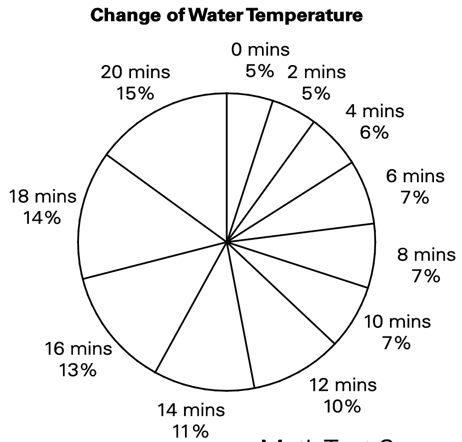
Student tasks, such as the following could be completed to assist in determining students' prior knowledge.

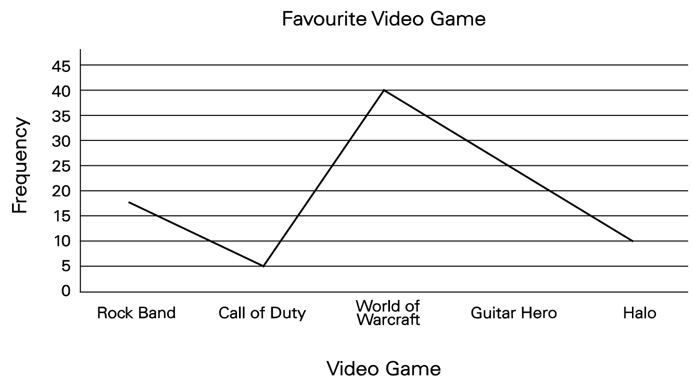
- With a partner discuss each of the following questions:
  - What are the three measures of central tendency?
  - What is meant by the terms mean, median and the mode?
  - What is meant by range? Why is it not a measure of central tendency?
  - What is an outlier? What impact does an outlier have on mean, median, mode, and range?
- Using the data set below, answer the following questions.

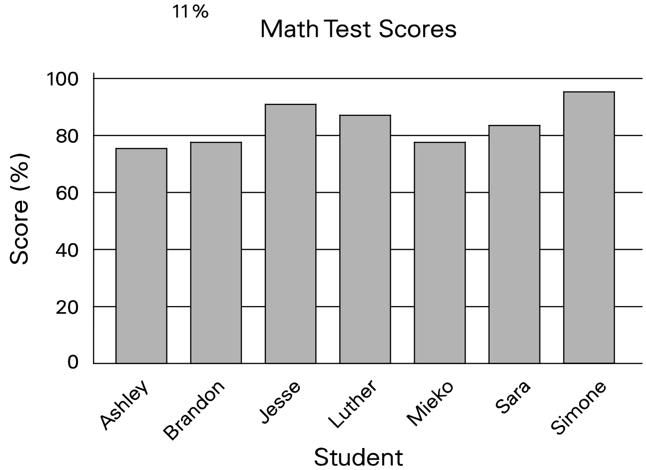
  14 19 30 15 22 25 12 18 23 17 19 31

  - What is the mean, median and mode?
  - Why is the mean different than median?
- In each scenario discuss with a partner whether the mean is less than or greater than the median. Explain.
  - In 2016 the mean price of a home in Canada was $509 000.
  - A total of 11.8 million people worked full time in 2014. The median employment income for these workers was $50 400.
  - The height of fifteen year old girls in Nova Scotia.

- With a partner look at the graphs below and consider the following questions.
    - o Is each graph a suitable way to represent the data. Explain.
    - o If you feel the graph is unsuitable, which type of graph would better represent the data. Explain.







Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- The data set below shows the number of text messages sent by a group of friends on Victoria Day.

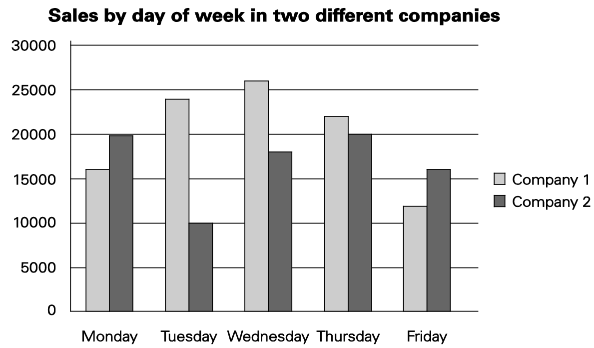
16	25	41	20	35	20	16	8	38
35	37	51	39	14	14	40	44	12

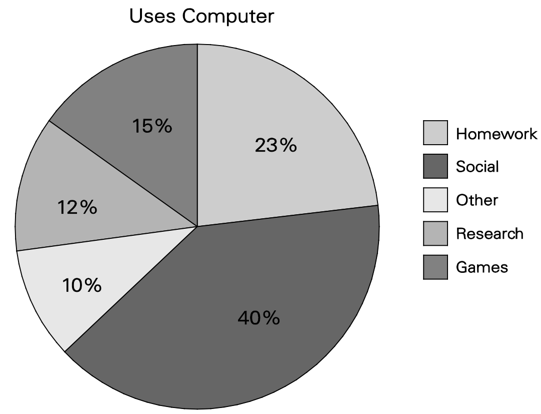
    - o Using technology, create a box-and-whisker illustrating the distribution shown above.
    - o Create a stem-and-leaf diagram illustrating the data set above.

- Using technology, calculate the mean, median, mode and range for the dataset below:

26	15	31	22	15	30	26	18	28

- Examine each diagram below. What can you interpret from each graph?

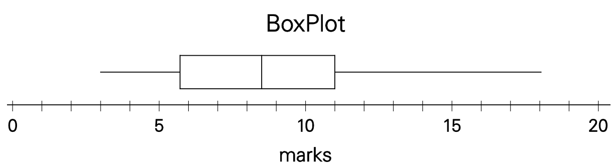




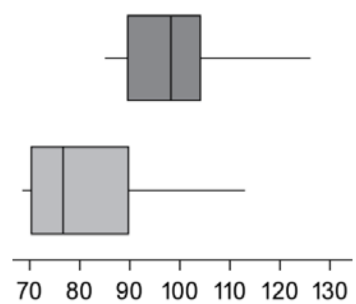
Phone Battery Life Comparison (in hours)

"Brand A" LEAF	STEM	"Brand B" LEAF
8 8 7 5	0	7
9 7 4 1 0	1	0 5 5 5 7 9
2 2 2 1	2	0 2 2 6 7
8 6 4 2 0	3	0 2 4 6 8
	4	
	5	6
1	6	
	7	5

Key: 6 | 1 = 61 hours



- The two box plots below represent the amount of time two classes spend, in minutes, watching videos at night. A class of grade 9 students is represented by the bottom graph, while a class of grade 11 students is shown on top.
    - o With a partner list at least three different things that you can you infer from the boxplots above?
    - o What percentage of students spend more than 90 minutes watching videos at night?
    - o Which quartile has the smallest interquartile range?
    - o A student spent more than 110 minutes watching videos each night. Which grade level is this student more likely to be in? Explain.



- The following data was collected in a survey. The data represents the ages of people who entered a new coffee shop within the first half hour of opening on its opening weekend. The C's in the data represent those ordering coffee and the T's represent those ordering Tea. Construct a back-to-back stem and leaf plot showing the ages of male customers and the ages of female customers. Compare the distributions.

12C	18T	15T	15C	10C	21C	25C	21C
26T	29T	29T	31C	33C	35C	36T	36T
41C	42T	42C	45C	46T	48T	51C	51C
55T	56T	58C	59C	61T	61T	62C	65C
65T	66C	70C	70C	72T	72C	75C	78C

- Fish are caught in two different lakes, Lake A and Lake B. Their lengths are recorded in the stem-and-leaf shown.
    - Calculate the mean and median length of the fish caught in each lake.
    - If one was interested in catching larger size fish, which lake would one choose based on this data? Why?
    - What percent of fish caught is Lake A are longer than 52 centimeters?

Lake A Leaf	Stem	Lake B Leaf
9	3	
945	4	3
988321	5	24
7641	6	0117789
2	7	011234
	8	24

- What can you conclude about a set of data where the mean is greater than the median?
- A marble is rolled down a ramp 5.0 cm in length and out onto the floor. Students measured how far, in cm, along the floor the marble rolled. The data below was collected by repeating this over and over again. Jamie predicts the next marble will roll 45.10 cm while Zara predicts it will roll between 43.20 cm and 45.65 cm. Which of these predictions are you more confident with?

  45.40, 42.80, 46.10, 45.30, 44.70, 43.20, 45.10, 46.40, 43.20, 47.10, 45.60, 42.80, 43.30, 45.70, 42.80, 44.60, 45.50

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

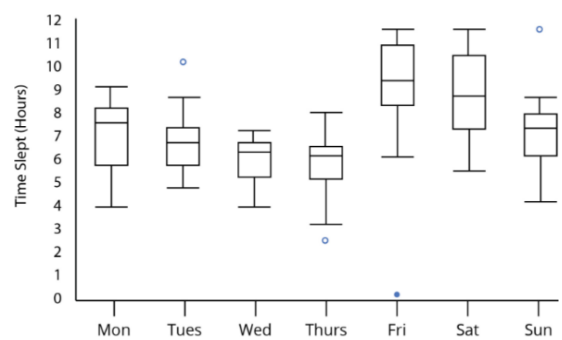
Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How will the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Effective instruction should consist of various strategies. Consider the following sample instructional strategies when planning lessons.

- Where and when possible use interesting data. For example rather than using data about grades in course at school – consider using sleep data such as that shown below.



- In order to emphasize that measures of central tendency are never sufficient to represent or compare data use the following Tag Team Wrestling Data:

Stats	Team 1	Team 2
Ave Weight	222.5 pounds	220 pounds
Ave Height	6 feet	6 feet

Team 1		
Name	Weight	Height
Mad Dog Max	320 pounds	5’ 2”
String Bean Sam	125 pounds	6’ 8”

Team 2		
Name	Weight	Height
Hulk Hogan	220 pounds	6’
Mr. X	220 pounds	6’

Make certain that students realize that to truly examine a set of data they must look at three things: center, spread and shape.

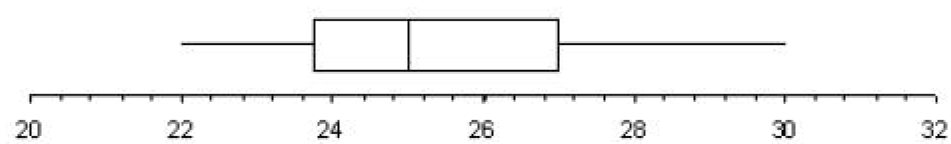
- Use a Think/Pair/Share strategy to have students compare and contrast two different sets of data like the one below.

  The following data represent the bowling scores of Lisa and Ruth in their last 10 games.
  Lisa: 65, 105, 90, 95, 72, 85, 110, 88, 92, 95
  Ruth: 125, 110, 81, 62, 98, 115, 68, 72, 118, 69
  - Which person is the better bowler?
  - What information can you use to support your position?
  - Create a display of data to help argue for your position.
  - Which person would you want on your team next year?
  - Which person would you want on your team for the play-off next week?

- Students should be given the opportunity to work with open ended questions that have more than one correct answer and/or more than one strategy to obtain this answer. One example of an open question is to have students create a set of five whole numbers from 1 to 20 that have the same mean, median, and range.

- Students should be aware of the advantages and disadvantages of each type of graph. For example, students could investigate whether it is possible to determine the sample size of a distribution from: a circle graph, a bar graph and a box plot.

- Use small group discussions to have students analyze a box plot such as the one below and record their findings to share with the class and/or another group. The box plot below represents the distribution of the weights of a group of two year old toddlers. What conclusions can be made from this graph?



- Students should be familiar with stem-and-leaf plots where the stems might be more than one digit, a negative or a split stem.

Multi Digit Stem

Race Running Times (in seconds)

12	2 6
13	0 2 5
14	1 2 4 6
15	2 3 7 8
16	1 2 4 6 8
17	5 7 8
18	1 3

Key: 14 | 2 = 14.2 sec

Negative Stems

Temperature (in °C)

4	5 7
3	6 8
2	3 5 7
1	1 1 2 9
0	1 2 3 5 7 7
–0	4 6 8
–1	2 1
–2	5 7

Key: 4 | 5 = 45°C

Split Stem

Height of Plant (in cm)

1L	0
1H	6 7
2L	0 1 3 3
2H	6 6 7 8
3L	0 1 2 3 4
3H	5 6 6 8
4L	0 3

Key: 2H | 6 = 26cm

- One purpose of a stem and leaf display is to clarify the shape of the distribution of data. In order to see this distribution clearly, choosing an appropriate stem is important. For example, to clearly see the data for the heights of Toronto Raptors players shown below, the stems might be both the hundreds and tens digits of the number and the leaves would be the units digit. The stem-and-leaf plot might be clarified further by splitting each stem into two parts. For example, there would be one row for the values from 200-204 and another row for the values from 205-209.

  Toronto Raptors 2016-17 Player Heights (in cm)
  183, 185, 191, 193, 196, 201, 201, 203, 206, 206, 206, 206, 208, 213, 213, 213

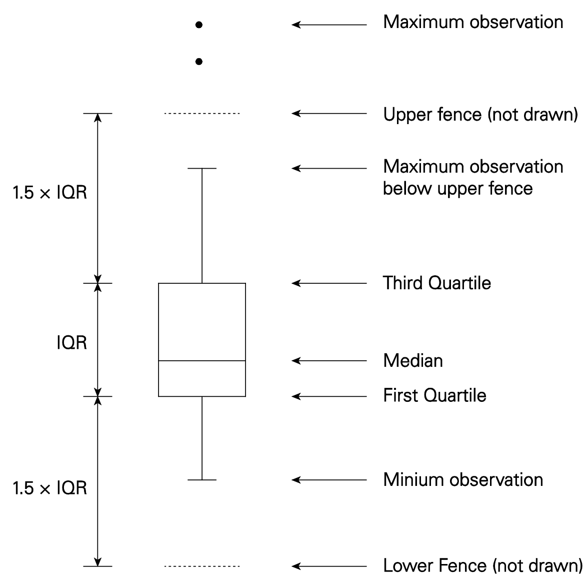
  Another example of a data set which would be clarified by splitting the stems is the total number of touchdowns made by each of the 32 NFL Football teams.

  2015 NFL Team Statistics - All Touchdowns Scored
  59, 58, 52, 51, 50, 49, 47, 45, 45, 45, 45, 44, 44, 44, 42, 42,
  42, 39, 39, 38, 38, 38, 37, 37, 37, 35, 34, 33, 31, 28, 26, 24

- Students should be familiar with situations where different measures of central tendency might be preferred depending upon the point of view. For example, in a labor dispute the employee union might report the average salary using a different measure of central tendency than the employer might use.

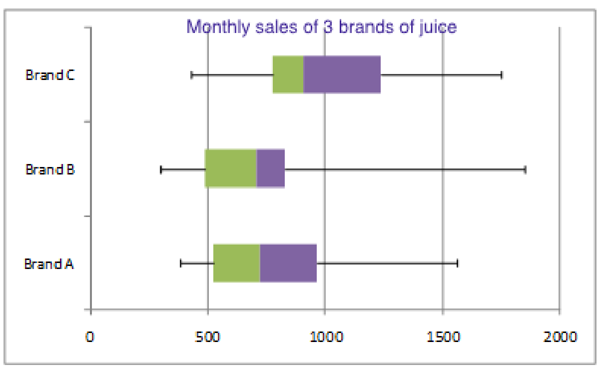
- Students should become familiar with situations where mode is a valuable measure. For example, a sandwich vendor would be most interested in knowing which type of sandwich sells the best in order to prepare a day's sandwiches, or a t-shirt vendor would want to know the size of the highest selling t-shirt.

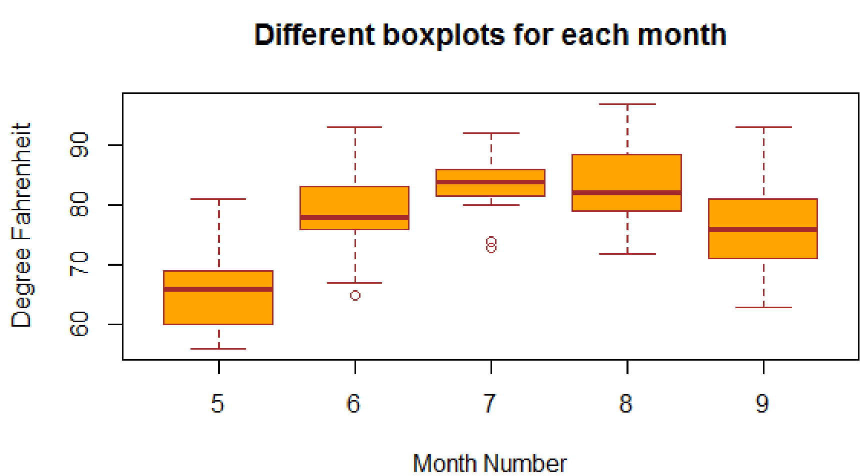
- Box plots are graphs used to show the distribution of a data set at a glance. There are a number of variations of how box plots are drawn and students will see many varieties in textbooks, journals and articles. Different styles of box plots are best for different situations, and there are no firm rules for which to use. Students should be aware that these different variations are acceptable. For example, sometimes the whiskers have a crosshatch placed at their ends (as on the TI-83/84) and sometimes they do not. Some box plots are drawn horizontally and some are drawn vertically. Outliers may be shown as additional data points beyond the ends of the whiskers. Sometimes the mean value of the data set is indicated with a mark such as a plus sign.



  A modified box plot has an asterisk or cross hatch point placed for any values considered to be outliers (defined as greater than 1.5 x IQR).

- Show diagrams where multiple box plots are represented on each graph such as the ones shown below and have students draw conclusions from each graph.





- Students should choose an appropriate type of graph to display a given set of data. Students should also carefully format the graph with the presence of title, labels, appropriate scales, and accurate data representation.
- Students should examine graphs found in the media and describe what they can interpret from it. Critically analyze what is the message conveyed through the choice of this data display.
- Data collection and analysis activity. Have students toss a manipulative (such as multi-link cubes, two-color counters or coins) into a defined area on the floor (a circle drawn with white board marker, a square outlined with tape, etc). Students will time how many items they can throw into the defined area from a fixed distance away in 1 minute. Students will collect, organize and consolidate this data to determine the average number of items a typical student would get into the defined area. Students should graphically represent the summary data. Students can then use this information to predict the effects of changing one of those variables (time, distance to target, size of target, etc.) on the number of items successfully thrown into the area. Students could revisit this activity for additional trials after completing outcome S03 to include additional statistical analysis with standard deviation and histograms. They could also see if additional practice made students better at this activity.

Suggested Models and Manipulatives

- grid paper
- prepared graphs from media such as newspapers or magazines

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- back to back stem and leaf plot
- bar graph
- box plot (box-and-whisker plot)
- circle graph
- frequency
- interquartile range
- mean
- measures of central tendency
- median
- mode
- modified box plot
- outlier
- range
- stem-and-leaf plots

Resources/Notes

Internet

- Statistics Canada (Government of Canada 2014) http://www.statcan.gc.ca/
- Online Statistics Education: An Interactive Multimedia Course of Study Developed by Rice University (Lead Developer), University of Houston Clear Lake, and Tufts University http://onlinestatbook.com/2/index.html

Print

- *Data Management 12 (*Erdman-McGrath et al. 2014, McGraw Hill Ryerson)
- Making Mathematics Meaningful to Canadian Students, K–8, Third Edition (Small 2017), Chapter 21, pp. 582–622
- The Visual Display of Quantitative Information, 2nd Edition (Tufte 2001), pp13-77.

SCO S02 Students will be expected to demonstrate an understanding of normal distribution, including standard deviation and z-scores. [CN, PS, T, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

S02.01 Explain, using examples, the meaning of standard deviation.
S02.02 Calculate, using technology, the population standard deviation of a data set.
S02.03 Explain, using examples, the properties of a normal curve, including the mean, median, mode, standard deviation, symmetry, and area under the curve.
S02.04 Determine if a data set approximates a normal distribution and explain the reasoning.
S02.05 Compare the properties of two or more normally distributed data sets.
S02.06 Explain, using examples that represent multiple perspectives, the application of standard deviation for making decisions in situations such as warranties, insurance, or opinion polls.
S02.07 Solve a contextual problem that involves the interpretation of standard deviation.
S02.08 Determine, with or without technology, and explain the z-score for a given value in a normally distributed data set.
S02.09 Solve a contextual problem that involves normal distribution.

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
	S02 Students will be expected to demonstrate an understanding of normal distribution, including standard deviation and *z*-scores.	

Background

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for Statistics is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

Descriptive statistics describe the population we are studying. The data could be collected from either a sample or a population, but the results help us organize and describe data. Descriptive statistics can only be used to describe the group that is being studied. That is, the results cannot be generalized to any larger group.

Descriptive statistics are useful and serviceable if you do not need to extend your results to any larger group. However, many uses of statistics in society tend to draw conclusions about segments of the population, such as all parents, all women, all victims, etc.

Inferential statistics are used to make predictions or inferences about a population from observations and analyses of a sample. That is, we can generalize the statistics from a sample to the larger population that the sample represents. In order to do this, however, it is imperative that the sample is representative of the group to which it is being generalized.

In this Statistics unit, students will also consider that the measure of dispersion they learned in Mathematics 7, range, does not provide any information about the variation within the data values themselves since it is based on the two extreme values of the data set. Students will be introduced to standard deviation, which takes into account every score in a distribution, as a useful way to compare two or more sets of data.

In this course, students will be calculating standard deviation of a population represented by the lower case Greek letter, sigma, σ. Note: Students will not be familiar with sigma notation for summation, Σ.

$$\sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$$

Students should be exposed to calculating standard deviation manually to provide an understanding of how standard deviation is determined. Importantly, this should be done using an example with a small amount of data to keep calculations simple. The benefit of using technology to solve contextual problems is that data sets are often large. Students are expected to calculate standard deviation using technology such as scientific calculators, graphing calculators, or computer software programs
(e.g., Microsoft Excel or Google Sheets).

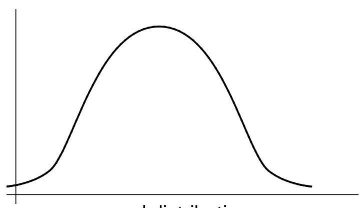
Students will be introduced to **histograms** and **frequency polygons**. Both provide a pictorial representation of the data. These will serve as the basis for the development of the normal curve.

In this Statistics unit, students will discover that, in many situations, if a controlled experiment is repeated and the data are collected to create a histogram, the histogram becomes more and more bell-shaped as the number of data in the distribution increases.

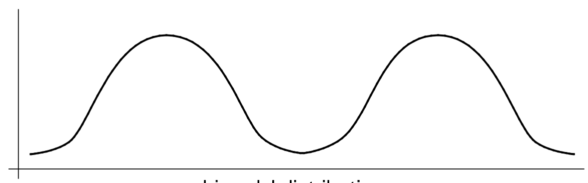
Throughout the unit, it is important to remind students that an attribute may be considered to be normally distributed in a population, but a random sample may not show a normal distribution. Doing a simulation on a calculator will often show students that sample size is relevant. That is, for a small sample size, you may repeat the experiment several times and the histogram will not become bell-shaped but for a very large sample the histogram may appear normal without the need to repeat the experiment multiple times to see the normal curve.

A rule of thumb often suggested is that a sample size of 100, that is obtained using a random sampling method, will be roughly normally distributed if the population it represents is normally distributed.

The heights of adult males are normally distributed. The heights of adult females are also normally distributed. When you consider the heights of adults without the gender differentiation, however, the data tend to be bimodal rather than normal in its distribution. Other examples of variables that are not normally distributed are the length of time that a person stays in a job and a soldier's age within an army platoon.



normal distribution



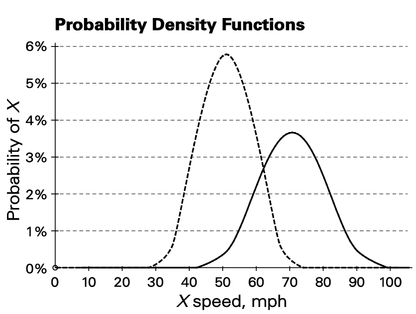
bimodal distribution

Many variables are normally distributed such as students' marks on examination tests, blood pressure, heights of people, and errors in measurement.

In this unit the focus is on populations that are normally distributed. Students will compare two or more normal distributions by comparing the mean and standard deviation of each normal distribution. The mean will determine the location of the centre of the curve on the horizontal axis. The standard deviation will determine the width and height of the curve for any density curve.

A **density curve** is a graph that shows probability. The area under the density curve is equal to 100 percent of all probabilities. As we usually use decimals in probabilities you can also say that the area is equal to 1 (because 100% as a decimal is 1).

To look at or compare two sets of data more equitably the scale is changed to create a density function.

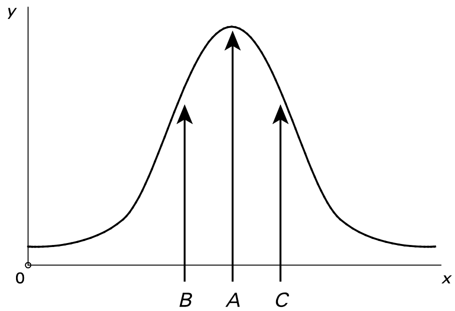


------ mean = 52. Standard Deviation = 7

_____ mean = 71. Standard Deviation = 11

For populations of data that are normally distributed, when enough data are plotted, the curve will be symmetrical, and it will be concave down for the section that is within one standard deviation of the mean and will be concave up for other areas.

This means that you can very roughly estimate the standard deviation by looking at a carefully constructed normal curve distribution. However, it can be difficult to accurately locate this point visually where the curve changes concavity so caution should be taken when using the graph to determine the standard deviation visually.



Point A: Mean

Point B: 1 standard deviation below mean

Point C: 1 standard deviation to the right of the mean

If two sets of data have the same mean, for example, but different standard deviations, the distribution of data described as a density curve with a lower standard deviation would result in a graph that is taller and narrower.

	Sample	Population
Mean	$\bar{x}$	μ
Standard Deviation	S_x	σ

A ***z*-score** is a standardized value that indicates the number of standard deviations a data value is from the mean. It can only be used when the sample is normally distributed. For example, if $z = 2$, it simply means that the score is 2 standard deviations to the right of the mean. For example, in a normally distributed population with a mean of 25 and a standard deviation of 3, a score of 22 would have a
z-score of –1.

A *z*-score can be calculated using the formula $z = \frac{x-\mu}{\sigma}$.

x – data input (value)

μ – population mean

σ – population standard deviation

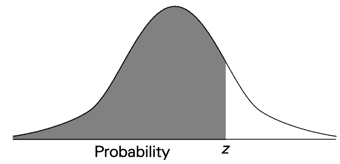
When students convert normally distributed scores into *z*-scores, they can determine the probabilities of obtaining specific ranges of scores using a *z*-score table or technology. There are several different *z*-score tables. Three possible tables are illustrated below. It is helpful for students to have exposure to a variety of these tables. It is also useful for students to use technology such as the graphing calculator to calculate the probabilities associated with *z*-scores.

Note: Students should recognize that the level of accuracy will increase as the precision of the tool they use increases. Z-score charts will give a better estimate than the 68%-95%-99.7% rule of thumb.

Standard Normal Cumulative Probability Tables

Cumulative probabilities for *positive* z-values are shown in the following table:

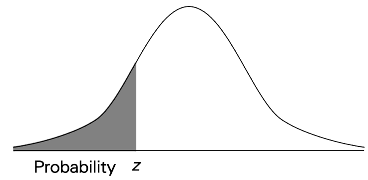
Table 1



z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.5	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Cumulative probabilities for *negative* z-values are shown in the following table:

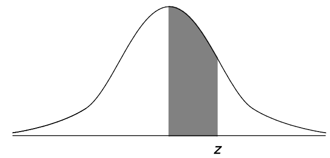
Table 2



z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Probability for values between mean and z-value are shown in the following table:

Table 3



Probability

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3236	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4367	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4809	0.4911	0.4913	0.4915
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4983	0.4989	0.4989	0.4989	0.4990	0.4990

For this Statistics unit, students will be exposed to situations that require the use of *z*-scores in the discussion of:

- particular scores within a set of data
- whether or not a score is above or below average
- how far away a particular score is from the average
- the comparison of scores from different sets of data and figuring out which score is better

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- How can you find the range of a set of data? Create an example to illustrate.
- Could a data set have more than one mean? One mode? One median? Explain, using an example.
- What is an outlier? How does a data set with an outlier differ from one with a gap?
- Give a context in which the mode would be the most suitable measure of central tendency.
- Give a context in which the median would be the most suitable measure of central tendency.
- Give a context in which the mean would be the most suitable measure of central tendency.
- In a science experiment, a group of students tested whether compost helped plants grow faster by counting the number of leaves on each plant. The following results were obtained:

Plant growth without compost (# of leaves per plant)	**Plant growth with compost (# of leaves per plant)**
6	6
4	11
5	1
4	6
8	2
3	4

(a) Calculate the measures central tendency for each group.

(b) Calculate the range for each group.

(c) Construct parallel box plots.

(d) Which group of plants grew better? Justify your decision.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Calculate the range of each group. Explain why the range, by itself, can be a misleading measure of dispersion.
  Group A: 8, 13, 13, 14, 14, 14, 15, 15, 20
  Group B: 7, 7, 8, 9, 11, 13, 15, 15, 17, 18

- The marks awarded for an assignment for a Grade 11 class of 20 students are given below.

6	7	5	7	7	8	7	6	9	7
4	10	6	8	8	9	5	6	4	8

  Display the data on a histogram and describe how the data are distributed.

- In December, the number of hours of bright sunshine recorded at 36 selected stations was as follows:

16	25	41	20	35	20	16	8	38	23	25	38
41	34	24	39	47	45	17	42	44	47	45	51
35	37	51	39	14	14	40	44	50	40	31	22

  Create a histogram, ensuring that it includes a title and all labels.

- Two obedience schools for dogs monitor the number of trials required for twenty puppies to learn the *sit and stay* command.

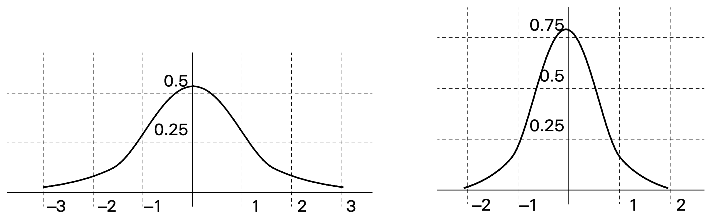
True Companion Dog School	
Number of trials	Number of puppies
7	1
8	2
9	5
10	4
11	4
12	4

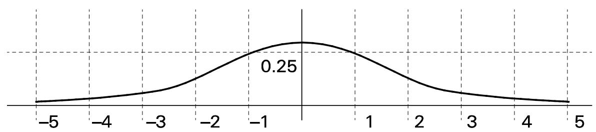
Top Dog School	
Number of trials	Number of puppies
7	4
8	3
9	2
10	3
11	4
12	4

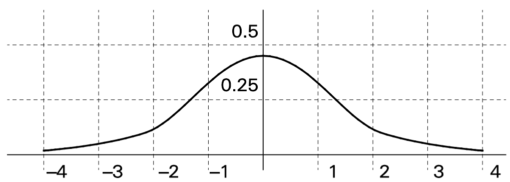
  (a) Determine the mean and standard deviation of the number of trials required to learn the “sit and stay” command.

  (b) Which school is more consistent at teaching puppies to learn the *sit and stay* command? Justify your reasoning.

- Which normal distribution curve in the graphics below has the largest standard deviation? Explain your reasoning.







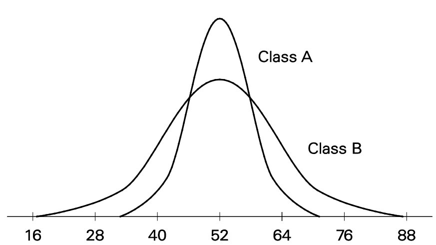
- A data set of 50 items is randomly selected from a normal population and shown below. The standard deviation of this sample is 1.8.

6	9	6	7	6	7	6	7	6	6
5	8	5	8	6	7	7	4	8	6
5	4	6	8	6	6	7	6	6	8
8	7	6	6	0	4	7	8	5	3
6	7	6	7	1	7	7	4	4	3

(a) What is the value of the mean, median, and mode?

(b) Is the data normally distributed? Explain your reasoning.

- The graph below shows two density curves describing the scores on a standardized test, normally distributed, for two classes.



(a) What do these graphs have in common? How are they different?
(b) Which graph has the greatest standard deviation? Why?
(c) If a student scored 42 on a test, which class are they most likely in? Explain your answer.

- Sally has a height of 1.75 m and lives in a city where the average height for women is 1.60 m and the standard deviation is 0.20 m. Leah has a height of 1.80 m and lives in a city where the average height for women is 1.70 m and the standard deviation is 0.15 m.

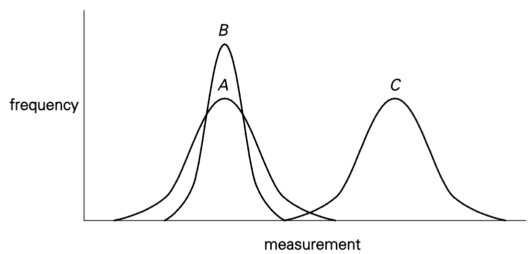
If heights are normally distributed, which of the two is considered to be taller compared to their fellow women. Explain their reasoning.

- The average life expectancy of greyhounds was determined to be 12.2 years with a standard deviation of 1.3 years.
  (a) What is the probability that a given greyhound will live less than 14 years?
  (b) In a sample of 200 greyhounds, how many would you expect to live between 10.9 years and 13.5 years?
  (c) What assumption did you make that made your calculation possible?

- On a mathematics placement test at Orchard-view University, the marks were normally distributed with a mean score that was 62 and a standard deviation that was 11. If Mark's z-score was 0.8, what was his actual exam mark?

- What does a z-score of 1.5 mean? How was it calculated?

- The data graphed below represents three samples of the same size. Use these distributions to answer the following questions.



  (a) What do graphs A and B have in common? How are they different?
  (b) What do graphs A and C have in common? How are they different?
  (c) Is there any relationship between graphs B and C?
  (d) Which graph has the smallest standard deviation? Why?
  (e) Why is the graph with the lowest standard deviation also the tallest?
  (f) Which graph has the greatest mean? How do you know?

- Battery Inc. claims that its batteries have a mean life of 50 hours, with a standard deviation of six hours. If these data are normally distributed, what is the probability, to two decimal places, that a battery will last 57 hours or longer?

- The speeds of cars on a highway have a mean of 80 km/h with a standard deviation of 8 kmh. If these data are normally distributed, what percent of the cars averaged more than 96 km/h?

- The assessed value of housing in a community in the Annapolis Valley is normally distributed. There are 1200 houses with a mean assessed value of $180,000. The standard deviation is $10,000.
  (a) What percentage of houses have an assessed value between $170,000 and $200,000?
  (b) How many houses are expected to have an assessed value between $170,000 and $200,000?

- Many human attributes are normally distributed. What statistics would a clothing manufacturer of women's sweaters want to know? Would the manufacturer be interested in standard deviation?
- Infant weight gain and growth are normally distributed. Why are these statistics valuable?
- What can you say about a data value if you know that its *z*-score is negative?
- What can you say about a data value if you know that its *z*-score is zero?
- The average playing time of a defense player in the NHL is 17.2 minutes per game with a standard deviation of 2.2 minutes.
  (a) What is the probability that a defense player will play between 15.6 and 19.2 minutes?
  (b) What assumption did you make that made your calculation possible?
- Create and solve a question using a real-world example that uses *z*-scores.
- Cars are undercoated as a protection against rust. A car dealer determines that the life of the undercoating is normally distributed with a mean life of 65 months and a standard deviation is 4.5 months. Answer the following questions.
  - What guarantee should the dealer give so that fewer than 15% of the customers could potentially return their cars?
  - The dealer creates a fund, based on the guarantee, from which refunds and repairs are made. It is estimated that about 2500 cars will be undercoated annually. The average repair on returned cars is about $165. How much money should be placed in the fund to cover customer returns?
  - What is the probability that an undercoated car, chosen at random, will be returned in five years?
- Consider two different automobiles: the Swiftcar and the Zippycar. For both cars, the mean value of repairs after an accident is $3500. The standard deviation for the Swiftcar is $1200, while the standard deviation for the Zippycar is only $800. If the cost of repairs is normally distributed, determine the probability that the repairs costs will be over $5000 for both cars, suggest how your answer might impact insurance premiums.

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

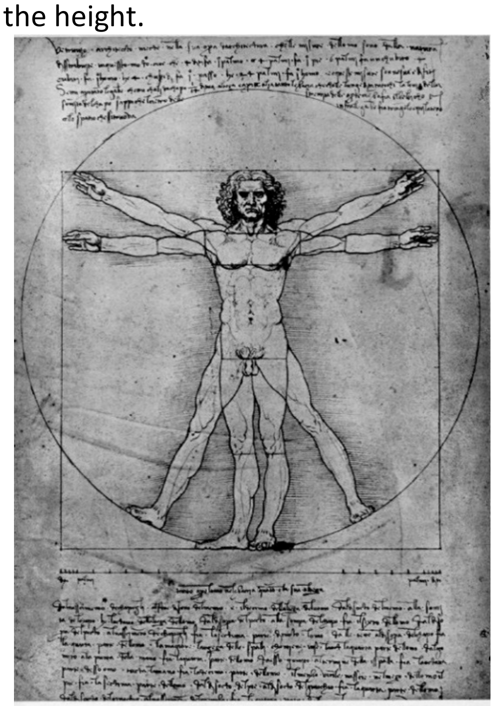
Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- One of Leonardo da Vinci's well-known illustrations shows the proportions of a human figure by enclosing it in a circle. It appears that the arm span of an adolescent is equal to the height.

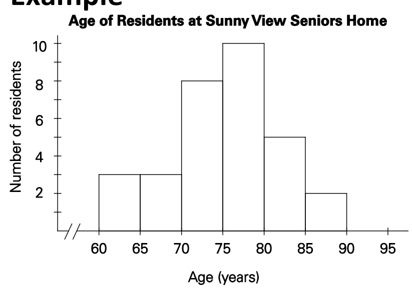


(a) Working with a partner, measure each other's arm span and height. The teacher will collect all the classroom data sheets and create two groups of data sets (one for arm span and one for height).

(b) Use the class data to calculate the mean and standard deviation for each of the two sets of data. Compare the values and explain what you notice.
(c) Calculate the difference between each person's height and arm span. From the data, what would they expect the mean and standard deviation of this data set to be? Check your prediction for ten students in the class.

- Graph data that has been organized into **bins** (also called **intervals** or **classes**) resulting in a **histogram**. Students should be made aware that in histograms, the data elements are grouped and form a continuous range from left to right. The data are arranged in a **frequency table**. The **range** of the interval is used to determine the width of the bars that should be neither too narrow nor too broad. Intervals are the same size, and the number of bins is usually kept to between 4 and 10. Often the bins and bin sizes will be determined by the context of the problem and the range of the data.

Example



The first bar in this graph begins at 60 on the horizontal axis and ends at 65. The second bar starts where the first bar ends, at 65–70. As the intervals are required to be non-overlapping, the convention is that the lower limit of each interval includes the number. For example, in the above graph, the data value 65 would be placed in the 65–70 interval, and 70 would be placed in the 70-75 interval.

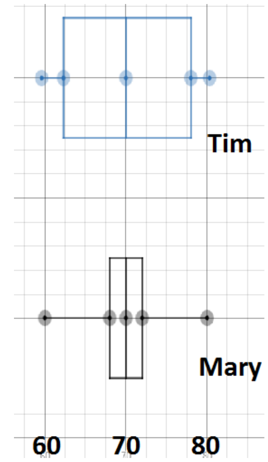
Note: While all bins must be the same width different conventions on where to put numbers that are on the boundary are possible. Most importantly a data piece should only be placed in one bin and the convention being used should be clearly be communicated.

To plot each vertex for the frequency polygon, students can determine the midpoint of the interval and then join the vertices with line segments. Ensure students connect the endpoints to the horizontal axis.

- Remind students of range as a measure of dispersion. They should recognize that range only measures how spread out the extreme values are, so it does not provide any information about the variation within the data values themselves. Introduce students to the **standard deviation**. It is useful for comparing two or more sets of data. Consider an example similar to the one shown below:

	Unit 1	**Unit 2**	**Unit 3**	**Unit 4**	**Unit 5**	**Unit 6**	**Unit 7**
Tim	60	65	70	75	80	79	61
Mary	60	69	70	71	80	72	68

The measures of central tendency and range for these two data sets are equal, but there are clearly differences in the two sets of marks.



Once students understand that this difference cannot be described by mean or by range, have the students create parallel box plots (as shown to the right). Then introduce the concept of standard deviation as a measurement of how far the data values lie from the mean.

Using the test marks as an example, prompt discussion around standard deviation by asking questions such as

- o Whose marks are more dispersed? What does this mean in terms of a high or low standard deviation?
- o If the data are clustered around the mean, what does this mean about the value of the standard deviation?
- o Who was more consistent over the five unit tests?

Students should make the connection that if the data are more clustered around the mean, there is less variation in the data and the standard deviation is lower. If the data are more spread out over a large range of values, there is greater variation and the standard deviation would be higher.

For this specific example:
While both Mary and Tim's marks have a mean of 70 and a range of 20, the data are not the same. This variation can be reflected using the standard deviation. Tim's marks have a standard deviation of 7.7 while Mary's marks have a standard deviation of 5.5.

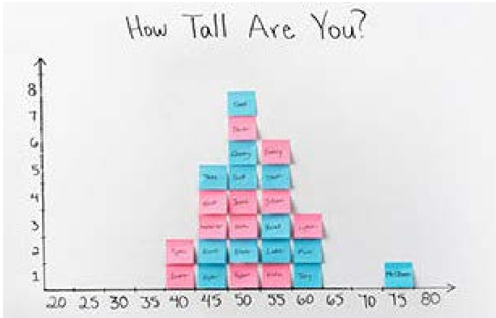
- Have students explore the meaning of standard deviation through relevant examples. Promote discussion around situations such as the following:
  - o *Sports Illustrated* is doing a story on the variation of player heights on NBA basketball teams. The heights, in cm, of the players on the starting lineups for two basketball teams are given in the table below.

Lakers	195	195	210	182	205
Celtics	193	208	195	182	180

  The team with the most variation in height will be selected for the cover of *Sports Illustrated* magazine. Ask students which team will appear on the cover.

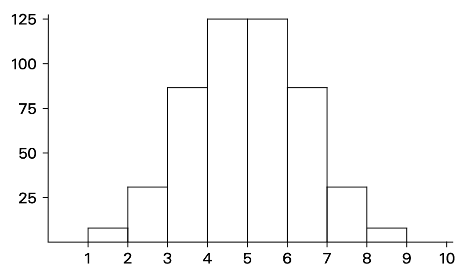
- It is also worthwhile to discuss how important it is in manufacturing and packaging to have very small standard deviations. For example, what do you think the standard deviation would be for the mass of a bag of chips or the mass of each cookie in a bag of cookies? It is also valuable to discuss whether the manufacturer would want such data to be normally distributed. If time allows, you could ask students to weigh (using a very accurate scale borrowed from the science department) small bags of chips or granola bars. The data could then be plotted using a histogram. (As you collect more and more data, it is likely that a normal distribution will begin to be revealed.) *It is important to note that standard deviation can only be accurately used on normally distributed data.*

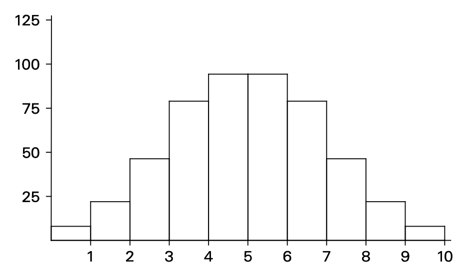
- A histogram can be created using Post-it-Notes such as illustrated below.



- Choose an attribute common to all students (number of siblings, number of weekly text messages sent or received, numbers of hours watching television, number of minutes spent eating lunch, number of hours spent sleeping, etc.). Ask students to collect the data from the class, select an appropriate bin size, and display the information in a frequency table. They should construct a histogram of the data and then describe the data distribution.

- Ask students which of the distribution of scores in the graphs below has the larger dispersion? They should justify their reasoning.



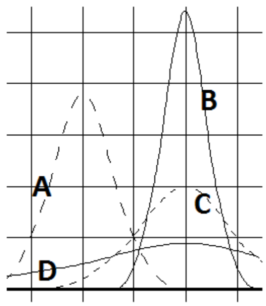


- Introduce students to the shape of the normal curve using an interactive Quincunx (MathIsFun.com 2010) (www.mathsisfun.com/data/quincunx.html).

  The general shape of the histogram will begin to approach a normal curve as more trials are added. Students should observe the following characteristics:
  - Normal distributions are symmetrical with a single peak at the mean of the data.
  - The curve is bell-shaped with the graph falling off evenly on either side of the mean.

- To probe understanding of standard deviation, ask students to answer the following questions:
  (a) Is it possible for a data set to have a standard deviation of 0? Can the standard deviation ever be negative? Explain why or why not.
  (b) If 5 was added to each number in a set of data, what effect would it have on the mean? On the standard deviation? What if each element in the data set was multiplied by –3?
  (c) Identify the characteristics of the data that make the standard deviation larger or smaller.

- Ask students to discuss several groups of density curves similar to those shown to the below. Students should be able to indicate the curve(s) with the largest mean as well as the curve(s) with the larger standard deviation and explain their reasoning.



- Graphing a normal distribution by hand can be tedious and time consuming. Using technology such as graphing calculators and spreadsheets, students can much more easily investigate the properties of a normal distribution and work with data that are specifically one, two, and three standard deviations from the mean. Consider the following data sample:

Number of hours playing video games in a week (15-year-old students)									
21.1	21.8	17.7	23.4	14.9	20.8	20.2	18.6	25.6	23.3
18.5	19.1	18.8	18.3	20.2	15.5	21.4	16.7	14.7	20.5
17.9	20.9	24.5	22.2	17.1	24.5	21.2	18.1	21.0	16.2
23.7	18.9	21.3	16.0	20.2	27.3	17.6	20.7	19.7	14.9

Guide students through the problem asking them to do the following:

(a) Determine the mean, standard deviation, median, and mode. Explain what you notice about the values.
(b) Construct a frequency table to generate a histogram. Use an interval width equal to the standard deviation since the spread of the normal distribution is controlled by the standard deviation.
(c) Discuss the symmetry of the histogram.
(d) Draw a frequency polygon and explain its shape.

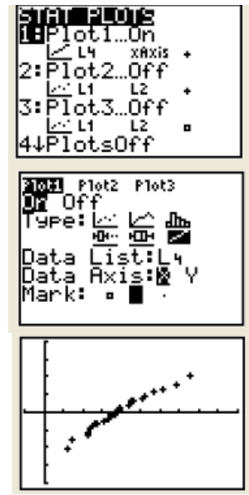
Students should understand that in a normal distribution the data are distributed symmetrically around the mean. Since the mean lies in the middle of the data, it is also the median. As the mean is located at the point at which the curve is highest, it is also the mode. Therefore the mean, median, and mode are equal for a perfect normal distribution and are approximately equal for the data that is approximately normal.

- Using data such as those in the previous chart, continue to investigate the area under the curve and the percentage of data that are one, two, and three standard deviations from the mean. Coach students along the investigation using the following directions, and discuss their observations.
    - Approximate the percentage of data that are to the left of the mean and then to the right of the mean.
    - Approximate the percentage of data within three standard deviations. Ask them what they notice.

- o Approximate the percentage of data within one standard deviation.
- o Approximate the percentage of data within two standard deviations.

As students analyze the data, they should recognize that fifty percent of the distribution lies to the left of the mean and fifty percent lies to the right of the mean. Furthermore, approximately 68% of the data lie within one standard deviation of the mean, 95% is within two standard deviations of the mean, and almost all the data are within three standard deviations of the mean.

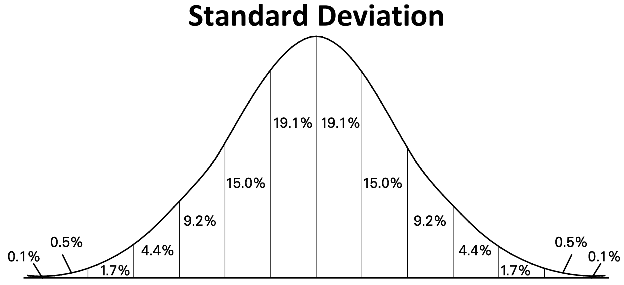
- Students will have to confirm whether a data set approximates a normal distribution. Observe students as they are working through examples to make sure they understand the concept of normal distribution. Using technology, students should consider the measures of central tendency, the appearance of the histogram, and the percent of data within one, two, and three standard deviations of the mean (i.e., are they close to 68%, 95%, and 99.7%). Furthermore there is a normal probability plot on the TI graphing calculator allow students to determine if data is approximately normal. If the data is approximately normally distributed the Normal line will be straight. If the data is normal then standard deviation is a meaningful statistic to use.



- If the data approximate a normal distribution, the properties of the normal distribution can be used to solve problems. Students should recognize that for normally distributed data, the area under the curve between any two values is equal to the probability that a given data value will fall between those two values. Consider a data set of 100 items normally distributed having a mean of 3.4 and a standard deviation of 0.2. If 68% of the area under the curve is within one standard deviation, then students should realize approximately 68 items should be between 3.2 and 3.6. Students should consider how many items would be between 3.2 and 3.6 if the data set has 200 items? 500 items? 750 items? 1000 items?

- Once students are comfortable working with applications where the numbers are one, two, or three standard deviations from the mean, you can introduce a problem where the score is 1.5 standard deviations from the mean. At this point you could use the diagram shown below.

Normal Curve
Standard Deviation

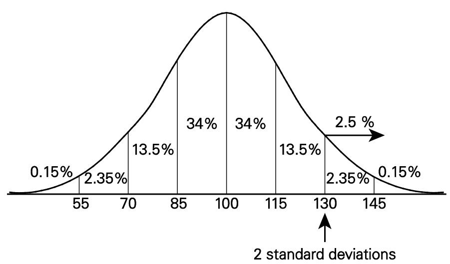


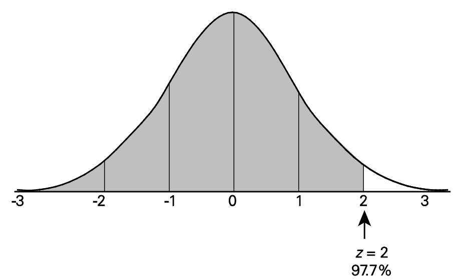
When students encounter a question where the numbers work so neatly, they quickly see the need for a table that is more flexible. This need creates the environment where they are ready to use a *z*-score table.

It is now time to introduce students to the *z*-score formula: $z = \dfrac{x-\mu}{\sigma}$.

- Students will use the properties of the normal curve to solve problems. Encourage students to sketch the normal distribution curve to help them visualize the information. Begin by having students solve problems that fall exactly one, two, or three standard deviations from the mean. Work with these questions and develop a comfort level with these questions prior to introducing *z*-scores and the *z*-score chart.

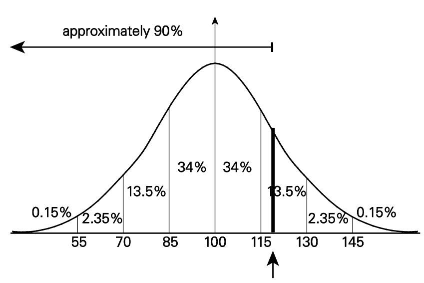
- To emphasize what a *z*-score is, present the following problem:
  The age of a sea turtle is normally distributed with a mean age of 100 years and a standard deviation of 15 years. What percentage of turtles live less than 130 years?
  - Ask students to first draw the normal distribution curve, labelling the mean and standard deviation, then verify their answer using *z*-scores.

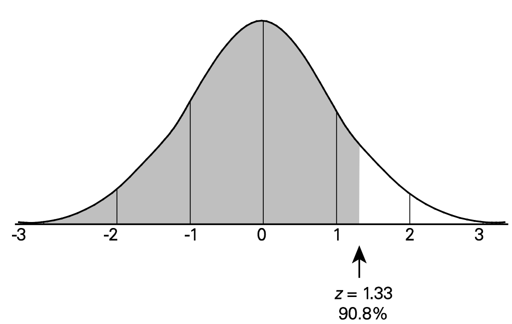




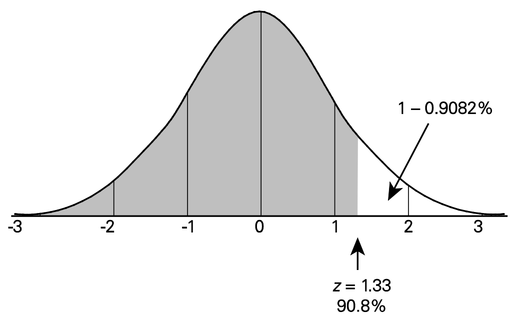
Using a visual representation, students should see that 97.5% of sea turtles would live less than 130 years. Since the score of 130 is two standard deviations away from the mean, students can use the *z*-score table to find the area under the curve to the left of the standard normal curve. This value is 0.9772, which is 97.7%. Discuss with students why there is a small error in the calculation when compared to standard deviation.

  - Using the example, ask students to consider a data value that is between one and two standard deviations from the mean such as an age of 120 years. Ask them to determine the percentage of sea turtles who live less than 120 years? Students should first estimate their answer (using standard deviation and the normal distribution curve) and then verify it using the *z*-score.





- Engage students in a discussion by asking the following questions:
  - Was your estimate reasonable when you compared it to the *z*-score?
  - Why is the *z*-score more reliable than estimating using standard deviation?
  - What percentage of sea turtles lived less than 120 years?
- Students should understand that the *z*-score table can be used to determine the area under the curve. If students were asked to determine the percentage of sea turtles that lived longer than 120 years, they would write 1 – 0.9082 , or 100% – 90.82%.



- The *z*-score can also provide a standard measure for comparing two different normal curves by transferring each to the standard normal distribution curve, having a mean of zero and a standard deviation of one. Students should consider examples such as the following:
  - An orange has an average mass of 141 g and a standard deviation of 12 g. A kiwi fruit has an average mass of 76 g and a standard deviation of 8 g. The masses of both of these fruits are normally distributed. When you are at the store you buy a “LARGE ORANGE” that has a mass of 150 g and a “LARGE KIWI” that has a mass of 86 g.

It appears that the Orange is larger and in some ways it is but can you make a case that the Kiwi is actually more accurately termed LARGE?

We might want to take into account the overall mass of each type of fruit. By calculating the orange’s and kiwi’s *z*-score values, we can compare the relative sizes of the fruit.

For the orange: $z = \frac{150-141}{12} = 0.75$, which means that this orange is only 0.75 standard deviations above the mean.

For the kiwi: $z = \frac{86-76}{8} = 1.25$, which means that this kiwi is 1.25 standard deviations above the mean.

Since both *z*-scores relate to a standard normal distribution, students can compare the values and should conclude that the kiwi is actually more worthy of the label “LARGE” than the orange.

Suggested Models and Manipulatives

- z-score tables (several different types)

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- σ and μ
- $\bar{x}$ and S_x
- bimodal
- bin width
- density curve
- dispersion
- frequency distribution
- frequency polygon
- histogram
- line plot
- mean
- median
- mode
- normal curve or bell curve
- normal distribution
- outlier
- range
- standard deviation
- z-score or standard score

Resources/Notes

Internet

- *Free Statistics Software* (Wessa.Net 2014)
  www.wessa.net
  Data generation site that can be used to produce histograms, count data values and develop normal curve. (Found only on US site.)

- *Kids'Zone: Learning with NCES*, "Chances" (US Department of Education 2014)
  http://nces.ed.gov/nceskids/chances/index.asp
  An interactive dice game that demonstrates the histogram as you input the number of rolls.

- *Math Is Fun*, "Standard Deviation Calculator" (MathIsFun.com 2012)
  www.mathsisfun.com/data/standard-deviation-calculator.html
  A standard deviation calculator showing all the steps in calculating the population standard deviation.

- *Measuring Usability*, "Interactive Graph of the Normal Curve" (Measuring Usability 2014)
  www.measuringusability.com/normal_curve.php
  An interactive graph of the normal curve.

- *Post-it Products*, "Activity Center: Histograms" (3M 2014)
  www.post-it.com/wps/portal/3M/en_US/PostItNA/Home/Ideas/Articles/Histogram

- *Statistics Calculator: Standard Deviation* (Arcidiacono 2011)
  www.alcula.com/calculators/statistics/standard-deviation
  This website calculates the mean and population standard deviation without showing any calculations.

- *Untitled* [Plinko game] (University of Colorado 2013)
  http://phet.colorado.edu/sims/plinko-probability/plinko-probability_en.html
  An interactive plinko game demonstrating the histogram.

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  Sections 5.1–5.6, pp. 210–275

SCO S03 Students will be expected to interpret statistical data, using confidence intervals, confidence levels, and margin of error. [C, CN, R]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

S03.01 Explain, using examples, how confidence levels, margin of error, and confidence intervals may vary depending on the size of the random sample.
S03.02 Explain, using examples, the significance of a confidence interval, margin of error, or confidence level.
S03.03 Make inferences about a population from sample data, using given confidence intervals, and explain the reasoning.
S03.04 Provide examples from print or electronic media in which confidence intervals and confidence levels are used to support a particular position.
S03.05 Interpret and explain confidence intervals and margin of error, using examples found in print or electronic media.
S03.06 Support a position by analyzing statistical data presented in the media.
S03.07 Explain, using examples, the significance of the Central Limit Theorem (CLT).

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
	S03 Students will be expected to interpret statistical data, using confidence intervals, confidence levels, and margin of error.	

Background

It is intended that the focus of this outcome be on the interpretation of data rather than on mathematical calculations.

In Mathematics 9 (SP02), students have had experience identifying whether a given situation represents the use of a **sample** or a **population**. In this Statistics unit, students will recognize that since an entire population is often difficult to study, we often use a representative population called a sample. They will recognize that, if the sample is truly representative, the statistics generated from the sample will be the same as the information gathered from the population as a whole. It is unlikely, however, that a truly representative sample will be selected. We need to make predictions of how confident we can be that the statistics from our sample are representative of the entire population.

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for Statistics is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

It is always important to keep straight whether a number describes a population or a sample.

A **parameter** is a number that describes a population. A parameter is a fixed number, but in practise we do not know its value.

A **statistic** is a number that describes a sample. The exact value of a statistic can be determined from a sample; but, is variable as it can change from sample to sample. This fact is called sampling variability. We use a statistic to estimate a parameter.

For example, a Gallup Poll asked a Simple Random Sample (SRS) or 515 adults if they believed in ghosts. 160 said "yes." So the proportion of the sample who said that they believed in ghosts is $\frac{160}{515} = 0.31$. This number, 0.31, is a statistic. We can use it to estimate the proportion of all adults who believe in ghosts. This is our parameter of interest.

Confidence Intervals are used to draw conclusions from a sample. How useful the sample is in representing the larger population depends on two important values, the **margin of error** and **confidence level**.

Students are expected to gain an understanding of the Central Limit Theorem (CLT). By sampling known populations multiple times they will discover that whether or not a population is normally distributed, the distribution of the sample means will be normal.

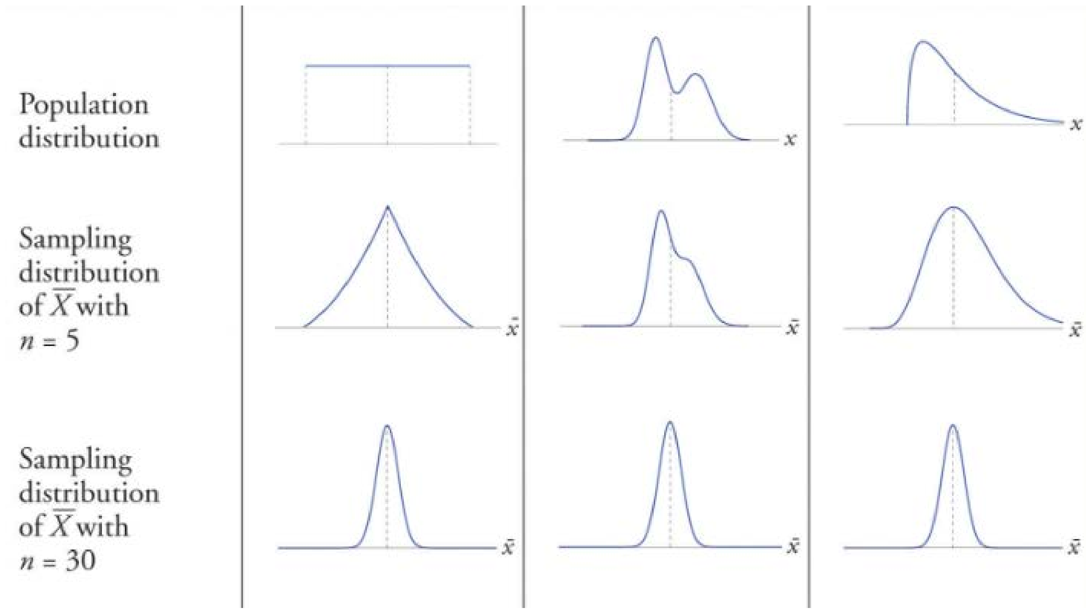
It is important that students understand the difference between a population distribution (distribution of all items in the population), a sample distribution (distribution of the items in a single sample) and a sampling distribution (distribution of the means of many similar samples).

It is expected that students will understand that it is reasonable to assume that when a sufficient number of randomly selected samples are chosen their means will form a normal distribution with a mean close to the mean of the population.

The link between z-scores and confidence intervals will be made. It should be noted that this calculation is a good estimate but, unless the population standard deviation is known, it is only a rough estimate. Students will use technology to more accurately determine the confidence interval.

To do this they will need to ask three questions:

1. Was the sample randomly selected (without bias)? If the answer to this is no then a confidence interval calculation will be meaningless.
2. Is the population distribution normal?
    - If yes then a confidence interval can be calculated even if the sample is relatively small.
    - If no, then the sample must be at least 25 – 30 to calculate the confidence interval with any degree of reliability as illustrated below.



3. Is the standard deviation of the population known?
    - If yes, use the z- interval to determine the confidence interval.
    - If no, use the t-interval to determine the confidence interval.

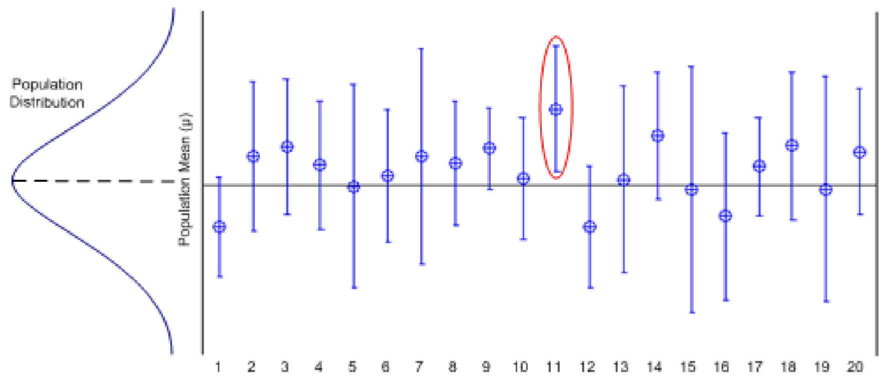
"What the Confidence Interval (CI) Is Not

There are a lot of things that the confidence interval is not. Unfortunately many of these are often used to define confidence interval.

- It is not the probability that the true value is in the confidence interval.
- It is not that we will obtain the true value 95% of the time.
- We are not 95% sure that the true value lies within the interval of the one sample.
- It is not the probability that we correct.
- It does not say anything about how accurate the current estimate is.
- It does not mean that if we calculate a 95% confidence interval then the true value is, with certainty, contained within that one interval."

(Source: *Statistical Research*, " When Discussing Confidence Level with Others ..." Blog by Wesley, August 13, 2013. http://statistical-research.com/some-issues-relating-to-margin-of-error/?utm_source=rss&utm_medium=rss&utm_campaign=some-issues-relating-to-margin-of-error)

"In statistics, creating confidence intervals is comparable to throwing nets over a target with an unknown, yet fixed, location."



The diagram above illustrates a confidence interval plot for 20 simulated sample sets. Each interval was calculated providing 95% confidence. Note that one interval (circled) out of twenty does not cover the known mean representing 5% of the total number of intervals. This is in line with the expectation that 95% of the intervals will contain the true population mean.

A 95% CI indicates that 19 out of 20 samples (95%) taken from the same population will produce CI's that contain the true population parameter. A 90% CI indicates that 18 out of 20 samples from the same population will produce CI's that contain the population parameter, two of the intervals will not contain the parameter and so on."

(Source: *The Mimtab Blog*, "What Do Confidence Intervals Have to Do with Rabies?" Carly Barry in Health Care Quality Improvement, September 27, 2012. http://blog.minitab.com/blog/real-world-quality-improvement/what-do-confidence-intervals-have-to-do-with-rabies)

Therefore, the Confidence Level of 95% does mean, in the long run, if we keep on computing these confidence intervals, then 95% of those intervals will contain the true value.

The Margin of Error

A direct component of the confidence interval is the margin of error. This is the number that is most widely seen in the news, whether it be print, TV, or otherwise.

For example the following infographic was published and the margin of error reported.

The survey was conducted among 1,213 randomly selected adults in California and 9 Northeast states who were licensed to drive and had driven a vehicle in the past 12 months. The survey was carried out from April 1 to April 8. The margin of error is 4 percentage points at a 95 percent confidence level for questions asked of all respondents.

Sometimes the margin of error has to be calculated from the confidence interval itself. For example in the media report "*The average weight loss for people on the new miracle drug was 4.6 pounds, with a 95 percent confidence interval of 2.2 pounds to 6.9 pounds.*"

If a population is known to be normal with the mean and standard deviation know this information can be abbreviated as N(μ,σ)

Sample size is often calculated based on the desired margin of error. For large populations sampling more than 1500 results in no significant gain in accuracy.

There are various on-line calculators to determine sample size based on population size, confidence level and desired margin of error. One such calculator can be found at http://fluidsurveys.com/university/survey-sample-size-calculator/

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Explain the difference between a population and a sample.
- When would the individuals in your math class be considered a sample? When would they be considered a population?
- Are the Canadian citizens who respond to the federal census a sample or a population? Explain your reasoning.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- A botanist collects a sample of 50 iris petals and measures the length of each. He finds that the mean is 5.55 cm and the standard deviation is 0.57 cm. The botanist then reports that he is 95% confident that the average petal length is between 5.39 cm and 5.71 cm, 19 times out of 20.
  (a) Identify the margin of error, the confidence interval, and the confidence level.
  (b) Explain what information the confidence interval gives about the population of iris petal length.
  (c) How would the length of a 99% confidence interval be different from that of a 95% confidence interval?
  (d) If you do not know the margin of error but you do know that the confidence interval is between 5.39 cm and 5.71 cm, how could you determine the margin of error?
  (e) Was his statement misleading? Explain.
- A report claims that the average family income in a large city is $32,000. It states the results are accurate 19 times out of 20 and have a margin of error of ±2500.
  (a) What is the confidence level in this situation? Explain what it means.
  (b) Explain the meaning of a margin of error of ±2500.
- What is meant by a 90%, a 95%, and a 99% confidence interval? How are these intervals similar? How are they different?
- In a national survey of 400 Canadians aged 20 to 35, 37.5% of those interviewed claimed they exercise for at least four hours a week. The results were considered accurate within 4%, 9 times out of 10.
  (a) Are you dealing with a 90%, 95%, or 99% confidence interval? How do you know?
  (b) How many people in the survey claimed to exercise at least four hours a week?
  (c) What is the margin of error?
  (d) What is the confidence interval? Explain its meaning.
  (e) What are some limitations of this survey?
  (f) If the writers of the article created a 99% confidence interval based on these data, how would it be different? How would it be the same?
  (g) How would the confidence interval change if the sample size were increased to 1000 but the sample proportion remained the same?

- In a recent survey, 355 respondents from a random sample of 500 first-year university students claim to be attending their first choice of university. If the sample size were decreased but the sample proportion remained the same, how would the confidence intervals change?
- The results of a survey show that 71% of residents in Yarmouth own cell phones. The margin of error for the survey was ±2.3%. If there are 7300 people in Yarmouth, determine the range of the number of people that own cell phones.
- When a sample of batteries was tested, it was determined that this brand of battery had a mean life expectancy of 12.6 hours with a margin of error of ±0.7 hours.
  (a) State the confidence interval for this brand of battery.
  (b) If a larger sample of batteries were tested and it was determined that this brand of battery had a mean left expectancy of 12.6 hours, how do you think that the margin of error would compare to 0.7 hours?
- A company that produces white bread is concerned about the distribution of the amount of sodium in its bread. The company takes a simple random sample of 50 slices of bread and computes the sample mean to be 98 milligrams of sodium per slice. Construct a 90% confidence interval for the unknown mean sodium level assuming that the population standard deviation is 8 milligrams.
- You want to rent an unfurnished one-bedroom apartment in Halifax next year. The mean monthly rent for a simple random sample of 40 apartments advertised in the local newspaper is $950 plus utilities. Assume that the standard deviation is known to be $200. Find a 99% confidence interval for the mean monthly rent for unfurnished one-bedroom apartments available for rent in this community.
- A news report stated that "The average weight loss for people on the new miracle drug was 4.6 pounds, with a 95 percent confidence interval of 2.2 pounds to 6.9 pounds."
  - What is the margin of error for this study?
  - How would the margin of error change if the same survey had reported a 90% confidence interval?
- The birth process of a newly discovered mammal is being studied, and the lengths of 18 observed pregnancies have been recorded. The mean gestation period was 97.3 days with a standard deviation of 2.2 days. Find a 95% confidence interval for the mean time of pregnancy for this mammal.
- We have IQ test scores of 31 seventh-grade girls in a Midwest school district. We have calculated that sample mean is 105.84 and the standard deviation is 14.27.
  - Give a 99% confidence interval for the average score in the population.
  - What is the margin of error?
  - In fact, these are the scores of 31 girls who volunteered to share their results with the researchers. Explain carefully why we cannot trust the confidence interval calculated from this sample.

- Create a poster that includes an example from print or electronic media that uses confidence intervals or confidence levels to support a position. Your poster should respond to each of the following:
    - Interpret the confidence interval and confidence levels for someone who has no knowledge of statistics.
    - How would the confidence interval change if the sample size used for the study doubled?
    - Do you agree or disagree with any concluding statements that were made about the data from the study? Explain.

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Find an example in the media that uses confidence intervals or confidence levels to support a position relevant to teenagers today. Write two short news articles on the topic. In one article, make the situation sound as disastrous as you can. In the other article, try to minimize the problem.

- Begin with an activity to develop the big idea of the Central Limit Theorem. Reinforce this by another activity where multiple samples are randomly selected from a known population. It is important to take time for the students to observe that, regardless of the shape of the population distribution, a sufficiently large sampling distribution will be normal in shape and that larger sample sizes will create sampling distributions that have less variation.
- Take time to make sure that students have made the connection between the z-score charts and the calculation of an estimate for the confidence interval. Note that if the population standard deviation is known then using a z-interval will be correct, if the population standard deviation is not known then a z-interval only provides a rough estimate. A t-interval would need to be used to obtain the confidence interval. At this point just have students obtain an estimate using the z-interval and the formula for 90%: $\bar{x} \pm 1.645\frac{\sigma}{\sqrt{n}}$; for 95%: $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$; and for 99% $\bar{x} \pm 2.576\frac{\sigma}{\sqrt{n}}$.

The goal is to encourage the understanding of the mean and margin of error. AFTER students have calculated confidence intervals by hand using z-intervals, introduce the use of technology to calculate the confidence intervals using the type of interval that is appropriate.

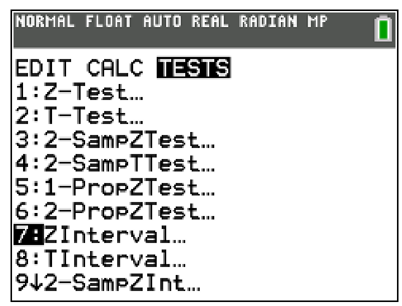
Example when to use z-interval: *(based on normal curve and z-scores)*

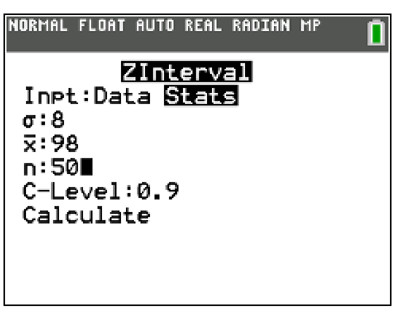
A company that produces white bread is concerned about the distribution of the amount of sodium in its bread. The company takes a simple random sample of 50 slices of bread and computes the sample mean to be 98 milligrams of sodium per slice. Construct a 90% confidence interval for the unknown mean sodium level assuming that the population standard deviation is 8 milligrams.

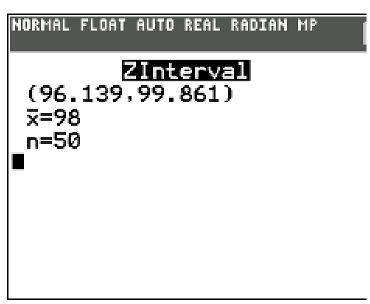
Sample was randomly selected.

Sample size is large enough even if the population distribution is not normal

Since we have been told the population standard deviation we can use a z-interval.







Therefore the confidence interval is 96.14 mg to 99.86 mg of salt in a slice of bread.

This means that we can state with 90% confidence that the average amount of salt is a slice of bread is between 96.14 mg and 99.86 mg.

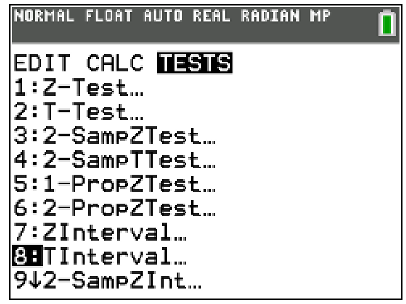
Example when to use a t-interval: *(based on t-distribution which is studied in later statistics courses)*

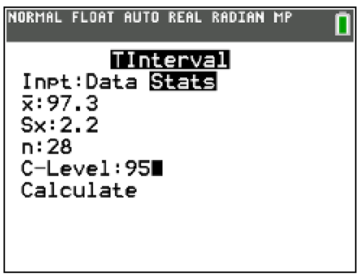
The birth process of a newly discovered mammal is being studied, and the lengths of 28 observed pregnancies have been recorded. The mean gestation period was 97.3 days with a standard deviation of 2.2 days. Find a 95% confidence interval for the mean time of pregnancy for this mammal.

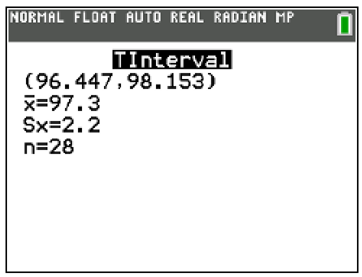
Sample was randomly selected.

Sample size is large enough even if the population distribution is not normal

Since we have NOT been told the population standard deviation we can use a t-interval.







Therefore the confidence interval is 96 days to 98 days.

This means that we can state with 95% confidence that the average gestation period for this mammal is between 96 days and 98 days.

- A relatively quick way for students to relate to confidence levels is to ask a student to choose a number between 0 and 100.
  - Once this choice has been made you can ask the student if the number is a specific number such as 46. It is unlikely that you have guessed it correctly. (This is a point estimate)
  - Then ask if the number is between 0 and 100? (This would be 100% confidence level).
  - Then ask if the number is between 10 and 90? (This could represent a 80% confidence level)

  This discussion leads students to understand that by using a wide interval, the chance of their guess including the actual number that the student selected increases. Discuss how a wider confidence interval is more likely to actually contain the number that the student chose. Students also need to understand that as you increase the confidence level, the interval becomes wider and may lose its usefulness. Basically, students need to understand that a balance is necessary; the interval selected needs to be narrow enough to be useful and wide enough to be accurate.

- Use a situation such as the one described below to link an intuitive understanding of sample size with the margin of error.
  The size of a random sample will affect the confidence interval and margin of error. Suppose that the town of Shelburne is trying to determine the location of a new

recreation center. Ask students to make a prediction about the confidence interval and margin of error if 100 people were surveyed, if 1000 people were surveyed, or if all the people in the town were surveyed and then investigate, the impact of sample size using the z-interval formula for 95% confidence intervals. $\pm 1.96\frac{\sigma}{\sqrt{n}}$

- If 100 people were surveyed, students should realize the results could be skewed one way or the other. In other words, there is no guarantee that the results will be accurate. This would result in a significant margin of error (due to a small sample size) and would produce a confidence interval with a large range.
- If 1000 people were surveyed, there would be a greater chance for more accurate results. The increase in sample size would decrease the margin of error and, in turn, decrease the range on the confidence interval, thus, getting us closer to the actual result.
- If all the people in the town were surveyed, students should recognize they would know the actual result and have no margin of error.

- Provide students with a mean value and a margin of error or the confidence interval and ask them to analyze the data to make inferences. Consider the following example:
  - A recent study reports that 71% of gamers own at least fifteen games, accurate to within 2.5%, 19 times out of 20. According to the study, if there are 1000 gamers at a conference, what is the range of gamers who own at least fifteen games?
  - Ask students the following questions:
    (a) What is the confidence level?
    (b) What is the confidence interval?
    (c) According to the study, if there are 1000 students, what is the range of students who could own at least 15 games?
- Provide students with an inference based on given confidence intervals and ask them to comment on the validity of the inference. Consider the following example:
  - The city of St. John's is trying to determine whether or not to continue the curbside recycling program. A survey indicated that 50% of residents wanted the program to continue. The survey was reported to be accurate 9 times out of 10, with a margin of error of 16.7%. Based on these results, what course of action should the city take with respect to the curbside recycling program? Explain your answer.

  Students need to learn that the confidence level is only 90%, and due to the large margin of error, the range of the confidence interval is also large (33.3% to 66.7%). The city is only 90% confident that between one-third and two-thirds of the population are in favor of the recycling program. Ask students if this is useful information. It should be suggested that the city increase its sample size to reduce the margin of error and use at least a 95% confidence interval to get a better picture of how the people actually feel about the program.

- Students should be able to determine the confidence interval and interpret the results. Consider using an example such as the following to illustrate this.

Suppose you're the manager of an ice cream shop, and you're training new employees to be able to fill the large-size cones with the proper amount of ice cream (10 ounces each). You want to estimate the average weight of the cones they make over a one-day period, including a margin of error. Instead of weighing every single cone made, you ask each of your new employees to randomly spot check the weights of a random sample of the large cones they make and record those weights on a notepad. For n = 50 cones sampled, the sample mean was found to be 10.3 ounces. Suppose the population standard deviation is 0.6 ounces.

(a) What is the 90% confidence interval?

(b) What advice should the manager tell his employees?

- Students could find a variety of examples, such as quality control or public opinion polls as reported in the newspaper or other news sources, in which confidence levels, confidence intervals, and margins of error are used to report results. They should interpret the meaning of the confidence intervals and levels and its implications in society. These examples could then be used to prompt discussions around interpreting and explaining statistical data leading to forming an opinion on the topic.

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- Central limit theorem
- confidence interval
- confidence level
- margin of error
- parameter
- population
- population distribution
- sample
- sample distribution
- sampling distribution
- sampling variability
- t-interval
- z-interval

Resources/Notes

Internet

- e! Science News, “Science news articles about ‘confidence interval’” (e! Science News 2014)
  http://esciencenews.com/dictionary/confidence.interval
  Confidence intervals in scientific research.

- Statistical Research, “When Discussing Confidence Level with Others ...” (Blog by Wesley, August 13, 2013)
  http://statistical-research.com/some-issues-relating-to-margin-of-error/?utm_source=rss&utm_medium=rss&utm_campaign=some-issues-relating-to-margin-of-error

- The Mimtab Blog, “What Do Confidence Intervals Have to Do with Rabies?” (Carly Barry in Health Care Quality Improvement, September 27, 2012)
  http://blog.minitab.com/blog/real-world-quality-improvement/what-do-confidence-intervals-have-to-do-with-rabies

  Sample Size Calculator given confidence level, population size and required margin or error.
  http://fluidsurveys.com/university/survey-sample-size-calculator/

  Video: Bunnies, Dragons and the 'Normal' World: Central Limit Theorem | The New York Times
  https://www.youtube.com/watch?v=jvoxEYmQHNM

- Video: Understanding Confidence Intervals: Statistics Help
  https://www.youtube.com/watch?v=tFWsuO9f74o

Print

- Foundations of Mathematics 11 (Canavan-McGrath et al. 2011, Nelson Canada)
  Section 5.6, pp. 267–275

Relations and Functions
55-60 hours

GCO: Students will be expected to develop algebraic and graphical reasoning through the study of relations.

SCO RF01 Students will be expected to model and solve problems that involve systems of linear inequalities in two variables. [CN, PS, T, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF01.01 Model a problem, using a system of linear inequalities in two variables.
RF01.02 Graph the boundary line between two half planes for each inequality in a system of linear inequalities, and justify the choice of solid or broken lines.
RF01.03 Determine and explain the solution region that satisfies a linear inequality, using a test point when given a boundary line.
RF01.04 Determine, graphically, the solution region for a system of linear inequalities, and verify the solution.
RF01.05 Explain, using examples, the significance of the shaded region in the graphical solution of a system of linear inequalities.
RF01.06 Solve an optimization problem, using linear programming.

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
RF01 Students will be expected to interpret and explain the relationships among data, graphs, and situations. **RF05** Students will expected to determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. **RF06** Students will be expected to relate linear relations to their graphs, expressed in • slope-intercept form ($y = mx + b$) • general form ($Ax + By + C = 0$) • slope-point form ($y - y_1) - m(x - x_1$) **RF09** Students will be expected to represent a linear function, using function notation. **RF10** Students will be expected to solve problems that involve systems of linear equations in two variables, graphically and algebraically.	**RF01** Students will be expected to model and solve problems that involve systems of linear inequalities in two variables.	**RF01** Students will be expected to represent data, using polynomial functions (of degree ≤ 3), to solve problems.

Background

In Mathematics 9, students solved single variable linear inequalities with rational coefficients (PR04). In Mathematics 10, students solved problems that involved systems of linear equations in two variables, both graphically and algebraically (using both elimination and substitution).

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for RF01 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

In Extended Mathematics 11, students will test points to decide if a given ordered pair satisfies a given inequality. For a given linear inequality, students will graph the appropriate region and verify. Students will be introduced to linear inequalities in two variables, extending the graphical model of a linear inequality from a one-dimensional linear number line to a two-dimensional coordinate plane and the solution set from points along a linear number line to points in a solution region.

Terms such as those in the table below are used with reference to inequalities.

greater than	less than	greater than or equal to	less than or equal to	at most
no more than	no less than	at least	more than	fewer than
maximum	minimum	optimal		

Students will learn how to represent a simple problem situation algebraically as a system of two linear inequalities and then graph the system to represent the solution set. They will graph systems of linear inequalities that include the possibility of equality, focusing on the intersection of the solution regions and how the common region represents the solution to the system.

Students must define any variable that they use so that its meaning is evident. For example, to say “let *d* represent distance” would be insufficient. They would rather need to say “let *d* represent the distance travelled since time, $t = 0$”. Otherwise d could represent the distance that you are from Halifax or home or school. This clear definition of a variable is very important to establish with students.

Students will represent problem situations algebraically as systems of two linear inequalities and then graph the systems to represent the solution sets. Students will verify their solution using test points and/or graphing technology.

New terminology is introduced and should be both developed and encouraged in student discussions. **Constraints** describe limiting conditions and restrictions on the problem situation and are represented by linear inequalities. The **feasible region** is the region for a system of linear inequalities within which any point is a feasible or valid solution. The **objective function** is the optimization equation in the linear programming model.

It is essential that students develop a solid understanding of constraints and their specific meaning to the context of a problem before putting them to use in the solution of linear programming questions.

The model car multi-link cube activity and/or the furniture Lego activity can be used to effectively build understanding of constraints, feasible region and objective function. These activities develop conceptual understanding of linear programming by walking students through the process of linear programming. Along the way, students are asked to explain what is happening and why, which allows them to internalize the procedural skill necessary to solve linear programming problems.

Once students have completed one of these activities they will be ready to explore the meaning of constraints for various manufacturing situations.

Once students have a good understanding of the meaning of the variables and the meaning of the constraints, they are ready to begin finding the constraints from the context of the question. It is important that students experience situations where they are determining a minimum as well as where they are determining a maximum.

Students will explore the feasible region by examining the coordinates of the intersection points and discover that optimal values of the objective function are found at the vertices of the feasible region, though the vertex of a feasible region is not necessarily an optimal solution. Although students should be aware of these situations, the level of sophistication that is required to locate the optimal point in these situations is beyond this grade level and is not an expectation in the course.

Technology can be used effectively to graph inequalities. While some software programs, such as DESMOS, allow in-equations in any form, students may need to rearrange an equation to obtain its slope intercept form in order to use technology such as the inequality application on the TI-84 calculator.

Students are expected to understand that linear programming is used to solve real world problems.

News From the World of Operations Research		
Chinese Farmers Plan Crop Production Using Linear Programming Chang Qing County Farmers Increase Profits, Improve Ecology, and Diversify Economy	**Nabisco Schedules Baking Operations** Scheduling an operation of bakeries is a difficult task. A realistic problem at Nabisco could involve 150 products, 218 facilities, 10 plants and 127 customer zones. A problem this size involves over 44,000 decision variables and almost 20,000 constraints. These problems were routinely solved in 1983 on an IBM 3033 computer in under 60 CPU seconds.	**Plywood Ponderosa de Mexico Optimizes Product Mix and Increases Profits**

It is expected that students showcase their understanding of RF01 by completing a project such as the Toy Shop Project or the Chapter Task found in the Student text.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

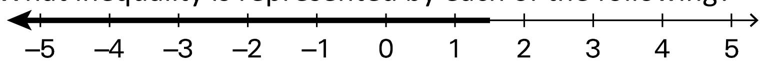
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

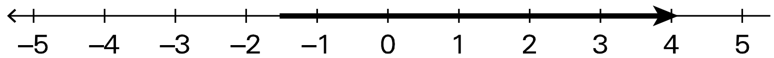
Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Explain to another student how you would solve the following system of linear equations. Justify the method you chose.
  - (a) $4x - 7y = -39$
  - (b) $3x + 5y = -19$
- Create a situation relating to coins that can be modelled by the linear system shown below and explain the meaning of each variable.
  - (a) $x + y = 24$
  - (b) $0.25x + 0.05y = 4.50$
- Solve each of the following inequalities:
  - (a) $2x + 5 \leq 8$
  - (b) $5 - 3x \leq 8$

- What inequality is represented by each of the following?

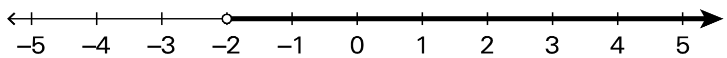




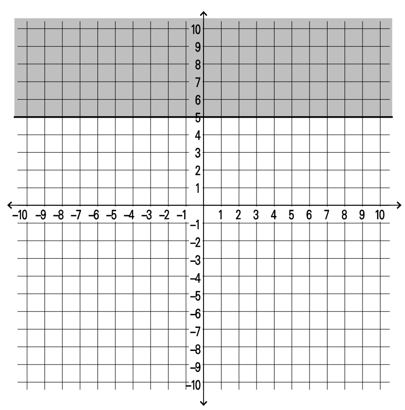
Whole-Class/Group/Individual Assessment Tasks

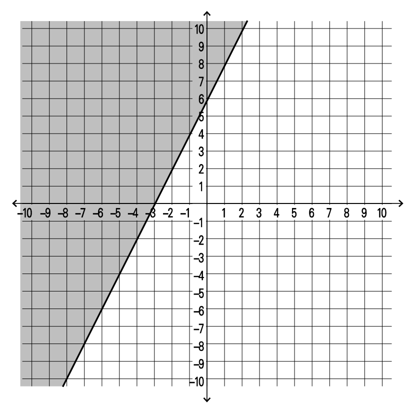
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- State the coordinates of a point that satisfies the inequality $3x + 2y \leq 4$?
- Is the point (2, 1) a solution for the in-equation $3x – 2y > 8$?
- Explain, using the number line, what is meant by the in-equation $x \geq 2$.
- What in-equation is represented by the following diagram?



- Explain, using the Cartesian coordinate system, what is meant by the in-equation $x \geq 2$.
- What in-equation is represented by each of the following diagrams?



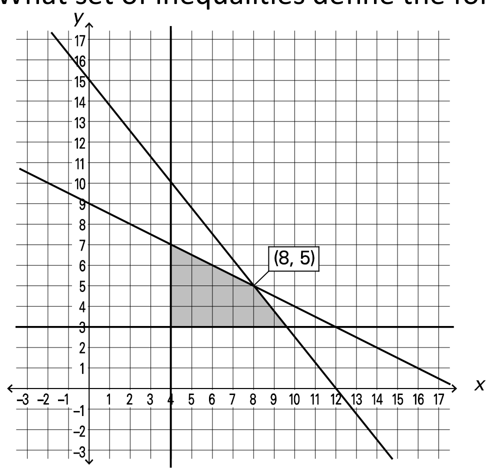


- A farmer has 100 acres on which to plant oats or corn. Each acre of oats requires \$18 to seed. Each acre of corn requires \$36 to seed. The farmer has \$2100 available for the purchase of seeds. If the revenue is \$55 from each acre of oats and \$125 from each acre of corn, what combination of plants will produce the greatest total profit?
- Midtown Manufacturing Company makes plastic plates and cups, each of which requires time on two different machines. Manufacturing a unit of plates requires one hour on machine A and two hours on machine B. Producing a unit of cups requires three hours on machine A and one hour on machine B. Each machine is operated for at most 15 hours per day. Using what you have learned about systems of inequalities, and the restrictions given above, determine the maximum number of plates and cups the company can produce each day.

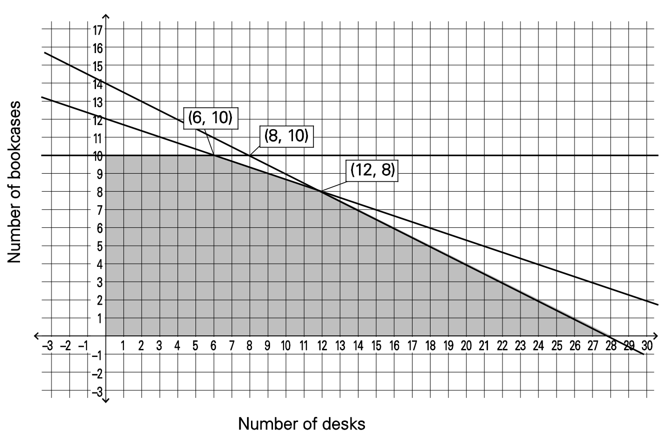
- Sketch the solution that corresponds to each set of inequalities.
  (a) $y \le 6 - x$ and $y \ge x - 2$

  (b) $3x - y < 4$ and $x - 2y > 3$

- What set of inequalities define the following shaded region?

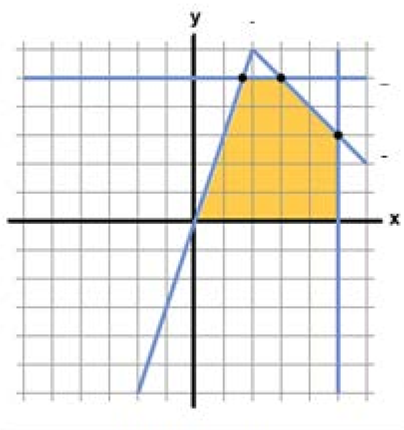


- A local furniture construction company has agreed to certain conditions in order to receive government funding. They also recognize that certain conditions exist due to the size of their workplace and the number of their employees. The shaded region on the graph below indicates all the possible number of desks and bookcases that the company can make each month.



  Each desk can be sold for a price that will yield a profit of \$120 and each bookcase can be sold and yield a profit of \$90. Write the objective function that could be used to determine the number of each type of item that this company should produce each month in order to maximize their profit and determine the number of desks and bookcases that should be produced in order to maximize their weekly profit.

- Jimmy is baking cookies for a bake sale. He is making chocolate chip and oatmeal raisin cookies. He gets 25 cents for each chocolate chip cookie and 30 cents for each oatmeal raisin cookie. He cannot make more than 500 cookies of each kind, and he cannot make more than 800 cookies total. He must make at least one-third as many chocolate chip cookies as oatmeal raisin cookies.
  (a) Define the variables and state the in-equalities that would represent this situation.
  (b) Use the graph shown below to determine how many of each kind of cookie should he make to get the most money.



Graph of Constraints

- An artist is producing handmade necklaces and bracelets. She makes more necklaces than she does bracelets. She can make up to a total of 24 bracelets and necklaces per week.
  (a) Define the variables in this situation.
  (b) Write the in-equations that describe this situation.
  (c) Graph the in-equations and determine the feasible region.
  (d) What additional information is needed if you are asked to determine how many bracelets and necklaces the artist should make in order to maximize her profit?
- Sean has two summer jobs. He works no more than a total of 34 hours per week. Both jobs allow him to have flexible hours. At one job, Sean works at least 12 hours and earns \$11.25 per hour. At the other job, he works no more than 25 hours and earns \$11.50 per hour.
  (a) Sean wrote the in-equations shown below. Explain what the variables represent and what each of the in-equations represents.
    (i) $A \geq 12$
    (ii) $B \leq 25$
    (iii) $B \geq 0$
    (iv) $A \leq 34$
    (v) $A + B \leq 34$
  (b) Sketch the feasible region.
  (c) What combination of numbers of hours will allow Sean to maximize his earnings? What can he expect to earn?

- A firm operates two types of aircraft:
  Aircraft A - carry a maximum of 40 passengers and 30 tons of cargo / Cost $1500 per Journey
  Aircraft B - carry a maximum 60 passengers and 15 tons of cargo/ Cost $1800 per Journey
  - The aircraft firm is contracted to carry at least 480 passengers and 180 tons of cargo each day.
  - What choice of aircraft will minimize overall cost?
- A calculator company produces both a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific calculators and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific calculators and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped every day. Each scientific calculator sold results in a $3 profit, but each graphing calculator produces a $5 profit. Let *S* represent the number of scientific calculator produced per day. Let *G* represent the number of graphing calculator produced per day.
  (a) Write the inequalities to represent the constraints for this company.
  (b) Determine the number of each type of calculator that this company should produce each day in order to maximize their daily profit.
- Complete the Dirt Bike activity found at: https://illuminations.nctm.org/Lesson.aspx?id=2355

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Students should begin this unit by completing activities such as the counter-basket, polygraph and other Desmos activities to ensure that they have a solid understanding of linear equations as well as the vocabulary of linear equations and in-equations.
- Insisting that students clearly define the variables that they choose to use can often reduce any confusion that they may experience when obtaining in-equations and interpreting their solutions.
- After students have reviewed the solution of linear systems use a situation that builds understanding of inequalities, such as the following:

  Two car rental agencies in the UK have the following rate structures for a subcompact car (prices in Canadian Dollars).
    - Ace Cars charges \$20 per day plus 15¢ per km.
    - Big Ben Cars charges \$18 per day plus 16¢ per km.

  (a) If you rent a car for one day, for what number of kilometres will the two companies have the same total charge?

  (b) Under what conditions, if any, should you rent from Ace Cars? Explain your reasoning.

  Since the answer to this second question is an inequality, this context will build on the students' understanding that while the solution of two linear equations is a point, the solution to a question may be conditional.

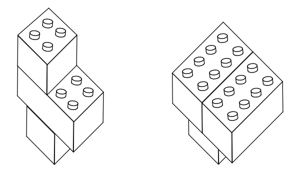
- Play a game where students have to match inequalities to their graphs. One such game can be found online at the Quia website at www.quia.com/rr/79715.html?AP_rand=571246363.

- Begin to introduce linear programming (the concept and vocabulary) with an activity such as the toy car multi-link cube or Lego furniture construction.

  Lego Furniture Activity:

  As an introduction to linear programming and constraints use Lego, sticky dots, and chart paper. Have students work in pairs. Each pair gets a Ziploc bag of eight small pieces and six large pieces and is given the following challenge.

  - You are the owner of a furniture shop, which specializes in full-size Lego-inspired furniture. Your supplier can deliver eight small components and six large components each day. A chair requires the use of two small and one large component and has a profit of \$10. A table requires the use of two small and two large components and has a profit of \$18. How do you make the most profit?

  Once students have determined how they would make the most profit for this situation, suggest that the chair be redesigned to have a back that is higher so that the chair now uses three small and one large component, the table remains unchanged. How would this design change impact the number of tables and chairs that should be made to maximize the profit?

  After students have worked with the Lego chairs and tables, present the following situation and ask students to agree on the variables to be used and together define these variables.

  - Several years later, your shop is now a large chain. You have updated your product line. Your supplier can deliver 1200 small components and 800 large components each day. How do you make the most profit?

  Allow groups time to think about this problem and how they would approach it.

  Once the class has agreed on the constraints of $2C+2T\le 1200$ and $1C+2T\le 800$ or $3C+2T\le 1200$ and $1C+2T\le 800$ depending on the furniture design being used, make sure that they are able to clearly explain what each of these equations represents. You might ask them what the constraints $2C+1T\le 1800$ and $3T\le 1200$ would represent in terms of a different furniture design.

  When you feel confident that the students understand the meaning of the constraints, choose a set of constraints for one particular furniture design. Give each group of students 8–12 points to sort and determine if they satisfy one of the constraints (red dot) or the other constraint (yellow dot). If the point satisfies both constraints use a (blue dot). If the point does not satisfy either equation, do not assign it a dot.

  The students will then place the appropriate coloured dots on a grid you have prepared. You can then project or draw the inequations that represent the constraints to illustrate the feasible region.

- After students have completed either the toy car or Lego activity you can use a manufacturing situation such as the following example.

  A potter is making cups and plates. It takes her 6 minutes to make a cup and 3 minutes to make a plate. Each cup uses $\frac{3}{4}$ *lb. of clay and each plate uses 1 lb. of clay. The potter has an order to fill for 10 cups. She has 20 hours available for making the cups and plates and has 250 lb. of clay on hand. She makes a profit of $2 on each cup and $1.50 on each plate. How many cups and how many plates should she make in order to maximize her profit?*

  Students would begin by clearly defining variables. Let *c* represent the number of cups the potter makes from the clay she has. Let *p* represent the number of plates the potter makes from the clay she has.

  Once the students have clearly defined variables, give them the constraints $c \geq 10$ and $p \geq 0$ and ask them what these represent in the context of the question.

  Give the students one of the additional constraints, such as $\frac{3}{4}c + 1p \leq 250$ and ask them to explain what each of the terms represents. We would expect them to say that
  - $\frac{3}{4}c$ represents that amount, in pounds, of clay per cup multiplied by the number of cups and, therefore, is the amount of clay used for the production of the cups
  - $1p$ represents that amount, in pounds, of clay per plate multiplied by the number of plates and, therefore, is the amount of clay used for the production of the plates
  - 250 represents the total amount of pounds of clay available

  Now ask students to explain why either $\frac{1}{10}c + \frac{1}{20}p \leq 20$ or $6c + 3p \leq 1200$ could be used for a constraint in this situation. We would expect them to say that $\frac{1}{10}c + \frac{1}{20}p \leq 20$ is a time constraint where time is in hours and $6c + 3p \leq 1200$ is the same time constraint where time is in minutes. It is important that they are able to clearly explain what each term represents in any constraint in-equation.

- Some students will find the use of a chart, as in the following example, will assist them in organizing their information for a linear programming problem.

  Example:
  Carmella spins yarn and then weaves it to produce handmade wall hangings and afghans. A wall hanging requires one hour of spinning and one hour of weaving; an afghan, two hours of spinning and four hours of weaving. Over several days, Carmella spends eight hours spinning and 14 hours weaving. Express this situation as a system of inequalities.

- It is important to ensure that students first clearly define the variables that they will be using. Let *w* represent the number of wall hangings that are created. Let *a* represent the number of afghans that are created. Once students have clearly defined the variables, they need to list any implied restrictions, such as the fact that it is impossible to create a negative number of items. We know that $w \geq 0$ and $a \geq 0$.
- Creating a chart such as the one below, simplifies obtaining the other in-equations.

	Spinning	**Weaving**
Wall hanging	1 hour	1 hour
Afghan	2 hours	4 hours
Total number of hours	8 hours	14 hours

Constraints

$w \geq 0$ $a \geq 0$ $1w + 2a \leq 8$ $1w + 4a \leq 14$

- Take time to ensure that students:

  clearly define variables and explain the meaning of what each term in the various constraints represents (For example, for the inequality $1w + 4a \leq 14$ from the question above, students should be to state that 1*w* represents the fact that a wall hanging takes one hour of weaving; 4*a* represents the fact that one afghan takes four hours of weaving and the 14 represents the total maximum number of hours weaving.)

- If you don't include examples where you are looking for a minimum value rather students may conclude that all linear programming questions deal with maximum values. Make sure both types are looked at.

For example:

The school cafeteria carries hot dogs and hamburgers. They cannot make more than 400 hot dogs in a single day. The cafeteria must make at least half as many hamburgers as hot dogs, but cannot make more than one-fourth the number of hot dogs plus 200. In addition, the number of hot dogs plus twice the number of hamburgers must be at least 400. Hot dogs cost 10 cents to make, and the cafeteria sells them for 20 cents each. Hamburgers cost 60 cents to make, and the cafeteria sells them for 50 cents each.

Problem (Maximizing Revenue) : How many of each item should the cafeteria make in order to have the highest revenue? What is the maximum amount of revenue the cafeteria can make?

[ANSWER: 400 hot dogs, 300 hamburgers. Revenue = $230]

Problem (Minimizing Cost) : How many of each item should the cafeteria make in order to minimize the cost? What is the minimum amount of money the cafeteria must spend?
[ANSWER: 200 hot dogs, 100 hamburgers. Cost = $80]

Problem (Maximizing Profit) : How many of each item should the cafeteria make in order to maximize the profit? What is the maximum profit?
[ANSWER: 400 hot dogs, 200 hamburgers. Profit = $20]

Suggested Models and Manipulatives

- chart paper
- Lego pieces
- multi-link cubes
- sticky dots

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- constraints
- feasible region
- inequality
- objective function
- profit equation
- verify
- vertex

Resources/Notes

Internet

- For students with netbooks, tablets, or access to the computer lab, use Internet sites such as the following for practice.
    - *Algebra 2 Online!,* "Module Solving Systems of Linear Equations and Inequalities" (Henrico County Public Schools 2006)
      http://teachers.henrico.k12.va.us/math/hcpsalgebra2/3-4.htm

      A PowerPoint lesson on how to solve systems of inequalities.
    - *Algebra-class.com*, "Systems of Inequalities Word Problems" (Karin Hutchinson 2013)
      www.algebra-class.com/systems-of-inequalities.html
      A good example of an inequality question with the solution worked out. At the bottom of the page there is a link to systems of inequalities practice problems.
- *Linear Inequalities Game* (Quia 2014)
  www.quia.com/rr/79715.html?AP_rand=571246363

- A possible review activity is the matching activity such as that found at: https://www.tes.com/teaching-resource/linear-programming-matching-activity-6147661
- Lego Linear Programming activity http://mchscc3.weebly.com/uploads/3/8/0/0/38001305/4.1_lego_1.pdf

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada) Sections 6.1–6.6, pp. 294–346

SCO RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. [CN, PS, T, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(It is intended that completion of the square not be required.)

RF02.01 Determine, with or without technology, the intercepts of the graph of a quadratic function.
RF02.02 Determine, by factoring, the roots of a quadratic equation, and verify by substitution.
RF02.03 Determine, using the quadratic formula, the roots of a quadratic equation.
RF02.04 Explain the relationships among the roots of an equation, the zeros of the corresponding function, and the x-intercepts of the graph of the function.
RF02.05 Explain, using examples, why the graph of a quadratic function may have zero, one, or two *x*-intercepts.
RF02.06 Express a quadratic equation in factored form, using the zeros of a corresponding function or the *x*-intercepts of its graph.
RF02.07 Determine, with or without technology, the coordinates of the vertex of the graph of a quadratic function.
RF02.08 Determine the equation of the axis of symmetry of the graph of a quadratic function, given *x*-intercepts of the graph.
RF02.09 Determine the coordinates of the vertex of the graph of a quadratic function, given the equation of the function and the axis of symmetry, and determine if the *y*-coordinate of the vertex is a maximum or a minimum.
RF02.10 Determine the domain and range of a quadratic function.
RF02.11 Sketch the graph of a quadratic function.
RF02.12 Solve a contextual problem that involves the characteristics of a quadratic function.

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
AN05 Students will be expected to demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially, and symbolically. **RF01** Students will be expected to interpret and explain the relationships among data, graphs, and situations. **RF02** Students will be expected to demonstrate an understanding of relations and functions. **RF04** Students will be expected to describe and represent linear relations, using words, ordered pairs, tables of values, graphs, and equations. **RF05** Students will be expected to determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. **RF06** Students will be expected to relate linear relations to their graphs, expressed in • slope-intercept form ($y = mx + b$) • general form ($Ax + By + C = 0$) • slope-point form $(y - y_1) - m(x - x_1)$ **RF09** Students will be expected to represent a linear function, using function notation.	**RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry.	**RF01** Students will be expected to represent data, using polynomial functions (of degree ≤ 3), to solve problems.

Background

In Extended Mathematics 11, students use technology to explore concepts, make and test inferences and reinforce learning. Additional time is also taken, prior to a topic being taught, to ensure that students have a solid understanding of prior knowledge necessary for the topic.

Constructivism is basically a **theory** -- based on observation and scientific study -- about how people **learn**. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. The additional time allocated to this course for RF02 is intended to allow students time to explore concepts through activities and labs, to complete projects that build connections, to make career connections, and to utilize technology as a tool for exploration and reinforcement.

It is important to note that students in Extended Mathematics 11 will **not** be expected to “complete the square” to change the form of a quadratic function.

In this outcome, students will be introduced to quadratic functions. They will examine quadratic functions expressed in the following three forms.

Standard Form: $f(x) = ax^2 + bx + c$
Factored Form: $f(x) = a(x - r)(x - s)$
Vertex Form: $f(x) = a(x - h)^2 + k$

Students will explore, using technology, the general impact that changes in **parameters** have on the graphs, and will sketch graphs using characteristics such as *x*- and *y*-intercepts, vertex (as a maximum or minimum point), axis of symmetry, and domain and range. Terminology such as “a vertical stretch of 2” is *not* an expectation. In this course students will talk about the shape of the quadratic in more general terms.

Before students are exposed to the standard form of a quadratic, they need to become familiar with the shape of a quadratic function and how one identifies a quadratic function. The terms **quadratic** and **parabola** are new to students. This will be their first exposure to functions that are non-linear. Students should have an opportunity to investigate what makes a function quadratic.

In Mathematics 9, students used a table of values to graph linear relations. They will now extend this strategy to quadratic equations to determine the vertex and its connection to the axis of symmetry. When graphing the quadratic function $y = x^2 + 4x + 3$, for example, students can create the following table of values.

x	–4	–3	–2	–1	0	1	2	3
y	3	0	–1	0	3	8	15	24

From the graph that students create, they can identify the characteristics of the quadratic function—the coordinates of the vertex, direction of opening, and the *x*- and *y*-intercepts.

Students will be required to determine the domain and range of the function. The concept of domain and range was introduced in Mathematics 10 using both set notation and interval notation (RF05). Students should recognize that all non-contextual quadratic functions have a domain of $\{x \mid x \in R\}$, whereas the range depends on the vertex and the direction of the opening.

Once students understand what type of equation is a quadratic they will begin to explore parametric changes to quadratic functions written in standard form, $y = ax^2 + bx + c$.

Students will generate graphs of quadratic functions using a table of values or with technology; they will determine the equation of its axis of symmetry. Students will observe that the axis of symmetry passes through the vertex and that this x-value is the x-coordinate of the vertex. Students will also observe that the y-coordinate of the vertex is the function's maximum or minimum value.

Students work with two basic ideas that both involve symmetry in a variety of settings.

- The first idea is that given two points with the same y-coordinate on a parabola, the equation of the axis of symmetry (and then the vertex) can be located by averaging the x-coordinates of the points.
- The second is that a table of values and/or a graph can be used to determine the parabola's axis of symmetry.

Students will also learn a technique for determining the vertex of a quadratic called **partial factoring**. The technique of removing a partial factor will make sense for students who understand that points that have the same y-value must be equidistant from the axis of symmetry. The **zero product property** will have to be introduced, to ensure that understanding is emphasized for this approach.

For example,
$f(x) = 2x^2 - 6x + 7$
$f(x) = 2x(x - 3) + 7$

When $2x(x - 3) = 0$, we know that $f(x) = 7$, and we can find two points on the quadratic that both have the same y value.
$2x = 0$ and $x - 3 = 0$
$x = 0$ and $x = 3$

The points (0, 7) and (3, 7) are on the parabola and the line of symmetry is $x = \frac{0+3}{2} = 1.5$.

Therefore the vertex is found where $x = 1.5$ and the vertex is $(1.5, f(1.5))$ or (1.5, 2.5).

Students should discover, *by exploration*, that the value of $x = \frac{-b}{2a}$ is the x-coordinate of the vertex, and the connection should be made to the equation of the axis of symmetry. The y-coordinate of the vertex can be found by substitution of the x-coordinate into the quadratic function.

Students should observe the following about a quadratic in standard form, $y = ax^2 + bx + c$.

- The value of the coefficient of the quadratic term will indicate whether the quadratic is opening up ($a > 0$) or opening down ($a < 0$)
- The value of the constant will be the y-intercept.
- The x-coordinate of the vertex can be determined using partial factoring.

- The values of a and c may allow the student to determine how many x-intercepts the quadratic will have. For example, if $a > 0$ and $c < 0$, then the y-intercept is negative and the quadratic opens up and there must be two x-intercepts.

After students have worked with the standard form of quadratic functions, they will work with quadratic functions in factored form. They should discover, by exploration, the connection between the factored form of a quadratic function and the x-intercepts of its graph. Thus the function $y = 6x^2 + x - 2$ would be factored as $y = (2x - 1)(3x + 2)$ and would have x-intercepts of $(\frac{1}{2},0)$ and $(-\frac{2}{3},0)$ when graphed. While its factored form of a quadratic is given as $y = a(x - r)(x - s)$, it is **not** expected that students would have to rearrange the equation to express it in this form as follows.

$y = 6x^2 + x - 2$

$y = (2x - 1)(3x + 2)$

$y = (2)(x - \frac{1}{2})(3)(x + \frac{2}{3})$

$y = 6(x - \frac{1}{2})(x + \frac{2}{3})$

Writing the quadratic as the product of two linear factors would be sufficient for students to determine the x-intercepts, vertex, and the axis of symmetry.

In Mathematics 10, when students were introduced to common factors and trinomial factoring, they modelled the factoring concretely and pictorially and recorded the process symbolically. Some students will continue to benefit from concrete representations. Manipulatives, such as algebra tiles, should therefore be available for use.

Students will be required to use factoring methods developed in Mathematics 10 to determine the zeros. Using games and puzzles to reinforce factoring skills is recommended.

It is important for students to distinguish between the terms **zeros**, **roots**, and ***x*-intercepts**, and to use the correct term in a given situation. The **zero product property** states that if the product of two real numbers is zero, then one or both of the numbers must be zero. This will be new for students. It is important for them to make the connection that the values of x at the zeros of the function are also the x-intercepts of the graph.

Students will relate the factored form of a quadratic function to the properties of its corresponding parabola (intercepts, axis of symmetry, vertex, and direction of opening). They will use these features to sketch the corresponding parabola and determine the function's domain and range.

Students should observe the following about a quadratic in factored form, $y = a(x - r)(x - s)$.

- The value of the coefficient of the quadratic term will indicate whether the quadratic is opening up ($a > 0$) or opening down ($a < 0$)
- The x-intercepts will be $(r, 0)$ and $(s, 0)$.

- The x-coordinate of the vertex and the axis of symmetry will be midway between the values of r and s.
- The y-intercept can be found by letting $x = 0$ and determining $y = a(-r)(-s) = ars$.

Once students are familiar with both the standard and factored forms of quadratics functions they will investigate the vertex form.

Students will graph a quadratic function in vertex form $y = a(x - h)^2 + k$, and will relate the characteristics of the graph to its function. Students were exposed to idea of a parameter earlier in this unit. They will now explore the parameters h and k and how they relate to the graph of the quadratic. This exploration is most efficiently done using graphing technology.

Students should observe the following about a quadratic in factored form, $y = a(x - h)^2 + k$.

- The value of the coefficient of the quadratic term will indicate whether the quadratic is opening up ($a > 0$) or opening down ($a < 0$)
- The vertex is the point (h, k).
- The maximum or minimum value for the quadratic is $y = k$.
- The axis of symmetry is $x = h$.
- The values of a together with the vertex will allow the student to determine how many x-intercepts the quadratic will have. For example, if $a > 0$ and the vertex is (2, 6), then the vertex is above the x-axis and the quadratic opens up and there are no x-intercepts.

Given a quadratic function in standard, vertex, or factored form, students will be able to sketch a graph of the function and then use that graph to solve contextual problems in which the equation of the quadratic is given.

Specifically,

- if the equation is in standard form, students will be able to get a rough sketch of its graph by plotting the y-intercept and using the direction of opening
  - If the context requires the vertex of the parabola, students can use the method of partial factoring or the formula $x = \frac{-b}{2a}$ to find its x-coordinate and then substitute into the equation to determine the y-coordinate of the vertex.
  - If the context requires the x-intercepts, the student will factor the quadratic to obtain its factored form.
- if the equation is in factored form, students will be able to get a rough sketch of its graph by plotting the x-intercepts and the direction of opening
  - If the context requires the y-intercept, students will let $x = 0$ and solve.

- If the context requires the vertex of the parabola, students will use the symmetry of the quadratic to obtain the *x*-coordinate and then substitute into the equation to determine the *y*-coordinate of the vertex.

- if the equation is in vertex form, students will be able to get a rough sketch of its graph by plotting the vertex and the direction of opening
    - If the context requires the *y*-intercept, students will let $x = 0$ and solve.
    - If the context requires the *x*-intercepts, students will let $y = 0$ and solve.

Students are also expected to work from a graph or data set with vertex or *x*-intercepts and one additional point given to obtain the equation of the quadratic function.

This is a process in which students work backwards to obtain the equation of the quadratic function in its factored form or to obtain the equation of a quadratic function in its vertex form.

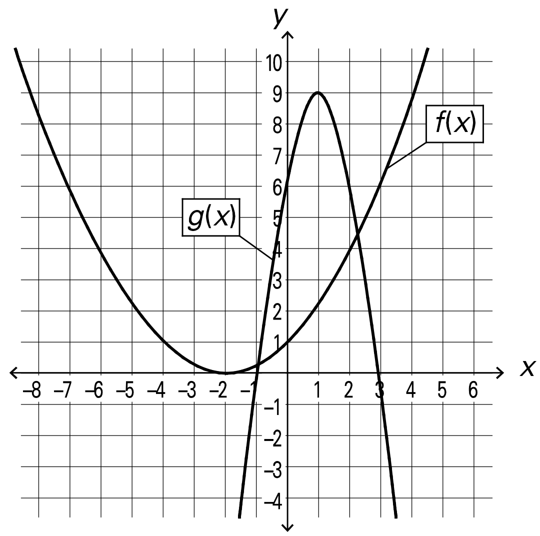
Examples:
From a data set where you can see the x-intercepts.

<table>
<tr><td>x</td><td>y</td><td rowspan="7">Since this has two x-intercepts at (2, 0) and (3, 0), you know that the equation is $y = (a)(x - 2)(x - 3)$; substitution of one of the other points yields the following:
(0, 6): $6 = (a)(0 - 2)(0 - 3)$
$6 = a(-2)(-3)$
$6 = 6a$
$1 = a$
$\therefore\ y = 1(x - 2)(x - 3)$</td></tr>
<tr><td>0</td><td>6</td></tr>
<tr><td>1</td><td>2</td></tr>
<tr><td>2</td><td>0</td></tr>
<tr><td>3</td><td>0</td></tr>
<tr><td>4</td><td>2</td></tr>
<tr><td>5</td><td>6</td></tr>
</table>

From a data set where you can see the vertex:

<table>
<tr><td>x</td><td>y</td><td rowspan="6">Since the vertex of (1, 4) can be read from this data set, we know that the equation is $y = a(x - 1)^2 + 4$; substitution of one of the other points yields the following:
(0, 6): $6 = (a)(0 - 1)^2 + 4$
$6 = a(-1)^2 + 4$
$6 = 1a + 4$
$2 = a$
$\therefore\ y = 2(x - 1)^2 + 4$</td></tr>
<tr><td>0</td><td>6</td></tr>
<tr><td>1</td><td>4</td></tr>
<tr><td>2</td><td>6</td></tr>
<tr><td>3</td><td>12</td></tr>
<tr><td>4</td><td>22</td></tr>
</table>

From a graph where both the *x*-intercepts and the vertex can be observed:



Factored Form	Vertex Form
$f(x) = k(x+2)^2$ $f(0) = 1$ $1 = k(0+2)^2$ $1 = k(4)$ $\frac{1}{4} = k$ $f(x) = \frac{1}{4}(x+2)^2$	$f(x) = k(x+2)^2 + 0$ $f(0) = 1$ $1 = k(0+2)^2 + 0$ $1 = k(2)^2 + 0$ $1 = 4k$ $\frac{1}{4} = k$ $f(x) = \frac{1}{4}(x+2)^2$
$g(x) = k(x+1)(x-3)$ $f(2) = 6$ $6 = k(2+1)(2-3)$ $6 = k(3)(-1)$ $6 = -3k$ $-2 = k$ $g(x) = -2(x+1)(x-3)$	$g(x) = k(x-1)^2 + 8$ $f(2) = 6$ $6 = k(2-1)^2 + 8$ $6 = k(1)^2 + 8$ $-2 = 1k$ $-2 = k$ $g(x) = -2(x-1)^2 + 8$

From a context:

An arrow is shot from a height of 1 m, it follows a parabolic path, and reaches a maximum height of 5 m two seconds after it is shot.	The vertex is (2, 5) and the initial height yields the point (0, 1). Therefore the equation in vertex form is $y = a(x-2)^2 + 5$; substituting in the point (0, 1) yields the following: (0, 1): $1 = (a)(0-2)^2 + 5$ $1 = a(-2)^2 + 5$ $1 = 4a + 5$ $-4 = 4a$ $-1 = a$ $\therefore\ y = -1(x-2)^2 + 5$

It is expected that students will be encouraged to work with and compare the various methods, decide when each of the three methods is most convenient to use, and explain why.

Students need to understand the characteristics of each quadratic form in addition to the limitations of each. Students should be provided with examples in which they have to select which form of the quadratic function best suits the information provided. For example, a key piece of information of the vertex form is knowing the vertex, and the benefit of the factored form is knowing the *x*-intercepts. Even though students initially start with the quadratic in vertex or factored form, they can always use their distributive property to rewrite the equation in standard form.

Using the form of the quadratic function which would most appropriately apply to the situation, students will solve contextual problems where

- the equation is given
- the vertex is evident and one other point is available
- the *x*-intercepts and one other point are given

Students will **not** be expected to generate a quadratic from data or a situation where the vertex or *x*-intercepts are not evident.

For example, although questions such as the following may be found in the selected student resource, in this course students will **not** be expected to solve questions such as the following:

- Ataneq takes tourists on dogsled rides. He needs to build a kennel to separate some of his dogs from the other dogs in his team. He has budgeted for 40 m of fence. He plans to place the kennel against part of this home, to save on materials. What dimensions should Ataneq use to maximize the area of the kennel?
- A career and technology class at a high school operates a small t-shirt business. Over the past few years, the shop has had monthly sales of 300 T-shirts at a price of $15 per T-shirt. The students have learned that for every $2 increase in price, they will sell 20 fewer T-shirts each month. What should they charge for their T-shirts to maximize revenue?

Once students have become familiar with working with the three forms of the quadratic function, they will be expected to solve a quadratic equation.

- It is expected that students will solve quadratic equations by factoring and will verify their solutions using substitution. Students will use their knowledge of common factoring, trinomial factoring, and difference-of-squares factoring from Mathematics 10, to solve quadratic equations algebraically.
- Students will also solve quadratic equations by using the quadratic formula. (The quadratic formula is presented to the students without derivation.) They will express their answers as either decimal approximations or exact values as the contexts warrant. In Mathematics 10, students have worked with radicals, and they have simplified entire radicals such as $\sqrt{98}$ to obtain $7\sqrt{2}$. While the term **discriminant** is not introduced in Mathematics 11, a discussion about how the radicand in the quadratic formula can be used to determine if a quadratic is factorable or not is expected. It is also expected that students can use the radicand to determine the number of *x*-intercepts the quadratic will have.
- If the equation is in factored form, students are most likely going to change it to standard form and then use the quadratic formula to solve the equation.
- If the equation is in vertex form and students are able to efficiently solve it without changing forms, they should solve it using that form.

Students will also examine the graphical methods for solving quadratic equations (with and without technology) and discuss the advantages and disadvantages of a graphical approach. They will make connections between the roots of the equation, the zeros of the corresponding function, and the *x*-intercepts of the corresponding parabola. Students will solve contextual problems that require the use of quadratic equations.

Students will also be expected to consider if the solution(s) are admissible or not. An **inadmissible** solution is a root of a quadratic equation that does not lead to a solution that satisfies the original problem. These will be tied to the context of the question. For example, if you determine that an arrow hits the ground at $t = -1$ sec. and $t = 3$ sec., where t was defined as the time since the arrow was shot, then the $t = -1$ is not admissible, since it could not hit the ground before the arrow was shot.

Students will **not** be expected to generate a quadratic from data or a situation where the vertex or *x*-intercepts are not evident. In these situations students, in this course, would be provided with the related equation before being expected to solve the quadratic equation that represents the situation. In Mathematics 12 students will be using Quadratic Regression to obtain the equation from a set of data points.

For example, although questions such as the following may be found in the selected resource, in this course students will not be expected to solve questions such as the following unless they were given the equation as well as the question.

- At noon, a sailboat leaves a harbour and travels due west at 10 kmh. Three hours later, another sailboat leaves the same harbour and travels due south at 15 kmh. At what time, *t*, to the nearest minute, will the sailboats be 40 km apart? [Note: Students could be given this problem to solve IF they were provided with the equation $(10x)^2 + (15(x-2))^2 = 40^2$ and x was defined as the number of hours since noon.]

Students are expected to demonstrate their understanding of this outcome through the completion of a project.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

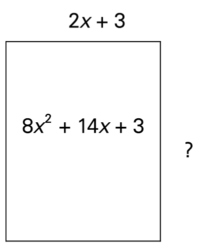
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Given $2x^2 + 6x - 40$, find and correct the mistake(s) in the factoring below.
  $2x^2 + 6x - 40$
  $= 2(x^2 + 3x - 40)$
  $= 2(x - 8)(x + 5)$

- Complete the following table of values and plot the points.

x	-2	-1	0	1	2	3
y = -2x + 1						

- The area of a rectangle is represented by the product $8x^2 + 14x + 3$ square units. If one dimension of the rectangle is $(2x + 3)$ units, determine the other dimension. How did knowing one of the factors help you determine the other factor?



Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Complete the following table of value and plot the points.

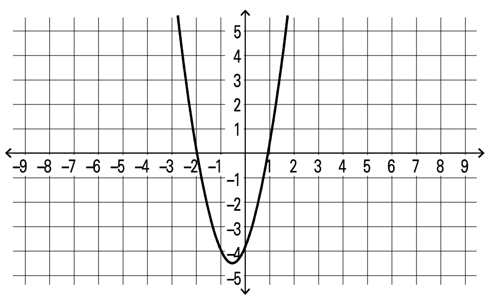
x	-2	-1	0	1	2	3
Y = x^2 - 2						

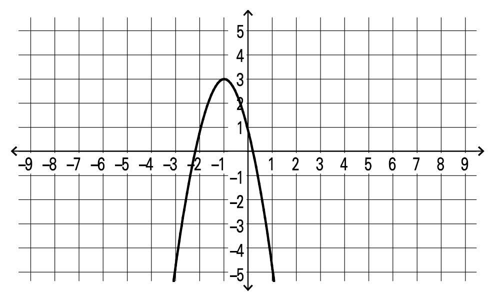
- Complete the chart:

Polynomial Function	**Classification**	**True or False**	**Explain/Justify**
$y = 5(x + 3)$	Linear		
$y = 5(x^2 + 3)$	Quadratic		
$y = 5^2(x + 3)$	Quadratic		
$y = 5x(x + 3)$	Linear		
$y = (5x + 1)(x + 3)$	Quadratic		
$y = 5(x + 3)^2 + 2$	Quadratic		

- The daily profit, P, (dollars) for Swift Shot, a company that makes tennis racquets, is given by $P = -n^2 + 240n - 5400$ where n is the number of racquets produced per day.
  - How many tennis racquets must be produced per day to have a maximum profit?
  - What is the maximum profit?
  - What profit is made when 75 racquets per day are produced?
  - How many tennis racquets must be produced to break even?
- Use algebra to write the quadratic function, $y = 5(x - 1)^2 - 8$, in the form $y = ax^2 + bx + c$.

  Show the work that leads to your solution.
- Sketch each of the following functions, labelling the values you use to draw each of them.
  $f(x) = -3x^2 + 24x - 50$
  $g(x) = 2(x - 3)(x + 1)$
  $p(x) = 4(x - 5)^2 - 2$
- State the equation that would describe each of the graphs shown below.

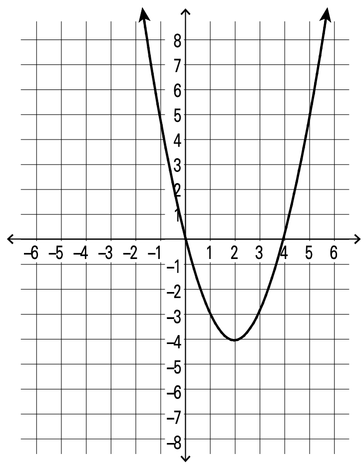




- Create a quadratic equation that has roots $x = -2$ and $x = 3$. Is it possible to have more than one quadratic that has these roots? Explain.
- What is the minimum value of $y = x^2 - 8x - 9$?
- Create a quadratic equation that has roots of 5 and 7 and a maximum value of 4. Is it possible to have more than one quadratic that satisfies these conditions? Explain.
- For the graph of $y = x^2 - 2x - 35$,
  (a) determine the coordinates of the x-intercepts
  (b) find the vertex
  (c) state the coordinates of the y-intercept
  (d) sketch a graph (Label completely clearly indicating all intercepts, the vertex, axis of symmetry, and scales of the axes.)
  (e) state the domain and the range of this function
- Explain how you could determine whether the y-coordinate of the vertex of a quadratic function is a maximum or a minimum without graphing.
- Lina was asked to describe how the graph of $y = 2x^2$ compares to the graph of $y = x^2$. She said that the graph of $y = 2x^2$ will be narrower than the graph of $y = x^2$. Is Lina is correct? Explain your reasoning.

- Explain how changing the parameter *a* affects the graph of the function form $y = ax^2 + bx + c$.
- State the vertex and the equation of the line of symmetry for each of the following:

(a)



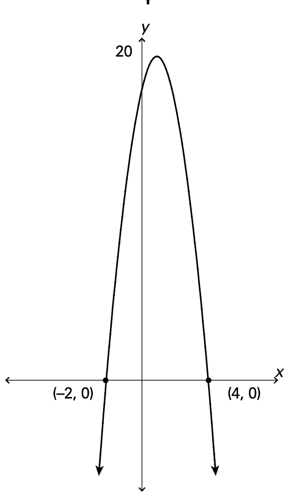
(b)

x	–1	0	1	2	3
y	10	1	–2	1	10

(c) $y = 2x^2 - 8x + 7$

(d) $y = 3(x - 1)(x + 5)$

- Given the graph of $y = -2x^2 + 8x - 3$, state
  - (a) the coordinates of the *y*-intercept
  - (b) the coordinates of the *x*-intercept(s)
  - (c) the direction of opening
  - (d) the vertex
  - (e) the equation of the axis of symmetry
- Find the equation of the axis of symmetry for the parabola shown in the graph below.



- How can you determine the coordinates of the vertex given the factored form of a quadratic function? Illustrate your answer using an example.
- How can you obtain the coordinates of the vertex given the standard form of a quadratic function? Illustrate your answer using an example.
- Determine the quadratic function with factors $(x + 3)$ and $(x – 5)$ and a y-intercept of –5.
- What are the minimum requirements to sketch a unique quadratic graph? Explain.
- If the factors of a quadratic function are identical, what does this information tell you about the equation and the graph?
- Based on the following information, which form of the quadratic equation would you prefer to write? *(Standard form, factored form or vertex form)* Why?
  (a) The vertex of the parabola is (5, –2) and passes through the point (–1, –4).
  (b) The factors are $(x – 1)$ and $(x + 3)$ and passes through the point (–2, 3).
  (c) The quadratic has a y-intercept of 10 and passes through the points (1, 12) and (2, 10).
- Find the equations represented by the following situations:
  (a) The vertex of the parabola is (5, –2) and passes through the point (–1, –4).
  (b) The factors are $(x – 1)$ and $(x + 3)$ and passes through the point (–2, 3).
  (c) The quadratic has a y-intercept of 10 and passes through the points (1, 12) and (2, 10).
  (d) A quarterback throws the ball from an initial height of six feet. It is caught by the receiver 50 feet away and at a height of six feet. The ball reaches a maximum height of 20 feet during its flight. Determine the quadratic function that models this situation, and state the domain and range.
- Is $x = –2$ a root of $x^2 + 5x + 6 = 0$?
- If $x = –3$ is one root of the equation $3x^2 + mx + 3 = 0$, what is the value of m, and what is the other root?
- The path of a model rocket can be described by the quadratic function $y = x^2 – 12x$, where y represents the height of the rocket, in metres, at time x seconds after takeoff. Identify the maximum height reached by the rocket, and determine the time at which the rocket reached its maximum height.
- What do you know about the function $y = x^2 + 6x + c$, if $c = 0$?
- For what values of c will the function $y = x^2 + 6x + c$ have
  (a) two x-intercepts
  (b) one x-intercept
  (c) no x-intercepts
- For what values of c is the function $y = x^2 + 6x + c$ factorable?
- Demonstrate, using alge-tiles or an area model, how you can determine the value(s) of b in the function $y = x^2 + bx + 12$ that create a factorable quadratic.

- Demonstrate, using alge-tiles or an area model, how you can determine the value(s) of b in the function $y = 2x^2 + bx + 6$ that create a factorable quadratic.

- The quadratic formula was used by a student to solve the equation $x^2 + x - 12 = 0$ as shown below. Identify and correct the error in the following:

  Step 1 $x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-12)}}{2(1)}$

  Step 2 $x = \dfrac{-1 \pm \sqrt{49}}{2}$

  Step 3 $x = \dfrac{-1 \pm 7}{2}$

  Step 4 $x = -1 \pm \dfrac{7}{2}$

- Solve the following quadratic equations using a variety methods:
  (a) $5x^2 + 5x + 5 = 35$
  (b) $2x^2 + 7x + 5 = 9$
  (c) $1 = x^2 - 6x$
  (d) $2(x - 3)^2 = 18$
  (e) $4(x - 2)(x + 1) = 0$
  (f) $(x + 2)(x - 1) = 4$

- The daily revenue of R dollars for a ski resort can be modelled by the equation $R = -16T^2 - 480T + 6400$, where T represents the temperature in degrees Celsius.
  (a) At what temperature does the daily revenue reach its maximum value? What is the maximum revenue?
  (b) What is the revenue when the temperature is –10°C?
  (c) What do you think the temperature was when the ski resort made $8704?

- A parabolic archway is 10 m wide. If it is 2 m high 1 m from where it touches the ground, what is the maximum height of the archway?

- A football is thrown so that the height of the ball above the ground, y, can be modelled by the function $y = -4.9x^2 + 24x + 1$, where x is the time in seconds after the ball was thrown.
  (a) State the range of this function to the nearest tenth of a metre, remembering the context of the question.
  (b) Assuming that the football is caught at a height of 1.5 m, how far was the ball thrown?
  (c) Sketch a graph of this situation and use the graph to check your answer to part (b).

- An arrow is shot into the air and its path given by the equation $h = -4.9(t - 1)^2 + 6$ where h is the height of the arrow above the ground, in metres, and t is the time since the arrow was shot, in seconds.
  (a) From what height was the arrow shot?
  (b) When does the arrow hit the ground?

- As part of an air show, an airplane is diving in a parabolic path. Its height above the ground, as recorded over a period of time, is shown below. What was the height of the plane at $t = 0$ seconds.

Time (sec)	1	5	9	13	17
Height above the ground (m)	685	205	45	205	685

(a) How low to the ground did the plane get in this dive?
(b) What was the height of the plane at $t = 0$ seconds?
(c) When does the plain reach a height of 100 m?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Before students are exposed to the standard form of a quadratic, they need to become familiar with the shape of a quadratic function and how to identify a quadratic function. The terms **quadratic** and **parabola** are new to students. As previously discussed, this will be their first exposure to functions that are non-linear. Technology is an effective tool for students to use to explore the graphs of quadratics and how changing the value of a parameter impacts the graph. Allowing students to explore and draw their own conclusions before summarizing is suggested. The use of sliders in a program such as DESMOS can facilitate this type of discovery.
- Students should be introduced to the quadratic term, linear term, and constant of a quadratic equation. They should distinguish between the quadratic term and the coefficient of the quadratic term. For example, students often mistake the quadratic term as the value of a instead of ax^2.
- Students should have an opportunity to investigate what makes a quadratic function. Have them multiply two linear equations or square a binomial of the form $ax + b$. Consider the examples $y = (x + 1)(x - 4)$ and $y = (3x - 2)^2$. Ask students what they notice in terms of the degree of the polynomial. (**Note:** You may need to define the degree of a polynomial function as the greatest exponent in a polynomial or equation.)
- Projectile motion can be used to explain the path of a baseball or a skier in flight. To help students visualize the motion of a projectile, toss a ball to a student. Ask students to describe the path of the ball to a partner and have them sketch the path of the height of the ball over time (the independent axis represents time and the dependent axis represents the height of the ball). Encourage them to share their graphs with other students. Ask students to think of other examples that might fit the diagrams of parabolas that open upward or downward.
- Characteristics of a parabola, such as vertex and axis of symmetry, should be discussed. It is important for students to recognize that, excepting the vertex, each point on a parabola has a corresponding point on its mirror image, which results in the parabolic shape. Discussions around everyday applications such as projectile motion give students an appreciation of the usefulness of quadratics.
- The characteristics of the quadratic function, $f(x) = ax^2 + bx + c$ should be developed through an investigation in which the parameters a, b, and c are manipulated individually. Technology like graphing calculators, or other suitable graphing software, should be used as students' graphing abilities for quadratics would be limited to the use of a table of values at this point.
  Note: Students are expected to notice, as they explore the effects of parameter changes, that a impacts direction and shape of the parabola, b changes the location of the line of symmetry and c gives us the y-intercept.

- Consider a task such as the following to examine the effects of manipulating the values of a, b, and c.

  (a) Students will first investigate the effect of changing the value of a by comparing quadratic functions of the form $y = ax^2$. As they compare the graphs of $y = 2x^2$ and $y = -2x^2$, $y=\frac{1}{2}x^2$ and $y=-\frac{1}{2}x^2$, use prompts such as the following to promote student discussion:

  (i) What happens to the direction of the opening of the quadratic if $a < 0$ and $a > 0$?

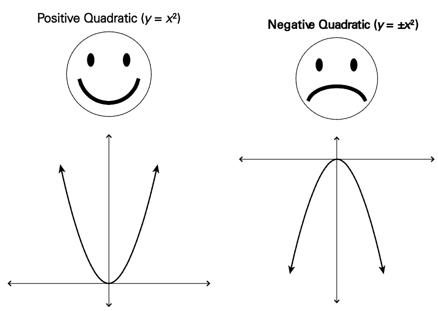
  (ii) If the quadratic opens upward, is the vertex a maximum or minimum point? Explain your reasoning. What if the quadratic opens downward?

  (iii) Is the shape of the parabola affected by the parameter a? Are some graphs wider or narrower compared to the graph of $y = x^2$?

  (iv) What happens to the x-intercepts as the value of a is manipulated?

  (v) What is the impact on the graph if $a = 0$?

  The idea of the vertex being a maximum or minimum point is determined by the value of a. One way to help students remember how to determine direction of opening is shown here.



  (b) Students should then proceed to examine the effects of manipulating the value of b and c in a similar manner. They should be given time to analyze the graphs of various quadratic functions, such as $y = x^2 + 2x$, $y = x^2 + 5x$, $y = x^2 + 1$, and $y = x^2 - 3$. Encourage them to look for connections between the changes in the graph and the function relative to $y = x^2$. Ask questions such as the following:

  (i) What is the effect of parameter b in $y = x^2 + bx$ on the graph of the quadratic? Is the parabola's line of symmetry changing?

  (ii) What is the effect of parameter c in $y = x^2 + c$ on the graph of the quadratic? How can you identify the y-intercept from the equation in standard form? Is the line of symmetry affected by the parameter c?

  (c) Once students have explored the characteristics of the quadratic function, $f(x) = ax^2 + bx + c$ as they manipulated the parameters a, b, and c, students can more easily identify the characteristics of the quadratic function—the coordinates of the vertex, direction of opening, and the x- and y-intercepts, for example—when they have a visual representation. When graphing the quadratic function $y = x^2 + 4x + 3$, for example, students can create the following table of values.

x	–4	–3	–2	–1	0	1	2	3
y	3	0	–1	0	3	8	15	24

Ask students, by looking at this table, if they can recognize the point at which the parabola changes direction. The goal is for students to be able to recognize the symmetry, as the points on the parabola that share the same *y*-coordinate are equidistant from the vertex. Whether students use a graph or a table of values, teachers should promote discussion by asking questions such as the following:

(i) What connection is there between the axis of symmetry and the coordinates of the vertex?

(ii) What is the equation of the axis of symmetry?

- Present students with the following questions:

  (a) If the axis of symmetry is provided in addition to the standard form of the quadratic function, how would you determine the *y*-coordinate of the vertex. (Teachers could start the discussion by reminding students of their work with linear relations in Mathematics 10. When given the slope of the line and the coordinates of a point on the line, students determined the y-intercept through the use of substitution. This algebraic method can also work with quadratics by substituting the known *x*-value into the function to find the corresponding *y*-value.)

  (b) If you knew the two *x*-intercepts of the quadratic how could you determine the coordinate of the vertex? (The axis of symmetry can also be linked to the *x*-intercepts of the graph of a quadratic function. Provide students with several graphs of quadratic functions and ask them how the location of the *x*-intercepts and the axis of symmetry are connected. The focus is for students to recognize that, due to symmetry, the axis of symmetry is exactly half-way between the two *x*-intercepts.)

  Be careful as students sometimes incorrectly state the equation of the axis of symmetry as a numerical value rather than an equation. It is important for them to recognize that, when they are graphing quadratic functions, the line of reflection is a vertical line. For the quadratic function $y = x^2 – 6x + 13$, students may mistakenly identify the equation of the axis of symmetry as 3 rather than $x = 3$.

- The characteristics of the quadratic function $f(x) = a(x – r)(x – s)$ should be developed through an investigation in which the parameters *a*, *r*, and *s* are manipulated individually. Technology like graphing calculators, or other suitable graphing software, should be used. Students could work as partners changing the parameters and making conjectures about the impact of those changes.

Note: Students are expected to notice, as they explore the effects of parameter changes, that a impacts direction and shape of the parabola, r and s will be the values of the x-intercepts of the graph.

It would be important for you to discuss with students why (r, 0) and (s, 0) are the x-intercepts of the quadratic when it is written in its factored form.

- The method of partial factoring can be used to determine the equation of the line of symmetry for a non-factorable quadratic. The quadratic function $f(x) = 2x^2 - 4x + 7$ is not factorable, but if we consider just factoring the first two terms, $f(x) = 2x^2 - 4x + 7$ and $f(x) = 2x\,(x - 2) + 7$, we can see that $f(0) = 7$ and $f(2) = 7$, since both these points have the same y-coordinate the line of symmetry must be half way between $x = 0$ and $x = 2$. The equation of the axis of symmetry, therefore, is $x = 1$.

  When using this method, remind students that they are not determining x-intercepts here as they would if they were factoring the quadratic function. A graphical representation may help students visualize the information this method provides.

  It is important for students to recognize that given any two points with the same y-coordinates in a table of values, the equation of the axis of symmetry can be determined by averaging the x-coordinates of the points.

- Students should also be given an opportunity to analyze several tables of values and their corresponding graphs. Use the following questions to help students make a connection between the axis of symmetry and x-intercepts:
  (a) How are the x-intercepts determined using a table of values?
  (b) Can you identify the vertex from the table? Explain.
  (c) How could you determine the x-coordinate of the vertex just from the x-intercepts?
  (d) What happens if the x-intercepts are not evident in a table of values? What strategy can be used to find the axis of symmetry?

- The characteristics of the quadratic function, $f(x) = a(x - h)^2 + k$ should be developed through an investigation in which the parameters a, h, and k are manipulated individually. Technology like graphing calculators, or other suitable graphing software, should be used.

  Note: Students are expected to notice, as they explore the effects of parameter changes, that a impacts direction of opening and the shape of the parabola, h is the x-coordinate of the vertex and k is the y-coordinate of the vertex.

- Yohaku and Tarsia puzzles can be used to review factoring, domain and range.

- Have students sort prepared index cards (Desmos) that have various quadratic expressions based on which factoring method best applies.

- To review factoring the bingo activity from M10 guide p. 125 or Old Poly game can be used. Directions for Old Poly Game are found at http://www.beaconlearningcenter.com/lessons/lesson.aspp.?ID=1178

- *Matching graphs task:* Working in groups of two, give one student the graph of a quadratic function. Ask them to turn to their partner and describe, using the characteristics of the quadratic function, the graph they see. The other student will then draw the graph based on the description from the first student. Both students will then check to see if the graphs match.

- To assist the students in making the connection between the factored form of a quadratic function and the x-intercepts of its graph, ask students to graph the following functions using technology and to determine the x-intercepts:
  (a) $y = x^2 - 6x + 9$
  (b) $y = 2x^2 - 4x - 6$
  (c) $y = (x - 2)(x + 3)$
  (d) $y = (x + 1)(x - 4)$

  Lead a discussion focusing on the following questions:
  (a) Can you determine the x-intercepts by looking at a quadratic function? Explain.
  (b) Which form of the quadratic function did you find the easiest to use when determining the x-intercepts?
  (c) What is the connection between the factors and the x-intercepts.
  (d) What is the value of the y-coordinate at the point where the graph crosses the x-axis?
  (e) What would happen if the factors of the quadratic function were identical?
  (f) How can you find the x-intercepts of a quadratic function without the graph or a table of values?

- Encourage students to think about what the graph of the quadratic function should look like before actually sketching the graph. They can discuss features such as the direction of the opening, whether the vertex is a maximum or minimum point, what the value of the y-intercept is, and how many x-intercepts there should be. Consider, for example, the equation $y = -x^2 + 5x + 4$. Since the parabola opens downward (a value is negative), the vertex is a maximum point. The y-intercept (c value) is 4, which is above the x-axis. This results in two x-intercepts.

- Ask students what the important characteristics are to consider when sketching the graph of a quadratic function. Students may initially use a table of values to draw their sketch. Encourage them to use the other methods addressed throughout this outcome. To draw a reasonably accurate sketch, students should plot the vertex and at least two other points on the graph.

- Each form of a quadratic has its own characteristics, and its own benefits.
  - If the quadratic is written is standard form, students can determine the y-intercept and the direction of the opening of the parabola directly from the equation.

  - If the equation is written in factored form, students can determine the x-intercepts of the graph and the direction of the opening of the parabola.
  - If the equation is written in vertex form, students can determine the vertex and the direction of the opening.

- Students should be given a quadratic in standard form and asked to put it in factored form so that they could determine the x-intercepts of its graph without using technology. A common student error occurs when students factor a quadratic and mistakenly ignore the signs when determining the x-intercepts. For example, when students factor $y = x^2 + 2x - 8$ as $y = (x + 4)(x - 2)$, they often state the x-intercepts are 4 and –2. Reinforce to students that they are solving equations $x + 4 = 0$ and $x - 2 = 0$ to obtain the x-intercepts –4 and 2.

- Promote student discussion around the following task:
  - Sketch a parabola that passes through (–4, 0) and (3, 0).
  - Make three more parabolas that are different from the first one, but still have x-intercepts –4 and 3.
  - How many parabolas do you think are possible? Explain.

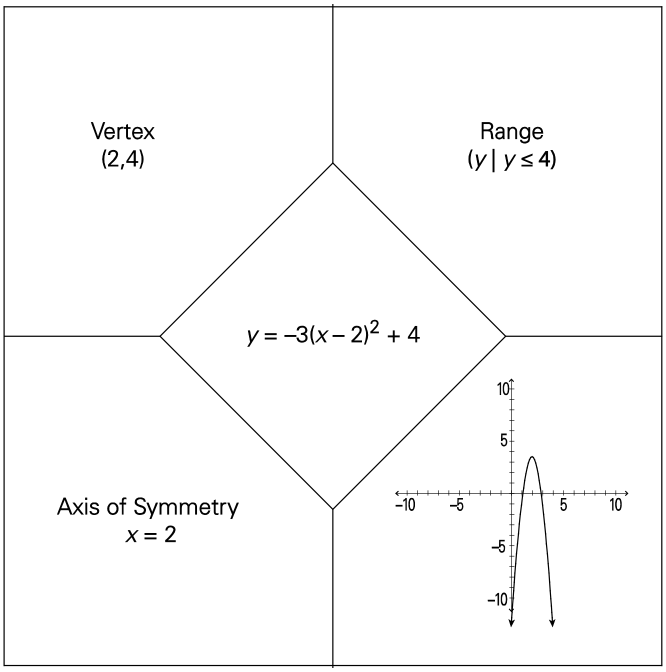
  The goal is for students to recognize that a family of parabolas are possible when the x-intercepts are given. When provided with an extra point, however, students can narrow down the exact formula for the quadratic equation. In order to determine the multiplier a in the factored form $y = a(x - r)(x - s)$, students will choose any point on the parabola and use substitution. They should compare their answers with the class to confirm that the points they chose resulted in the same quadratic function, and then have a discussion about whether some points are easier to work with than others. Using the distributive property, students may write the equation from factored form to standard form.

- *Find your partner:* Provide half the students with cards that show graphs labelled with the vertex or x-intercepts. The other half of the class will be given cards that have the corresponding quadratic functions in a variety of forms. Students move around the classroom trying to locate their matching card.

- *Frayer Model Puzzle:* This task will involve a group of three students. Provide students with a completed Frayer model cut into five puzzle pieces. To make this more challenging, keep some of the puzzle pieces blank. The titles would be Graph, Table of Values, Equation, Characteristics, and Domain and Range. Each student is randomly given one piece of the quadratic function. He or she should move about the classroom locating the corresponding pieces to put the puzzle back together and complete the Frayer model. Once the group is formed, they will have three pieces of the puzzle and as a group will have to complete the remaining two pieces. Teachers can first use this model with standard form and then apply it to the vertex and factored form of a quadratic.

- Students can work in pairs to complete the following quadratic puzzle, investigating the characteristics and graphs of various quadratic functions of the form $y = a(x - h)^2 + k$. They should work with 20 puzzle pieces (four complete puzzles consisting of a function and four related characteristics) to correctly match the characteristics with each function.

A sample is shown below.



- Students should be provided with examples where characteristics of quadratic functions are applied to conceptual problems. Consider the following example:
    - A ball is thrown from an initial height of 1 m and follows a parabolic path. After 2 seconds, the ball reaches a maximum height of 21 m. Using algebra, determine the quadratic function that models the path followed by the ball, and use it to determine the approximate height of the ball at 3 seconds.

  Encourage students to discuss the following:
    - How is the shape of the graph connected to the situation?
    - What do the coordinates of the vertex represent?
    - What do the *x*- and *y*-intercepts represent?
    - Why isn't the domain all real numbers in this situation?

- Students should be provided with a variety of contextual problems that place restrictions on the domain and range. To help them distinguish between situations where restrictions on the domain are necessary, students could be exposed to questions such as the following:
    - State the domain and range for the function $f(x) = -0.15x^2 + 6x$.
    - Now suppose the function $h(t) = -0.15t^2 + 6t$ represents the height of a ball, in metres, above the ground as a function of time, in seconds. What impact does this context have on domain and range?

- *Fishing for Quadratic Functions:* Fill two bags with intercepts (in coordinate form) written on pieces of paper that are shaped like fish.
    - **Version A**
      One bag would contain only x-intercepts, and the other bag would contain only y-intercepts. Ask students to fish out two x-intercepts and one y-intercept, and then determine the quadratic function that passes through the three points. For this version of the game, students would write the quadratic in its factored form. (**Note:** It will be necessary to have twice as many x-intercepts as y-intercepts.)
    - **Version B**
      One bag would contain only vertices and the other bag contain a point. For this version of the game, students would write the quadratic function in its vertex form.
- Ask students to play Parabola Math. Provide students with a deck of cards containing the graph of the parabola, the function of the parabola, the vertex of the parabola, and the equation of the axis of symmetry. Each characteristic could be on different colour paper. Students should work in groups to match the various components of the parabola.
- You can challenge students to use the method of partial factoring to show that $x = -\frac{-b}{2a}$ is the x-coordinate of the vertex for the general quadratic $y = ax^2 + bx + c$.
  After you have done this, students may use the formula $x = -\frac{-b}{2a}$ to find the x-coordinate of the vertex for quadratic functions of the form $y = ax^2 + bx + c$. This value can then be used to determine the equation of the axis of symmetry.
- Teachers should promote discussion when students are graphing quadratic functions by asking the following questions:
    - Why is the domain the set of all real numbers when only some points are plotted from the table of values?
    - How is the range related to the direction of the opening?
- As a review, students can play the following game. Each group should be given a pair of dice (or they can create their own).
    - For $y = ax^2 + bx + c$: On the first die, two of the sides will be labelled a, two sides will be labelled b, and two labelled c. On the second die, there will be various rational numbers. Students should roll the dice and state the effect on the graph of $y = x^2$ of changing the indicated parameter to the given number.
    - For $y = a(x - h)^2 + k$: On the first die, two of the sides will be labelled a, two sides will be labelled h, and two labelled k. On the second die, there will be various rational numbers. Students should roll the dice and state the effect on the graph of $y = x^2$ of changing the indicated parameter to the given number.

- For $y = a(x - r)(x - s)$: On the first die, two of the sides will be labelled a, two sides will be labelled r, and two labelled s. On the second die, there will be various rational numbers. Students should roll the dice and state the effect on the graph of $y = x^2$ of changing the indicated parameter to the given number.

- A project should provide students with the opportunity to explore quadratics and their applications while reinforcing what they learned as part of this outcome.

Suggested Models and Manipulatives

- area model
- alge-tiles

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- axis of symmetry
- break-even point
- degree of polynomial
- factored form
- parabola
- parameter
- partial factoring
- quadratic
- roots
- standard form
- vertex form
- vertex
- *x*-intercepts
- *y*-intercept
- zeros

Resources/Notes

Internet

- National Library of Virtual Manipulatives (Utah State University 2010)
  http://nlvm.usu.edu/en/nav/vlibrary.html
  This website allows students to review factoring using algebra tiles.

- Algebra-Class.com, "Solving a Quadratic Equation" (Karin Hutchinson 2013)
  www.algebra-class.com/quadratic-equation.html
  This website has good examples for solving quadratic equations using different methods and graphing quadratic functions.

- Graphic Calculator (Houghton Mifflin Harcourt 2013)
  http://my.hrw.com/math06_07/nsmedia/tools/Graph_Calculator/graphCalc.html
  This website contains an online graphing calculator.
- Old Poly – card game that provides factoring practice
  http://www.beaconlearningcenter.com/lessons/lesson.asp?ID=1178
- Quadratic Illuminations activities
  https://illuminations.nctm.org/Lesson.aspx?id=2650
  https://illuminations.nctm.org/Lesson.aspx?id=2105

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  Sections 7.1–7.8, pp. 358–438

Data Analytics
Time: 30 - 35 hours

GCO: Students will be expected to develop the ability to reason with and draw conclusions from data.

SCO DA01 Students will be expected to analyze, interpret, and draw conclusions from two-variable data using numerical, graphical, and algebraic summaries. [C, CN, R, T, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

DA01.01 Recognize that the analysis of two-variable data involves the relationship between two attributes.

DA01.02 Distinguish between situations that involve one variable and situations that involve more than one variable.

DA01.03 Generate scatter plots of two-variable data, by hand and using technology

DA01.04 Determine, by performing a linear regression using technology, the equation of a line that models a suitable two-variable data set

DA01.05 Determine, using technology, the correlation coefficient, and recognize it as a measure of the fit of the data to a linear model

DA01.06 Determine the fit of an individual data point to the linear model by determining its residual, and recognize how a residual plot can be used to determine if a linear equation is a good model for a two-variable data set.

DA01.07 Make inferences, and make and justify conclusions, from statistical summaries of two-variable data orally and in writing, using convincing arguments

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
RF07 Students will be expected to determine the equation of a linear relation to solve problems, given a graph, a point and the slope, two points, and a point and the equation of a parallel or perpendicular line. **RF04** Students will be expected to describe and represent linear relations, using words, ordered pairs, tables of values, graphs, and equations.	**DA01** Students will be expected to analyze, interpret, and draw conclusions from two variable data using numerical, graphical, and algebraic summaries.	**RF01** Students will be expected to represent data, using polynomial functions (of degree ≤ 3), to solve problems. **RF02** Students will be expected to represent data, using exponential and logarithmic functions, to solve problems. **RF03** Students will be expected to represent data, using sinusoidal functions, to solve problems.

Background

In Mathematics 9 students were expected to graph linear relations, analyze the graph, and interpolate or extrapolate to solve problems. (PR02)

In Mathematics 10 students were expected to identify independent and dependent variables in a given context (RF04.01). They have also used mathematical models to extrapolate and interpolate.

In Mathematics 10 students used technology to create a scatter plot, determine the line of best fit and the correlation coefficient (RF07.06). Earlier in Extended Mathematics 11 students explored one-variable statistics using both numerical and graphical summaries including the comparison of two related one-variable data sets (S01).

It is important that students understand the difference between one-variable and two-variable data. For example:

If data has been collected on the final weight of giant pumpkins during a given year this data would be one-variable data and could be represented by a 5-point summary and a box plot.

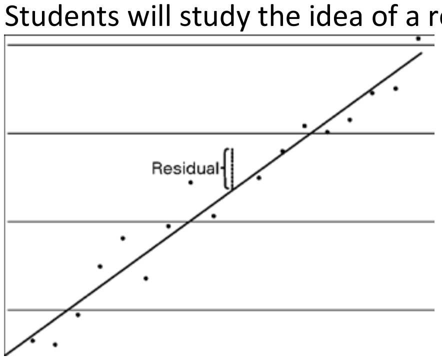
If data has been collected on the final weight of giant pumpkins during the past few years this data would be one-variable data and could be several stacked box plots.

If the data collected compared the final weight of giant pumpkins to the final circumference of the giant pumpkins, then this would be two-variable data that could be represented by a scatterplot.

Students will be expected to generate scatter plots both by hand and using technology. They would be expected to determine independent and dependent variables (where applicable), pick an appropriate scale, and determine if a linear model is appropriate. Students should realize that a scatterplot can look quite different depending on the scale. Changing the scale can make the correlation look stronger or weaker. Determining the correlation coefficient can assist the student in assessing the strength of the data's correlation.

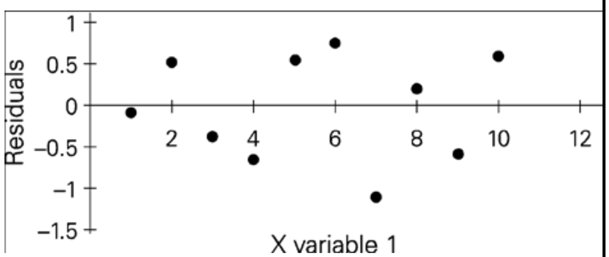
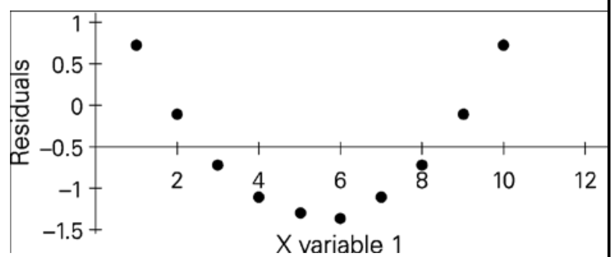
When a linear model seems reasonable, students will be expected to use technology to determine the equation of the line of best fit using regression.

Students will study the idea of a residual and a residual plot for the first time in this course.



When conducting any statistical analysis, it is important to evaluate how well the model fits the data and that a **residual** is the difference between the observed y-value (from scatter plot) and the predicted y-value (from regression equation line). The predicted value is called y-hat and is symbolized as $\hat{y}$. It is the vertical distance from the actual plotted point to the point on the regression line as illustrated.

A **residual plot** is a scatter plot of (x, residuals). A plot of residuals can be helpful to show whether linear regression was the right choice. If the residuals are more or less evenly distributed above and below the axis and show no particular trend, the linear regression model is likely a good one. But if there is a trend, a linear regression is likely being used to describe non-linear data and therefore is not a good choice. If the data points look like they fit a straight line but the residuals show a trend, it probably means that the data falls along a small part of a non-linear curve.

<table>
<tr><td>Residual plot indicating a good mathematical model.</td><td>Residual plot indicating that the linear model is not a good representation for the data
(that is the data may not be linear).</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

The table below gives data on height (in inches) and hand span (in cm) for 12 students enrolled in Extended Mathematics 11. The line of best fit for this data is $y = 0.5426x - 16.72$. Below is a sample of the calculation of a residual.

Height x	Hand span y	Predicted Value $\hat{y} = 0.5426x - 16.72$	Hand span $\hat{y} - 7$
66.0	20.0	19.092	−0.9084
69.0	21.1	20.719	−0.3806
69.0	17.6	20.719	3.1194
61.5	16.5	16.65	0.1499
63.0	17.5	17.464	−0.0362
68.0	19.0	20.177	1.1768
67.5	20.8	19.906	−0.8945
71.0	22.5	21.805	−0.6954
73.0	25.0	22.89	−2.11
69.0	23.0	20.719	−2.281
72.0	20.2	22.347	2.1472
71.0	21.1	21.805	0.7046

A graphing calculator can be used to create a residual plot although students should understand what a residual is and how to calculate it before using technology to generate a residual graph. The residuals are automatically calculated, when using a graphing calculator, during the regression; the student can plot them on the y axis against their existing x data.

Once students understand how a residual plot is generated it can be done using a spreadsheet such as Google sheets. [Note that it is important that students learn the basics of using Google Sheets at this point so that students can build on this basic understanding to analyze large data sets as a part of a final data analytics project.]

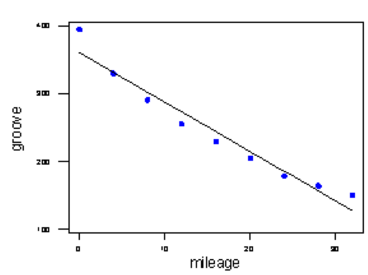
Students are expected to make and justify conclusions from statistical summaries and residual plots. They are expected to recognize when a linear model can be used to effectively represent two variable data. (In Mathematics 12 students will explore other types of regression.)

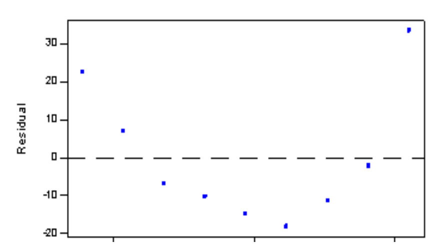
It is not an expectation of this course that students understand the importance of the value of r^2 . Students may, however, ask about the value of r^2 and what it means.

The **coefficient of determination** is symbolized by r^2. While the correlation coefficient, r, measures how closely a set of data can be represented by a linear model, the coefficient of determination can be thought of as a percent. It gives you an idea of how many data points fall within the results of the line formed by the regression equation.

For example, if $r = 0.922$, then $r^2 = 0.850$, which means that 85% of the points should fall within the regression line.

Example:
Is tire tread wear linearly related to mileage? A laboratory (*Smith Scientific Services*, Akron, OH) conducted an experiment in order to answer this research question. As a result of the experiment, the researchers obtained a data set containing the mileage (*x*, in 1000 miles) driven and the depth of the remaining groove (*y*, in mils). The fitted line plot of the resulting data:



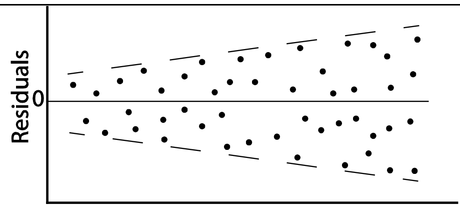
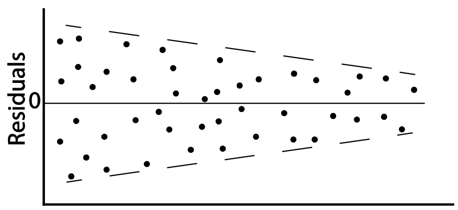
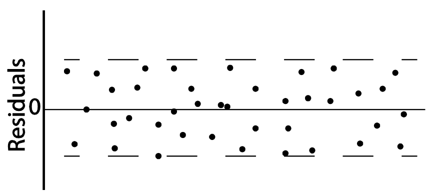


Note that the residuals depart from 0 in a *systematic manner*. They are positive for small *x* values, negative for medium *x* values, and positive again for large *x* values. Clearly, a non-linear model would better describe the relationship between the two variables.

The r^2 value is very high, 95.3% for this data. This is an excellent example of the caution "a large r^2 value should not be interpreted as meaning that the estimated regression line fits the data well." The large r^2 value tells you that if you wanted to predict groove depth, you'd be better off taking into account mileage than not. The residuals vs. fits plot tells you, though, that your prediction would be better if you formulated a non-linear model rather than a linear one.

Non-constant error variance shows up on a residuals vs. fits (or predictor) plot in any of the following ways:

- The plot has a "**fanning**" effect. That is, the residuals are close to 0 for small *x* values and are more spread out for large *x* values.
- The plot has a "**funneling**" effect. That is, the residuals are spread out for small *x* values and close to 0 for large *x* values.

	When residual plots reveal the “fanning” effect the linear model can be used to predict smaller values of x with more confidence than predicting larger values of x.
	When residual plots reveal the “funneling” effect the linear model can be used to predict larger values of x with more confidence the predicting smaller values of x.
	When residual plots reveal a constant variance the linear model can be used to predict any value of x with the same level of confidence.

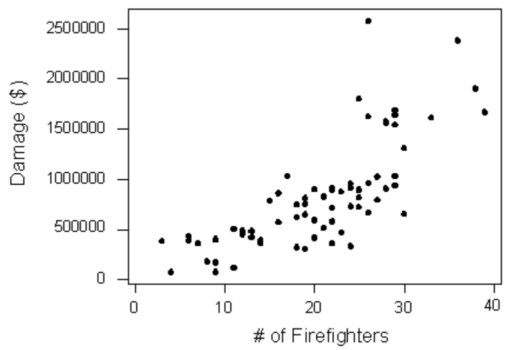
[https://www.originlab.com/doc/Origin-Help/Residual-Plot-Analysis]

Please note the following caution about Correlation and Regression.

In many studies of the relationship between two variables, the goal is to establish that changes in the explanatory variable cause changes in the response variable. Even when a strong association is present, the conclusion that this association is due to a causal link between the variable is often elusive.

Association, however strong, does NOT imply causation. Only careful experimentation can prove causation. A strong correlation simply points out the need for this scientific experimentation.

For example: The scatterplot below illustrates how the number of firefighters sent to fires (X) is related to the amount of damage caused by fires (Y) in a certain city.

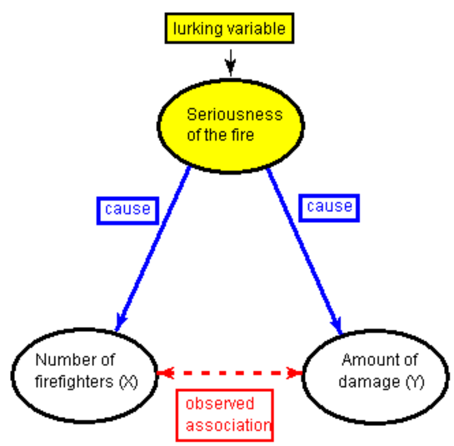


The scatterplot clearly displays a fairly strong (slightly curved) **positive** relationship between the two variables. Would it, then, be reasonable to conclude that sending more firefighters to a fire causes more damage, or that the city should send fewer firefighters to a fire, in order to decrease the amount of damage done by the fire? Of course not! So what is going on here?

There is a **third variable in the background**—the seriousness of the fire—that is responsible for the observed relationship. More serious fires require more firefighters, and also cause more damage.

Here, the seriousness of the fire is a **lurking variable**. A **lurking variable** is a variable that is not among the explanatory or response variables in a study, but could substantially affect your interpretation of the relationship among those variables.

(http://frewin.weebly.com/ap-statistics-causation-and-lurking-variables.html)



Assessment, Teaching, and Learning

Assessment Strategies

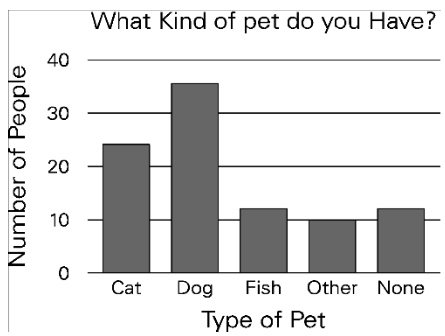
Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing all students—as a class, in groups, and individually.

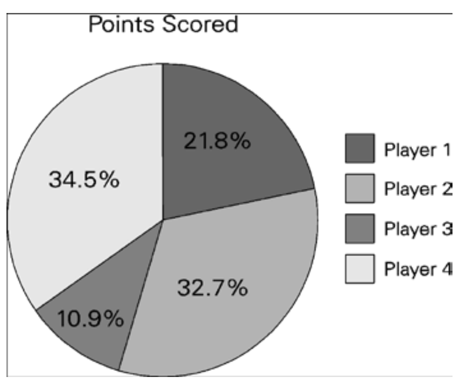
Guiding Questions

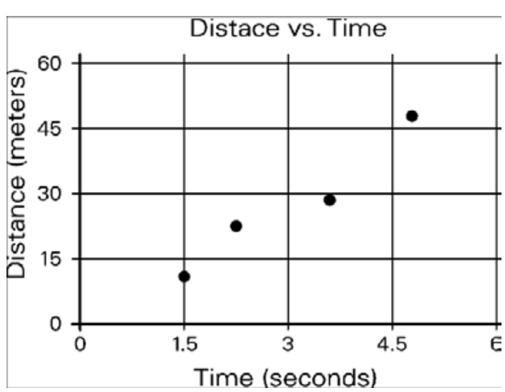
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Identify the independent variable when comparing the amount of gas remaining in a tank to the distance travelled.
- Which situation/graph is classified as one-variable data?
  - (a) A student researched average annual rainfall in various Canadian communities.
  - (b) Local doctors compared the number of hours teenagers spent exercising with the number of hours using technology.
  - (c) A telecommunications company investigated the number of cell phones per household in Nova Scotia.
  - (d) 

  - (e) 

  - (f) 

- A bird bath is filled with water. Over time, the water evaporates as shown in the table below:

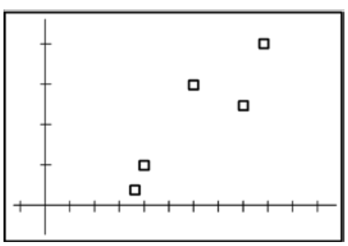
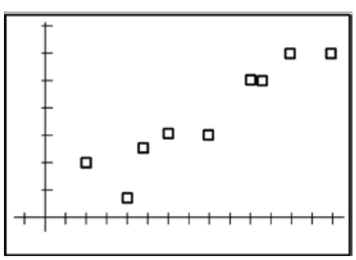
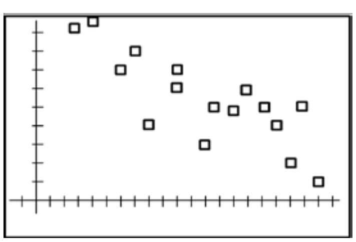
Time (h)	3	6	9	12	15	18	21	24
Water remaining (l)	7.7	4.6	3	2.1	1.6	1.3	1.2	1

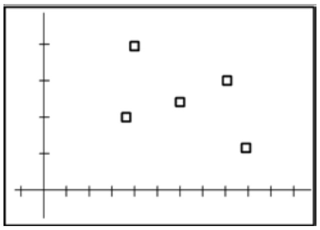
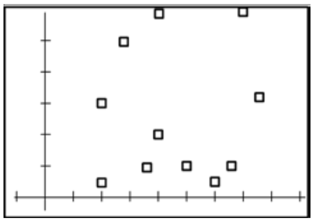
Susan used linear regression to obtain the model y = - 0.18x + 4.82.

(a) Use this model to determine the amount of water in the bird bath when it was filled. Was this interpolation or extrapolation? Explain.

(b) Use this model to determine the amount of water in the bird bath after 10 hours. Was this interpolation or extrapolation? Explain.

- Match the following graphs with the linear regression displays and then draw in an estimate for the line of best fit on the scatterplot. *(All graphs shown are using the same scale.)*

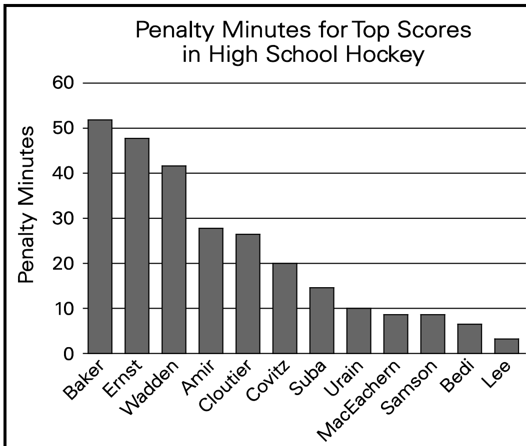
 A	1 y=ax+b a=-.759745721 b=9.989144828 r^2=.715274051 r=-.8457387623
 B	2 y=ax+b a=.8748966088 b=.2184656741 r^2=.8729708871 r=.9343291107
 C	3 y=ax+b a=.4522058824 b=.5522058824 r^2=.2992117308 r=.5470024962

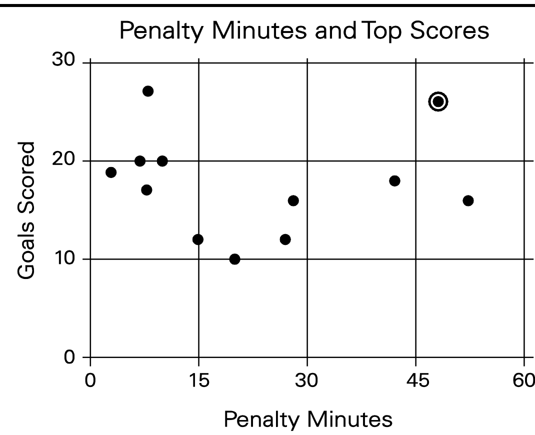
D 	4 y=ax+b a=-.474789916 b=3.96092437 r^2=.4465124411 r=-.6682158642
E 	5 y=ax+b a=.162601626 b=2.280561715 r^2=.0056279029 r=-.0750793501

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Which of the following graphical displays show two-variable data? Explain.





- Use information from the following table to describe a representation of one variable data and then to describe a representation of two variable data.

Player	Games Played	Goals	Assists	Points	Penalty Minutes
Baker	21	25	21	47	52
Ernst	22	19	31	50	48
Wadden	21	19	16	35	42
Amir	17	15	16	31	28
Cloutier	24	11	19	30	27

- The following table compares the age in years of a child to his or her mass.

Age (years)	2	4	6	8	10
Mass (kg)	13.5	18.1	27.6	36.3	42.8

(a) Using technology, create a scatterplot.
(b) Using the scatterplot, describe the strength and direction of the correlation.
(c) Perform a linear regression and record the correlation coefficient. Does the correlation coefficient support your answer to part b)? Explain.

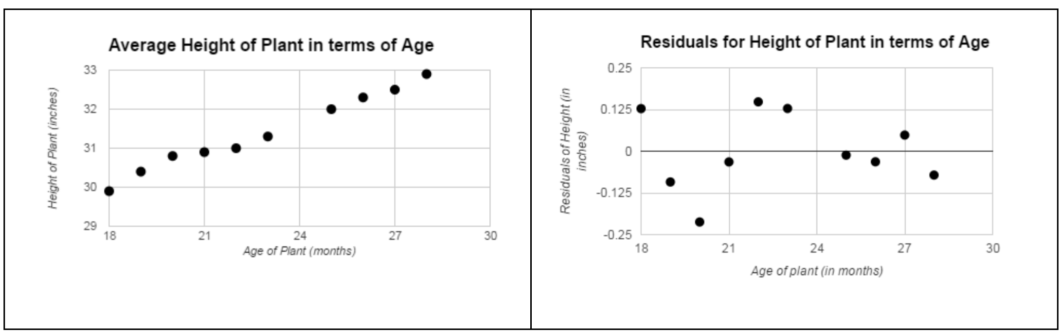
- After performing a linear regression you obtain a correlation coefficient of -0.85. Explain to another student the meaning of the correlation coefficient as a measure of how well the linear model represents the data. In addition, explain the significance of the correlation coefficient being negative.

- The data below shows the average height of a plant in terms of its age.

Age (m)	18	19	20	21	22	23	25	26	27	28
Height (in.)	29.9	30.4	30.8	30.9	31.0	31.3	32.0	32.3	32.5	32.9

(a) Using technology determine the equation of line of best fit.

(b) Based on the graphs below, would your linear model likely accurately predict the height of this plant after 2 years? Explain.



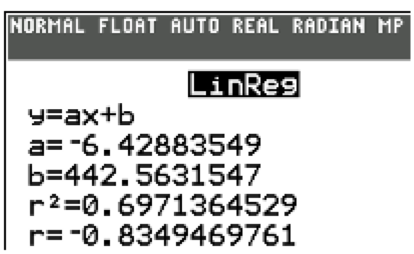
(c) Is this interpolation or extrapolation? Explain.

- Real Estate data for the areas surrounding New York City is shown below.

Distance from New York City (miles)	Median home price (thousands of dollars)
12	320
15	300
28	310
20	290
5	410
9	400
25	300
2	490
13	360
10	350
18	320
8	400

(a) Create a scatterplot for this data using technology.

(b) Technology determined a linear model with a correlation coefficient of -0.835, as shown below.



Use the linear model to predict the median housing prices in a village 33 miles from New City.

(c) Use technology to create a residual plot for this data.

(d) Is a linear model a good model to use for this data? Explain your reasoning.

- Considering the following data comparing shoe size and height in inches.

Shoe Size (x)	Height (y)	Predicted Height	Residual (Actual-Predicted)
8.5	66.0		
9.0	68.5		
9.0	67.5		
9.5	70.0		
10	70.0		
10	72.0		
10.5	71.5		
10.5	69.5		
11.0	71.5		
11.0	72		
11.0	73		
12.0	73.5		
12.0	74		
12.5	74		

(a) Create a scatterplot for this data by hand.

(b) If the linear regression model for this data is y = 1.87x + 51.36, then predict the shoe size of a person who is 60 inches tall.

(c) Using y = 1.87x + 51.36, plot the residuals after filling in the table.

(d) Based on the residuals plot, was your answer to part a) a reasonable answer. Justify.

- For each data set:
  (i) determine the linear regression equation.
  (ii) construct a scatter plot and a corresponding residual plot.
  (iii) state if a linear model is appropriate for the data and explain your reasoning.

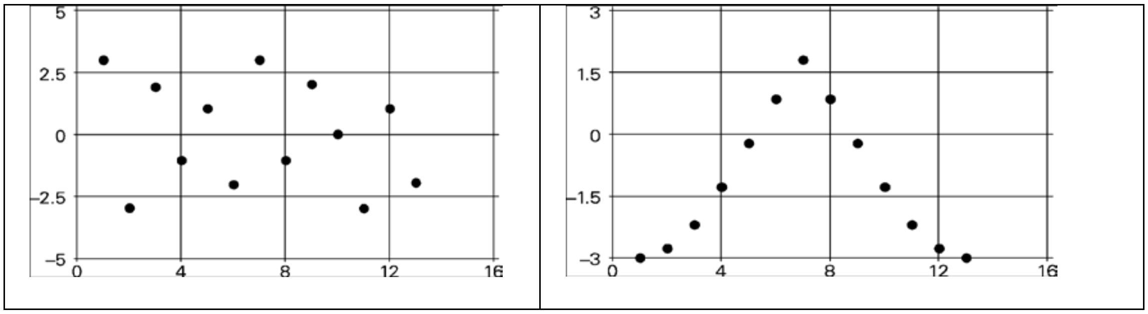
(a)

x	10	20	30	40	50	60	70	80
y	351	601	849	1099	1351	1601	1849	2099
Prediction								
Residual								

(b)

x	1	3	5	7	9	11	13	15
y	2	10	26	50	82	122	170	226
Prediction								
Residual								

- Which of the following residual plots indicate a good fit for a linear model? Explain your answer in terms of what you have learned about residuals.



- Does TV make you live longer? When the number of television sets per person *x* and the average life expectancy *y* for the world's nations are plotted, there is a high positive correlation. That is nations with many TV sets have high life expectancies. Could we lengthen the lives of people in Rwanda by shipping them TV sets? Justify your answer.

- A survey, as reported in a British newspaper, involved questioning a group of teenagers about their behavior, and establishing whether their parents smoked. The newspaper reported, as fact, that children whose parents smoked were more likely to exhibit delinquent behavior. The results seemed to show a correlation between the two variables, so the paper printed the headline; "Parental smoking causes children to misbehave." The Professor leading the investigation stated that cigarette packets should carry warnings about social issues alongside the prominent health warnings.

  Is there sufficient evidence to make the conclusion that was reached by the Professor? Explain.

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How will the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Effective instruction should consist of various strategies.

Guiding Question

Consider the following sample instructional strategies when planning lessons.

- Although students have learned about independent and dependent variables in the past, some students will still have difficulties determining when a variable is independent or dependent. Ask the students to complete a think-pair-share and decide which of the following variables is independent and which is dependent and explain why.
  (a) the time to finish a race and the running speed
  (b) height of a plant and rainfall
  (c) hours worked and amount of money earned
  (d) price of insurance and the number of accidents

- Ask two students to rank a list of 10 summer activities according to their preference or interest. Students then create scatter plots using points. Example: for Swimming Students A ranks it #3 Student B ranks it #1 Plot point (3,1) Students look at various created scatterplots to identify strength of correlation and what that means.

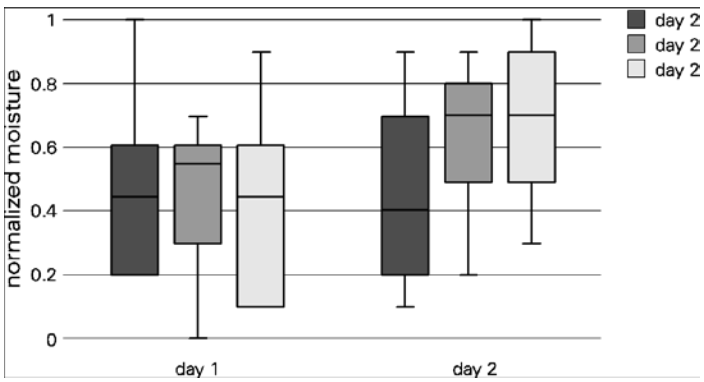
- Technology such as Desmos activities are available to use where students make predictions based on data and then are able to look at the actual reveal.

- While creating scatterplots by hand or with technology some students will struggle on choosing an appropriate scale others might automatically include (0,0). Some time might need to be spent on developing strategies for selecting a proper scale and discussing why (0,0) might not be an appropriate choice. These strategies could be student or teacher generated. To help students graph with TI graphing calculator technology it is important for students to understand the Window menu and how adjusting the window settings affects the graph.

- Some students may be tempted to calculate the correlation coefficient and determine the line of best fit for a set of data without graphing the data. For example, suppose a student calculated r = 0.542. This tells the student that the line of best fit does not fit the data very well, and so many of the points do not lie on or near the regression line. If the student had graphed the data, he or she might have discovered that only one point does not fit the trend of the graph, maybe as the result of a recording error or due to an outlier. If this point were ignored, the correlation coefficient would be much closer to 1 and the line of best fit a much better model for the data. For this reason, encourage students to start by graphing the data.

- Students might experience difficulties when they are trying to decide if a scenario is best described by two variable data or multiple sets of one variable data. Presenting students with various types of graphs will serve to clarify this for students.

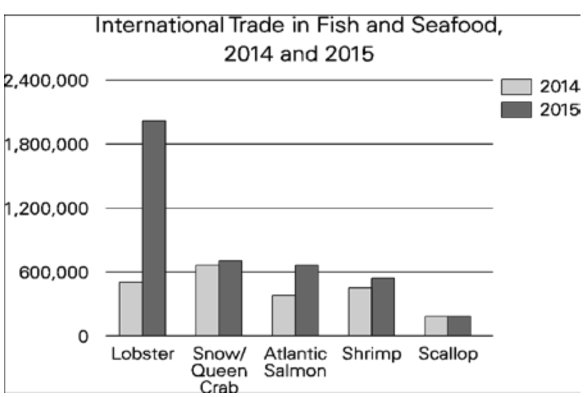
  (a) The graph below could be used to help further deepen the students understanding of multiple one variable data being described by box plots.



https://plot.ly/matlab/box-plots/

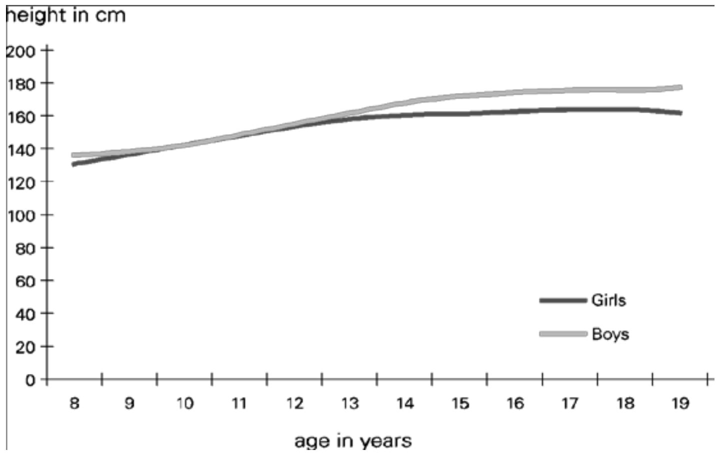
[**Note:** a couple of these box plots don't have a lower whisker. This means that all of the lowest 25% of the data was a single value represented by the lowest end of the box.]

  (b) The graph below could be used to help further deepen the students understanding of multiple one variable data being described by double bar graphs.



http://www.dfo-mpo.gc.ca/stats/FastFacts_15-eng.pdf

(c) The graph below could be used to help further deepen the students understanding of two-variable line graphs.. www.statscan.gc.ca



(d) The table below can be displayed as multiple one variable data as well as two variable data. If a student compared a player and his home runs this would be considered one variable data. If a student compared home runs and batting averages (AVG) then this would be considered two variable data.

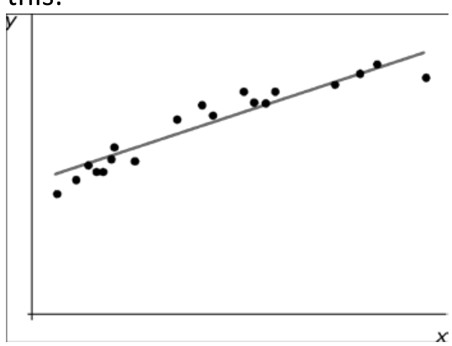
PLAYER	AB	R	H	HR	AVG
Mark Trumbo	487	74	125	38	0.257
Edwin Encarnacion	468	78	127	35	0.271
Khris Davis	444	67	115	33	0.259
Nelson Cruz	460	74	130	32	0.283
Todd Frazier	449	66	94	31	0.209
David Ortiz	419	62	135	30	0.322
Evan Longoria	487	67	140	30	0.287
Chris Davis	441	82	99	30	0.224
Brian Dozier	470	79	127	30	0.27
Mike Napoli	441	77	114	29	0.259

AB=At Bats, R=Runs, H=Hits, HR=Home Runs, AVG=Batting Averages
http://www.espn.com/mlb/stats/batting/_/sort/homeRuns/league/al/year/2016/seasontype/2

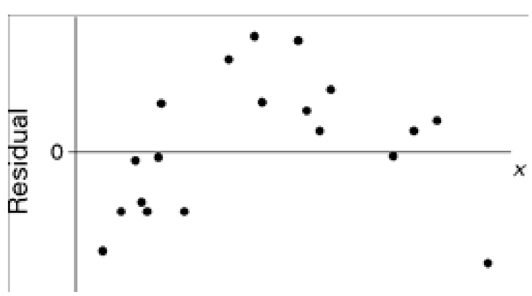
- Introduce the idea of a residual plot by having the students learn how to use the pattern in the scatter plot to predict what the residual plot will look like. Vertical distances from the regression line in the scatter plot are plotted as the y-values in the residual plot, with the x-coordinates remaining the same. (Note that the scales on the vertical axes of the two graphs may be different, and as a result, the sizes of the displacements can be greatly exaggerated in the residual plot.)

(a) Give the students a scatter plot with the line of best fit included and ask students what will the residual plot look like? If students have trouble with this, explain each point in the scatter plot one by one. For each point, look at whether it has a positive or negative residual and whether the residual is large or small relative to the other residuals. Once they have seen two or three points in the scatter plot and how they translate to points in the residual plot, students should see how the pattern translates as a whole. Students look at the residual plot as an indication of the fit of the points to the line.

For example, suppose you are given a scatterplot and regression line that looks like this:



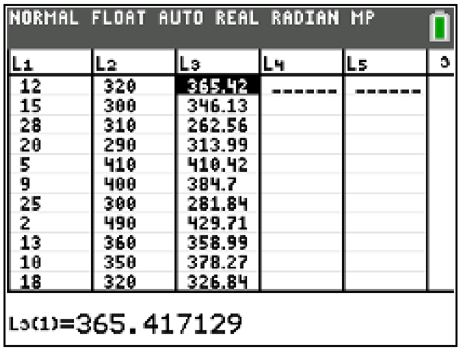
Then the residual plot would look like this:



(b) If students still struggle, have them look carefully at the scatter plot. Point out that moving from left to right, the points initially tend to be below the regression line, then move above it, and then below it. The residuals are negative, then positive, and then negative again. This means that the points in the residual plot will be below the horizontal axis, then above it, and then below it again.

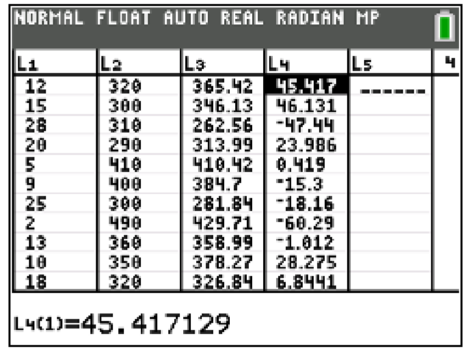
- Once the students have a solid understanding of what a residual is and what a residual plot represents they can then use technology to calculate residuals and to create a residual plot.

  On excel, google sheets or a graphing calculator students would create a column using the regression equation (in the example below: y=-6.4288355x + 442.563155) and then create another column, the residuals, by subtracting the predicted from the actual data value. The residual plot would be created using L1, L4 in the example below.



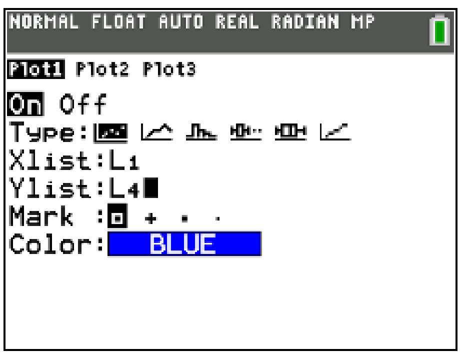
L1	L2	L3	L4	L5
12	320	365.42	------	------
15	300	346.13		
28	310	262.56		
20	290	313.99		
5	410	410.42		
9	400	384.7		
25	300	281.84		
2	490	429.71		
13	360	358.99		
10	350	378.27		
18	320	326.84		

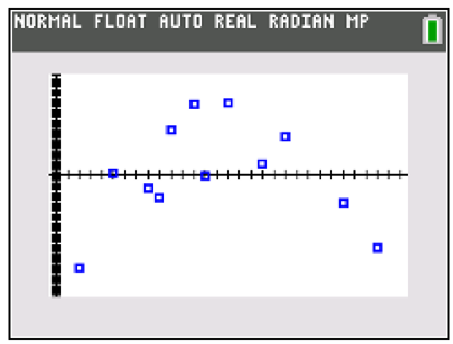
L3(1)=365.417129



L1	L2	L3	L4	L5
12	320	365.42	45.417	------
15	300	346.13	46.131	
28	310	262.56	-47.44	
20	290	313.99	23.986	
5	410	410.42	0.419	
9	400	384.7	-15.3	
25	300	281.84	-18.16	
2	490	429.71	-60.29	
13	360	358.99	-1.012	
10	350	378.27	28.275	
18	320	326.84	6.8441	

L4(1)=45.417129

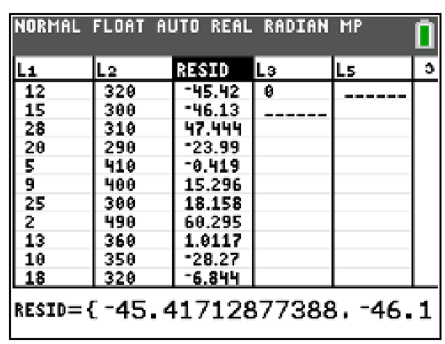




Note: On the TI graphing calculator the residuals are automatically calculated during the regression so a short cut is also possible and all you have to do is plot them on the y axis against your existing x data. Show line with original data points. Make the residuals visible in the statistics editor.

[STAT] [1] brings up the editor. Cursor to the column heading of [L3] and press INS to open up a new list. You see the NAME= indicator at the bottom of the screen, with the blinking A to indicate alpha mode.

Press LIST, then scroll to RESID and press [ENTER]. The list of residuals appears.



L1	L2	RESID	L3	L5
12	320	-45.42	0	------
15	300	-46.13	------	
28	310	47.444		
20	290	-23.99		
5	410	-0.419		
9	400	15.296		
25	300	18.158		
2	490	60.295		
13	360	1.0117		
10	350	-28.27		
18	320	-6.844		

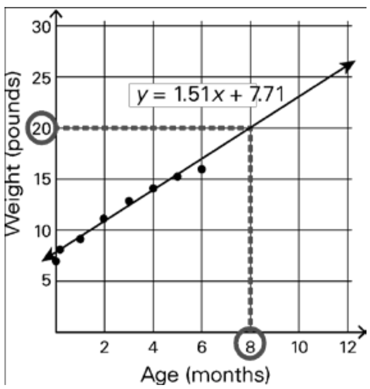
RESID={-45.41712877388, -46.1

Note that, to save time, you can use ZOOM 9 to have calculator set an appropriate window.

Don't worry about the magnitude of the residuals, what you want to look at is whether there's a trend in the residuals. Here there is no trend, so you conclude that a linear regression was the right choice.

- It is important to use real data such as:

  The following graph represents the median expected weights for girls from birth to six months of age. And ask that students to determine if using the line of best fit indicated to extrapolate would likely provide a reasonable answer.



  In order to answer this question students would consider the residual plot – looking to see if there is any apparent pattern (there is) that would indicate that the linear line of best fit would not be the best model and therefore that the indicated extrapolation for eight months would not likely be accurate.

- A good summary activity for DA01 is to ask students work in groups to collect data and then use a linear regression model to extrapolate and discuss the validity of such an extrapolation. Students may then present their data, graphs and conclusions.

  Examples:
  Cube-a-links activity - Have a student time how long it takes to construct a 2x2 square using 4 multi-link cubes; then a 3x3 square using 9 multi-link cubes; then a 4x4 square using 16 multi-link cubes; and then a 5x5 square using 25 multi-link cubes. Before constructing a larger square have students use their times, a scatterplot and line of best fit to predict how long it would take them to make a 7x7 square using 49 multi-link cubes. A discussion of residuals and residual plots should then be used as part of the discussion of the validity of a linear model.

  Number of letters activity - Have a student count how many times a student can write the word RED in a 30 second period; write the word BLUE in a 30 second period; write the word BLACK in a 30 second period; and the word MAGENTA in a 30-minute period. Before writing a longer color multiple times in a 30 second period have students use the number of times the word was written in 30 seconds, a scatterplot and line of best fit to predict how many times the student could write the words YELLOW and AQUAMARINE in 30 seconds. A discussion of residuals and residual plots should then be used as part of the discussion of the validity of a linear model.

- Later in the course students will learn about big data in terms of how it is used and how to analyze it. The basic ideas of how to analyze big data can be introduced at this point by analyzing a small amount of data such as the table below. The table has been constructed such that several patterns will emerge depending on which variables students compare.

Player	Games Played	Goals	Assists	Points	Penalty Minutes
Baker	21	25	21	47	52
Ernst	22	19	31	50	48
Wadden	21	19	16	35	42
Amir	17	15	16	31	28
Cloutier	24	11	19	30	27
Covitz	18	9	35	33	20
Saba	24	11	20	31	15
Ursin	24	20	20	40	10
MacEachern	24	27	30	57	8
Samson	10	10	25	25	8
Bedi	22	21	19	30	7
Lee	16	11	17	32	3

  To begin this activity, break students into small groups and introduce the table to the class. In their groups, ask students to make a list of different types of two-variable data and one-variable data they see. For instance, students might say goals and games for the two-variable and players and goals for one-variable data. After the groups have made their lists, create a list of the student generated two-variable data and a list for the student generated one-variable data on the board. Assign each group one of the two-variable and/or one-variable data sets, depending on the size of the class. Get students to create a graphical display they think will best display their data and ask them to look for patterns or trends that they notice. Once they have completed their graphical display, students can add it to a teacher generated Google Slides document. It is helpful if each group of students is assigned a particular slide. When all students have added their graph review all of the graphs with the class and get students to discuss the observations they made. In addition, ask the class to note any different observations they see.

- A project could be constructed for the end of this outcome where students construct a small set of data (20-30 items) with information about a minimum of three parameters. For example age, mileage and asking price for used Honda Civics. The students would then use Google Sheets to construct scatter plots, identify outliers, determine the line of best fit, construct residual plots and draw conclusions about their data set.

Suggested Models and Manipulatives

- graphing utilities
- grid paper
- multi-link cubes

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- causation
- correlation
- correlation coefficient
- dependent variable
- extrapolate
- independent variable
- interpolate
- lurking variable
- predicted value ($\hat{y}$)
- residual
- residual plot
- scatterplot
- trend
- two-variable data

Resources/Notes

Internet

- guessthecorrelation.com
  http://www.statcan.gc.ca/edu/power-pouvoir/glossary-glossaire/5214842-eng.htm
  http://www.statcan.gc.ca/edu/power-pouvoir/ch9/scatter-nuages/5214827-eng.htm
- video that explains how to use google sheets to explore data using linear regression and residual plots https://www.youtube.com/watch?v=9S2pb2-3Ll0
- PDF with examples of causation and lurking variables:
  http://www.csun.edu/~an73773/Causation_LurkingVariables.pdf
- Linear Regression Desmos Activity (student predict the time taken to charge a cell phone battery)
  https://teacher.desmos.com/activitybuilder/custom/584b173046693c8c1a090c93
- Linear Regression Desmos Activity (student predict average age at which people marry)
  https://teacher.desmos.com/activitybuilder/custom/56e077b20133822106a07ded#

Print

- Data Management 12 (McGraw Hill)

SCO DA02 Students will be expected to critically analyze society's use of inferential statistics. [C, CN, R, T, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

DA02.01 Investigate examples of the use of inferential statistics in society
DA02.02 Assess the accuracy, reliability, and relevance of statistical claims in the media by
- identifying examples of bias and points of view, including the use and misuse of statistics to promote a certain point of view
- identifying and describing the data collection methods, including the characteristics of a good sample, some sampling techniques, and principles of primary data collection
- determining if the data is relevant

DA02.03 Recognize and explain why conclusions drawn from statistical studies of the same relationship may differ.
DA02.04 Recognize and explain how the collection and analysis of data has impacted and continues to impact our world.
DA02.05 Create infographics / data visualizations using the design principles of good data visualization.
DA02.06 Identify, discuss, and present multiple sides of the issues with supporting data.

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
	DA02 Students will be expected to critically analyze society's use of inferential Statistics.	

Background

Data Analytics/Inferential Statistics is the art of analyzing and drawing conclusions from data. In this outcome students will be expected to analyze and draw conclusions from data that has already been collected and organized in tables or graphs.

In Mathematics 8, the focus of instruction was to critique ways in which data were presented (8SP01). The emphasis in Mathematics 9 (9SP01) was to analyze and critique the data collection process.

There are many factors within the data collection process that have the potential to influence the results. In Mathematics 9, students also considered factors such as the method used, the reliability and usefulness of data, and the ability to make generalizations about the population from a sample.

In Mathematics 6 (6SP03), students had experiences creating and analyzing pictographs, line plots, Venn diagrams, Carroll diagrams, bar graphs, double bar graphs and line graphs. In Mathematics 7 (7SP03) students constructed, labeled, and interpreted circle graphs. Students will not have had experience with creating more complex infographics and data visualizations. In this outcome, students will be introduced to the general design principles of good data visualization.

Descriptive statistics describe the population we are studying. The data could be collected from either a sample or a population, but the results help us organize and describe data. Descriptive statistics can only be used to describe the group that is actually being studied. That is, the results cannot be generalized to any larger group.

Descriptive statistics are useful and serviceable if you do not need to extend your results to any larger group. However, many uses of statistics in society tend to want to draw conclusions about segments of the population, such as all parents, all women, all victims, etc.

Inferential statistics are used to make predictions or inferences about a population from observations and analyses of a sample. That is, we can generalize the statistics from a sample to the larger population that the sample represents. In order to do this, however, it is imperative that the sample is representative of the group to which it is being generalized.

For this outcome, students will collect examples of inferential statistics, working in groups to consider articles that make a statistical claim that is valid as well as those that make claims that may be invalid or questionable.

Students will extend their understanding of bias, points of view, and objectivity for various common data collection methods. While it is not expected that students memorize the different types of sampling or bias they should recognize when samples are representative and when they are not.

A **sample** is a group of items or people selected from the population. Sample types that will be considered are:

- **Simple random sampling -** a sample in which each individual in the population of interest has an equal likelihood of selection.
- **Systematic random sampling -** Chosen on the basis of an ordered system. For example every 100th item is tested.
- **Stratified random sampling -** Before sampling, the population is divided into characteristics of importance for the research. For example, by gender, social class, education level, religion, etc. Then the population is randomly sampled *within* each category or stratum. If 38% of the population is college-educated, then 38% of the sample is randomly selected from the college-educated population.
- **Cluster random sampling -** A simple random sample of evident groups that the population contains. Then every item in each selected group is tested. For example from

the 80 high schools in Nova Scotia a random sample of 20 schools are selected, then all students in each of the selected schools are questioned.

- **Multi-stage random sampling -** Multistage sampling refers to sampling plans where the sampling is carried out in stages using smaller and smaller sampling units at each stage. For example a sample of 10 regions in Canada might be randomly selected to be surveyed. Then in each of the 10 selected regions sub regions would be randomly selected to be tested. This subdividing might have two or more stages.
- **Convenience sampling -** Selection of easy to obtain members from the population. This is typically not a worthwhile style for a sampling technique
- **Voluntary sampling -** Subjects from the population determine whether they will be members of the sample or not. This type of sample is not reliable to do meaningful statistical work.
- **Bias** occurs when there is a prejudice for or against an idea or response. Biased samples can result from problems with either the sampling technique or the data collection method. A statistical bias is a method of collecting data where one subset of data is either underrepresented or overrepresented within the total set.

Types of sampling bias that will be considered are:

- **Sampling bias -** Sampling bias occurs when the sample does not accurately represent the population. For example: You want to find the average age of the students at your school, and choose 2 grade 9 classes and a grade 11 class as your sample.
- **Non-response bias -** Sometimes, individuals chosen for the sample are unwilling or unable to participate in the survey. Nonresponse bias is the bias that results when respondents differ in meaningful ways from non-respondents.
- **Household bias -** Household bias occurs when strata (divisions) from the sample group are not equally represented. For example: A survey to determine the average speed on a highway would be biased if taken only during rush hour. Other strata (in this case, time intervals) need to be included (such as late evenings, etc.).
- **Response bias -** Voluntary response bias occurs when sample members are self-selected volunteers, as in voluntary samples. An example would be call-in radio shows that solicit audience participation in surveys on controversial topics (abortion, affirmative action, gun control, etc.). The resulting sample tends to over-represent individuals who have strong opinions.

Some of these types of media bias are:

- **Bias by omission**
- **Bias by selection of sources**
- **Bias by spin**
- **Bias by story selection**
- **Bias by placement**
- **Bias by labelling**

In this outcome students will check for whether information has been presented in an objective manner to decide whether or not if it is free from bias. Some indications that the information is reasonably objective are as follows:

- All relevant data are presented even when it does not support the preferred point of view.
- All views of an issue are presented and none are preferred.
- All views of an issue are presented even though one is preferred.
- The topic is presented in a clear and logical manner.
- Assertions, statements, opinions, etc., are documented.
- A variety of reliable sources are used to support the point being made.
- The purpose is clearly stated.

Some indications that information may not be objective are as follows:

- Only one view of an issue is presented.
- Other views of an issue are attacked or ignored.
- Not all data are presented; only data supporting the preferred point of view are presented.
- Assertions, statements, and opinions are presented as facts without adequate documentation.
- Emotion-arousing language is used to persuade the audience of a point without any accompanying documentation.
- Derogatory language is used.
- The presentation is illogical or contains logical fallacies.
- The purpose is not clearly stated or is hidden.
- Converting the audience to a particular point of view is the primary purpose.

Infographics are data visualizations that present complex information quickly and clearly. Large data sets can be difficult to sort through and make sense of. Infographics can help the reveal patterns and trends to the reader and make complex ideas clear. An easy-to-read data visualization can help the data tell a compelling story.

Infographics, when well-constructed, are data visualizations that:

- present complex information quickly and clearly
- integrate words and graphics to reveal information, patterns or trends
- are easier to understand than words or numbers alone
- are visually pleasing and engaging

Data analysis is a process that starts with collecting data from various sources, and then analyzing it to discover useful information, suggest conclusions, and supporting decision-making. As a part of this outcome, students should be aware that data analysis has historically had significant impact on our society (e.g. medical research, census results). In today's world the volume of collected data means that connections and questions can be explored that were previously not accessible. These large data sets are sometimes referred to as Big Data.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing all students—as a class, in groups, and individually.

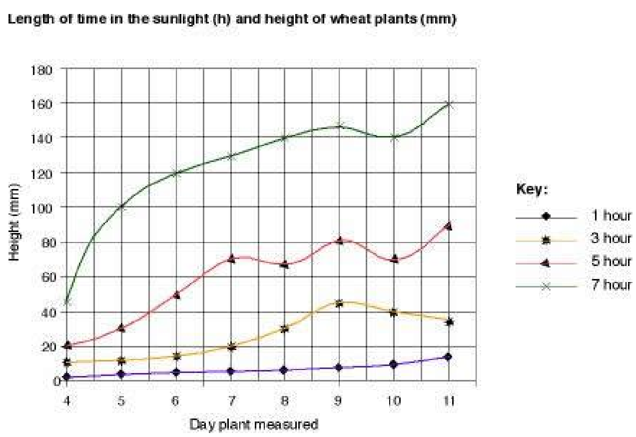
Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- For the following, identify the sources of bias and suggest ways to remove it.
    - At a soccer game, a survey was given and the results showed that when asked to give their favourite sport, 85% of the youth responded it was soccer.
    - Do you think that small dogs make good pets even though they are yappy?
- Examine the following graph and answer the questions below.



http://lrrpublic.cli.det.nsw.edu.au/lrrSecure/Sites/Web/research/student_research/3_graph/graph_05.htm

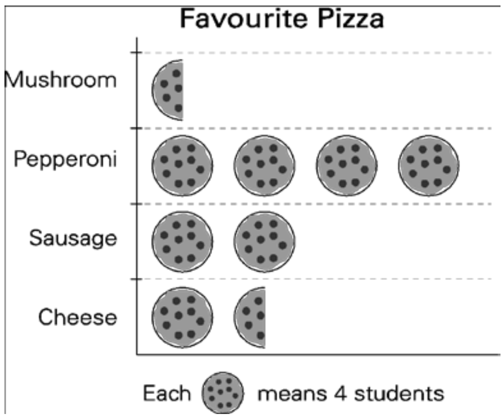
(a) What plants grew faster?

(b) Does sunlight affect plant growth?

(c) Plot points that you think might represent plants that are in the sunlight for 6 hours a day.

(d) At the end of a two-week period, how tall do you think the plants will be for each of the different hours of sunlight?

- Ask the class a survey question and collect and organize the data on the board with a tally (e.g. “What is your favourite genre of music”). Ask students to construct an appropriate graph to display this data.
- Use the data displayed on the pictograph below to create a different type of graph.
    - What type of graph did you choose?
    - Why did you make that choice?



Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

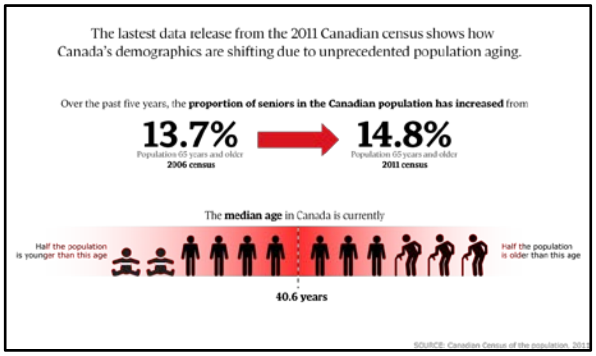
- According to a survey, 88% of residents have confidence in their cities police service. The survey was completed over the phone (average time 18 minutes) by approximately 400 residents aged 18 and over. How confident are you in the results of the survey? Explain your reasoning.
- Nearly half of Canadians recently surveyed said they were working in their dream jobs, according to an online poll. The survey results listed 47% of respondents as saying they had already found their dream jobs, while 65% said they looked forward to going into the office each day. The survey used results from an online sample of 1005 Canadians between July 12 and July 16. Is this survey a random sample of the population? What other questions might you ask about the reliability of this survey?
- Explain how it is possible that both the following statements about global warming could be correct.
    - “Global mean surface temperature over the past 20 years (1993–2012) rose at a rate of 0.14 ± 0.06°C per decade (95% confidence interval).” (Curry 2013, *Climate Etc.* blog.)
    - “According to the U.S. Global Change Research Program, the temperature in the U.S. has increased by 2 degrees in the last 50 years and precipitation by 5 percent.” (DoSomething.org 2014, *11 Facts About Global Warming*)

- You may have heard someone talk about how our ancestors died so much younger than we do today. Sometimes, for example, it's used to help explain why many women and men got married in their teens centuries ago; after all, they had to get started with families early since they would be dead by 40. According to the National Center for Health Statistics, life expectancy for American men in 1907 was only 45 years, though by 1957 it rose to 66. However, this does not mean that our great-grandfathers rarely lived into their fifties. The fact is that maximum human lifespan - a term that is often confused with life expectancy - has remained virtually unchanged. How is this possible?
- Identify the type of sampling that was used in each instance below.
    - Samjai visited each ATV dealership in a city and randomly selected 12 customers from each in order to understand what features were most important when a person decided to purchase an ATV.
    - Nan checked to see that the mushrooms packaged for sale in her store are the appropriate weight. To do this she randomly selected packages from a large shipment and weighed those she had selected.
    - To determine how new members were responding to fitness programs, the director Janine selected a sample of 20 people to survey. When looking at the overall list of new members, she noticed that 60% were female and 40% male. She selected 12 female and 8 males for her survey.
    - Erica randomly selected five counties in Nova Scotia to survey to determine their views on the construction of wind farms as an alternate source of energy in Nova Scotia.
- Explain why it is important for statisticians to use unbiased, rather than biased, data.
- Explain one example of how collected and analyzed data impacts decisions in Nova Scotia.
- Create an infographic based on the following data table about Nova Scotia Energy Sources.

Energy Source	**2007**	**2016**	**2020 Forecast**
Wind	1%	12%	18%
Hydro and Tidal	7%	14%	20%
Natural Gas and Oil	13%	14%	6%
Biomass	1%	2%	1%
Coal	76%	53%	38%
Imports	3%	4%	16%

Source: http://www.nspower.ca/

- What is the following infographic communicating?



Globe and Mail Infographic Aging Population

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How will the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Use a jigsaw activity with students to discuss the various types of sampling, sampling bias, and media bias.
- Have students, in small groups, create different points of view based on the same data and debate it.
- Use a TED Talk or other relevant video to interest students in statistical studies, their impact on our lives, and how they are reported in the media. A variety of these can be found at the TED Talk website at http://new.ted.com.
- Talk to the school's yearbook staff to see what sort of infographics about your school (student population, staff, sports teams, clubs) might be desired. Have students collect data about your school and create an infographic to give to the yearbook staff.
- Look at an article from the newspaper that has a lot of data in it but no data visualizations. Student highlight the data being communicated and then working in pairs create an infographic to help the author tell their story.
- Ask students to find an article based on an analysis of data. They would then analyze the article to determine type of sample used, the reliability of the reported data, and whether the author had a specific viewpoint or bias.
- Infographics also can emphasize a point of view. Students need to be aware of any agenda that the producer of the infographic may have.
- Show a video that illustrates the use of data historically such as this 2016 BBC Documentary: The Joy of Data [from approx min. 8 to min. 15]: and then ask students discuss where data collection has been / is being used to find answers to questions.
- Have students examine several easy-to-read statistical graphics that present information on various issues and trends in a visually appealing way such as a selection of USA Today Snapshots (do a Google image search of "USA Today Snapshots"). Students could be asked to explain what the info graphic is communicating. Students could also be asked to create an alternate infographic using the same data.

- Begin by discussing what make a good infographic. Co-construct with students what they should look for in a well-constructed infographic. Students should look at some infographics and discuss how they could be improved. A couple of sources of bad infographics is: http://mammothinfographics.com/blog/the-top-10-worst-infographics-of-all-time or https://www.theguardian.com/news/datablog/gallery/2013/aug/01/16-useless-infographics
- Before students begin to use technology to create an infographic it is important that they have an idea of what they want to say and what graphic they could use effectively.

- Have students create an infographic. Post their infographics around the classroom and have them do a gallery walk to examine students work and reflect on point of view.
- Have students create infographics and post using Google Slides. Share these slides and students can make comments for various slides created by others and/or some selected by you.
- To introduce the impact that well designed infographics can have the infographic showing the change in earth temperature from 20,000 BC to present (http://xkcd.com/1732/) can be used with the class as part of a discussion.

Suggested Models and Manipulatives

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- bias
- bias by omission
- cluster Random Sample
- convenience Sample
- descriptive statistics
- inferential statistics
- infographic
- multi-stage Random Sample
- population
- sample
- sampling bias
- simple random sample (SRS)
- stratified random sample
- surveys and polls
- systematic random sample
- voluntary sample

Resources/Notes

Internet

Use some of the free online resources available:

- Current examples of media bias: https://www.studentnewsdaily.com/archive/example-of-media-bias/
- *BBC Six-Part Primer on Understanding Statistics in the News* (American Statistical Association 2014) www.amstat.org/news/blastland_bbcprimer.cfm

"Survey shows significant increase in Canadian Netflix subscribers" (*Canadian Reviewer*, Monday, September 23, 2013; original source, *Maclean's*) www.canadianreviewer.com/cr/2013/9/23/survey-shows-significant-increase-in-canadian-netflix-subscr.html

- Free online infographics tools and templates can be found at:
  - Canva (https://www.canva.com/)
  - Piktochart (https://piktochart.com/)
- Articles containing information on the different types of data scientist jobs and the skills needed to do these jobs.
  - http://blog.udacity.com/2014/11/data-science-job-skills.html
  - https://adtmag.com/articles/2016/01/08/data-science-skills.aspx
- Sampling slide show: https://www.slideshare.net/swatiluthra5/sampling-ppt
- Robots and jobs from different perspectives - students could use to create infographics
  - https://www.fastcompany.com/3067279/you-didnt-see-this-coming-10-jobs-that-will-be-replaced-by-robots
  - https://qz.com/943073/compelling-new-evidence-that-robots-are-taking-jobs-and-cutting-wages/
  - http://adage.com/article/digitalnext/5-jobs-robots/308094/
  - http://www.cnbc.com/2017/02/03/warren-buffett-and-bill-gates-think-its-crazy-to-view-robots-as-bad.html
- The Feltron Report - Nicholas Felton is the author of numerous personal annual reports that condense the events of a year into a tapestry of maps, graphs and statistics. http://feltron.com/

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Supplementary Resource
  - Data Management 12 (McGraw Hill)

Video

- *RSS Presidential Address 2013* (Royal Statistical Society 2013) www.youtube.com/watch?feature=player_embedded&v=tsYUXBr4CEg#t=4
- Infographics Demo Video using Canva - https://youtu.be/RB2bjq-Wc4Y
- If The World Were 100 People video - https://www.youtube.com/watch?v=QFrqTFRy-LU
- Simpson's paradox and understanding data - examples that emphasize how initial results of data may be misleading when data is looked at in more detail https://www.youtube.com/watch?v=JJO4J_tJC2s

SCO DA03 Students will be expected to analyze data, identify patterns, and extract useful information and meaning from large, professionally collected data sets. [C, CN, R, T, V]
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics and Estimation **[T]** Technology **[V]** Visualization **[R]** Reasoning

Performance Indicators

DA03.01 Explore and analyze large sets of open data using technology.
DA03.02 Investigate what data is available and open and why some data is open and other data not.
DA03.03 Pose questions that might be answered or further explored with large open data sets.
DA03.04 Present their findings from an investigation of a big data set.

Scope and Sequence

Mathematics 10	Extended Mathematics 11	Mathematics 12
	DA03 Students will be expected to analyze data, identify patterns, and extract useful information and meaning from large, professionally collected data sets.	

Background

Data analysis is the process of examining and evaluating data in order to obtain constructive information, draw inferences, formulate conclusions and support decision-making.

In Mathematics 9 (9SP03), students planned and carried out a data project to answer a question. The project included formulating an appropriate question, collecting data from a sample or population, displaying the data, and drawing conclusions. In Extended Math 11, students are expected to draw conclusions from much larger sets of professionally collected data using technology. **Students are not expected to collect or clean up data sets.**

Hal Varian, Google's chief economist, states "The ability to take data – to be able to understand it, to process it, to extract value from it, to visualize it, to communicate it is going to be a hugely important skill in the next decades, not only at the professional level but even at the educational level for elementary school kids, for high school kids, for college kids. Because now we really do have essentially free and ubiquitous data. So the complimentary scarce factor is the ability to understand that data and extract value from it."

In this course students will explore examples to examine how data analytics is being used by companies, governments and scientists to guide research and decision making.

Open data is the idea that some data should be freely available to everyone to use as they wish, without restrictions. The debate on the value and benefits of open data is still evolving. Open government data portals seek to empower citizens, to help small businesses, or to create value in some other positive, constructive way. There are concerns regarding privacy and unintended impacts of open data.

For this outcome, students will be expected to take a given data set, organize it and determine what meaningful questions might be answered using this data. Using technology to organize and examine the data, students will attempt to draw conclusions that are relevant to the data they are working with. Data sets that students will be using should be large enough so that organizing the data by hand is not efficient.

The use of technology is a focus of this course and it is expected that students will apply computing technology in order to look at the data and interpret the results.

It is expected that students will explore and understand the difference between data analytics and data visualization.

- **Data Analytics** – process of analyzing data or making decisions and finding information from the data.
- **Data Visualization** – process of displaying data so the reader can clearly understand information from the data.

Students will be expected to use data visualization tools to explore and to create data visualizations as well as to communicate patterns and relationships discovered in large sets of data.

In this course, it is expected that students will gain experience using a spreadsheet computer application, such as Google Sheets, to analyze data.

Specifically they should learn how to do the following:

(a) sort and filter data

(b) use functions such as:
- MAX(range)
- MIN(range)
- COUNT(range)
- COUNTA(range)
- COUNTIF(criteria-range, criterion)
- COUNTIFS(criteria-range1, criterion1, criteria_range2, criterion2,…)
- SUM(range)
- SUMIF(range, criterion, sum_range)
- SUMIFS(sum-range, criteria-range1, criterion1, criteria-range2, criterion2,…)
- AVERAGE(range)

  - AVEREAGEIF(range, criterion, average-range)
  - AVERAGEIFS(average-range, criteria_range1, criterion1, criteria_range2, criterion2,…)
  - COUNTUNIQUE(range)
  - UNIQUE(range)

(c) use absolute and relative addressing

This outcome is to be assessed using an individualized or group project. Problem solving should permeate the whole process, as students investigate topics of personal interest, formulate questions, and analyze results.

The following are guidelines for project-based learning:

- Students may work in groups or independently.
- Allow students a choice on the topic and methods of presentation.
- Plan the project with drafts and timeline benchmarks.
- Provide clear expectations on success criteria (e.g. rubrics, exemplars, etc)

Students who have had experience coding should be encouraged to write a program to explore large data sets that might otherwise be too large to explore. Note that Google Sheets has a limit of 2 million cells. It can process no more than 256 columns of data (the same as Excel).

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Answer True or False for each statement below:
  - The purpose of a graph is to display data in a way that is easier to interpret.
  - All graphs do not require a legend.
  - All graphs must have a vertical scale and a horizontal scale.
  - All graphs must have a title.
  - A circle graph is always the best way to represent data

- List one conclusion you might make from the data shown below.

	A	B	C	D
1	Year	Both Sexes Pop	Male Population	Female Population
2	2010	309347057	152089484	157257573
3	2011	311721632	153294635	158426997
4	2012	314112078	154528573	159583505
5	2013	316497531	155741368	160756163
6	2014	318857056	156936487	161920569
7	2015	321368864	158345038	163023826
8				

- From the data illustrated below answer the following questions:

Table 1: World Cup Appearances

Flag	Appearances	Country Name	Flag	Appearances	Country Name
	18	Germany		1	Canada
	18	Italy		14	France
	10	United States		20	Brazil
	10	Switzerland		12	Uruguay
	16	Argentina		14	England
	4	Peru		11	Sweden
	12	Belgium		15	Mexico
	10	Netherlands		14	Spain

a) Which country has appeared in the most World Cups?

b) Which country has appeared in the least World Cups?

c) What is the mode of this data?

d) What is the mean number of appearances?

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Explain what is meant by open data. Why isn't all data open?
- List three questions that might be answered by analyzing the data gathered through a survey of 500 000 randomly selected Canadians. Data was collected for each of these categories: age, number of siblings, city or rural, educational level, and annual income.
- Work with a data set to discover patterns and trends, draw conclusions and present the findings.
- Demonstrate what you have learned from presentations of others by completing a questionnaire that focuses on the highlights of a presentation.
- Keep a journal entry describing what you have learned from each of the data analytics projects that have been presented in class.

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How will the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- When working with data it is always a good idea for each student to duplicate the original data to create a copy. Students then work with the copy only. That way they will always have access to a clean data set should they need it.
- Spreadsheets are intended to process and work with moderate sets of data. Less than 10 lines of data might just as easily be dealt with by hand or on a calculator, more than 400000 or 500000 lines of data would need to be dealt with by programming rather than a spread sheet.
- In the function bar of a spread sheet you can see the actual data stored in that cell. The data displayed in the cell may be formatted differently. Formatting options allow control over how the data is displayed.

- Please note that correct language for the following signs of aggregation is as follows:

  [brackets]
  (parenthesis)
  {braces}

- Students should brainstorm possible questions, ideas, and/or issues relating to their data set.
- A rubric should be used to assess the project, and students should be aware of the criteria before they start their project. Encourage them to participate in the development of the rubric and to work out the appropriate categories and criteria for specific tasks. This involvement will improve student motivation, interest, and performance in the project. Content, organization, sources, and layout are critical components used to evaluate projects. Illustrations, images, and graphics are also important features that should be included in assessment. Remind students that there are many different ways to deliver a project. Therefore, the rubric may have to be modified to fit the format of the presentation.
- If time allows, teachers may invite students to share the results of their data analytic projects to other students and for the students to respond to these projects.
- Looking at data such as that generated by Environment Canada - 'think', 'pair', 'share' could be used by pairs of students to analyze the data given to them.

Suggested Models and Manipulatives

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- aggregate data
- Closed data
- Data analysis
- Data visualization
- Open data
- primary source data
- secondary source data

Resources/Notes

Internet

(Note: Teachers should be aware that some of these open data set may contain information that could cause conflict in the classroom, such as divorce records for the Province of Nova Scotia, etc. Care should be taken when choosing data sets to be used in the classroom.)

- Statistics Canada (http://statcan.gc.ca) provides an excellent source of data.
- Government of Canada Open Data portal. This site contains data about Government of Canada services, financials, and national demographic information. - http://open.canada.ca/en/open-data
- Province of Nova Scotia Open Data portal. This site contains data from the Province of Nova Scotia. - https://data.novascotia.ca
- Open Source Sports - http://www.opensourcesports.com/olympic-medals-database/
- Gapminder World - A teacher guide explains how you can use Gapminder World to lecture about global development from 1800 until today. https://www.gapminder.org/downloads/200-years/
- Use of big data article: https://gigaom.com/2011/07/17/5-real-world-uses-of-big-data/

Print

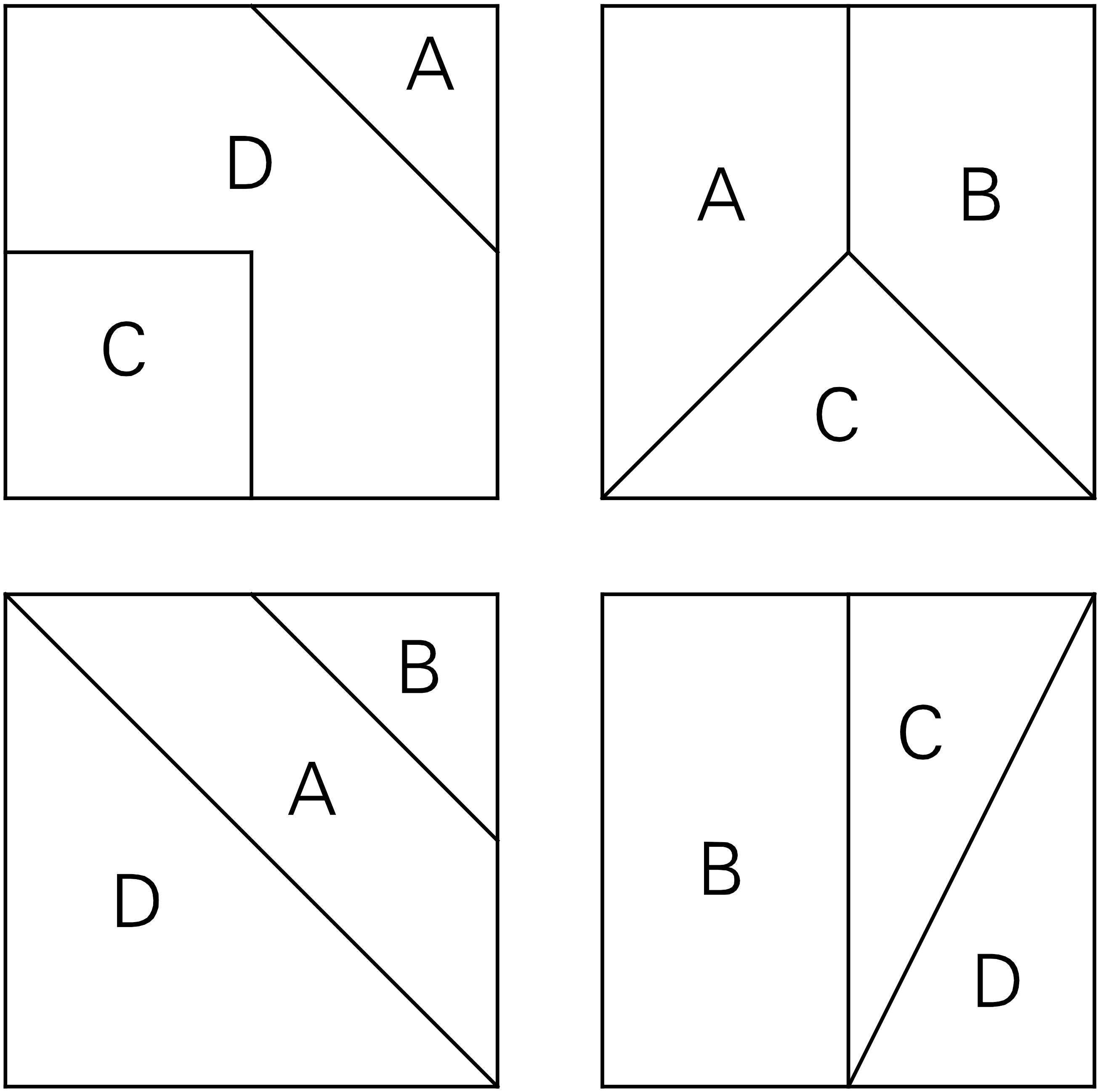
- Data Management 12 (McGraw Hill)

Video

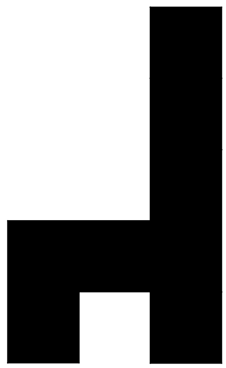
- Data analytics introduced with examples where used (<10 min; (start at 2:08) https://www.youtube.com/watch?v=SSRBDO0J5UI&feature=youtu.be&t=128
- Science of Big Data - The human face a data analytics (excellent talk about the power enabled by large sets of data) (approx 1 hour) https://www.youtube.com/watch?v=ahNdJdf867A
- Hans Rosling on CNN: US in a converging world. A 5 min. video clip to show power of Gapminder https://www.gapminder.org/videos/hans-rosling-on-cnn-us-in-a-converging-world/

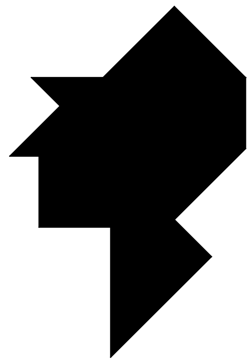
Appendices

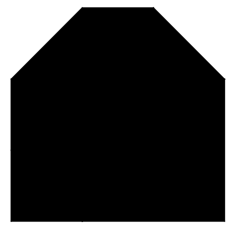
Square Puzzles

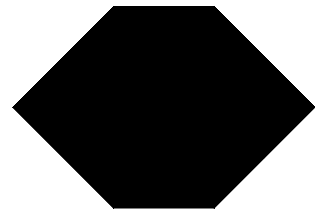


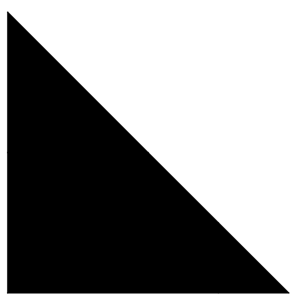
Tangrams

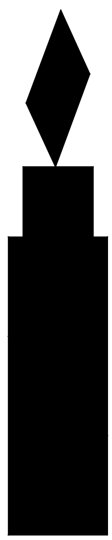

Chair

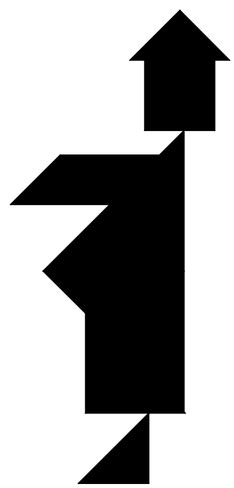

Head

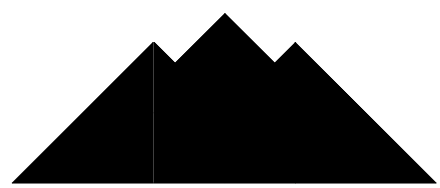

Barn

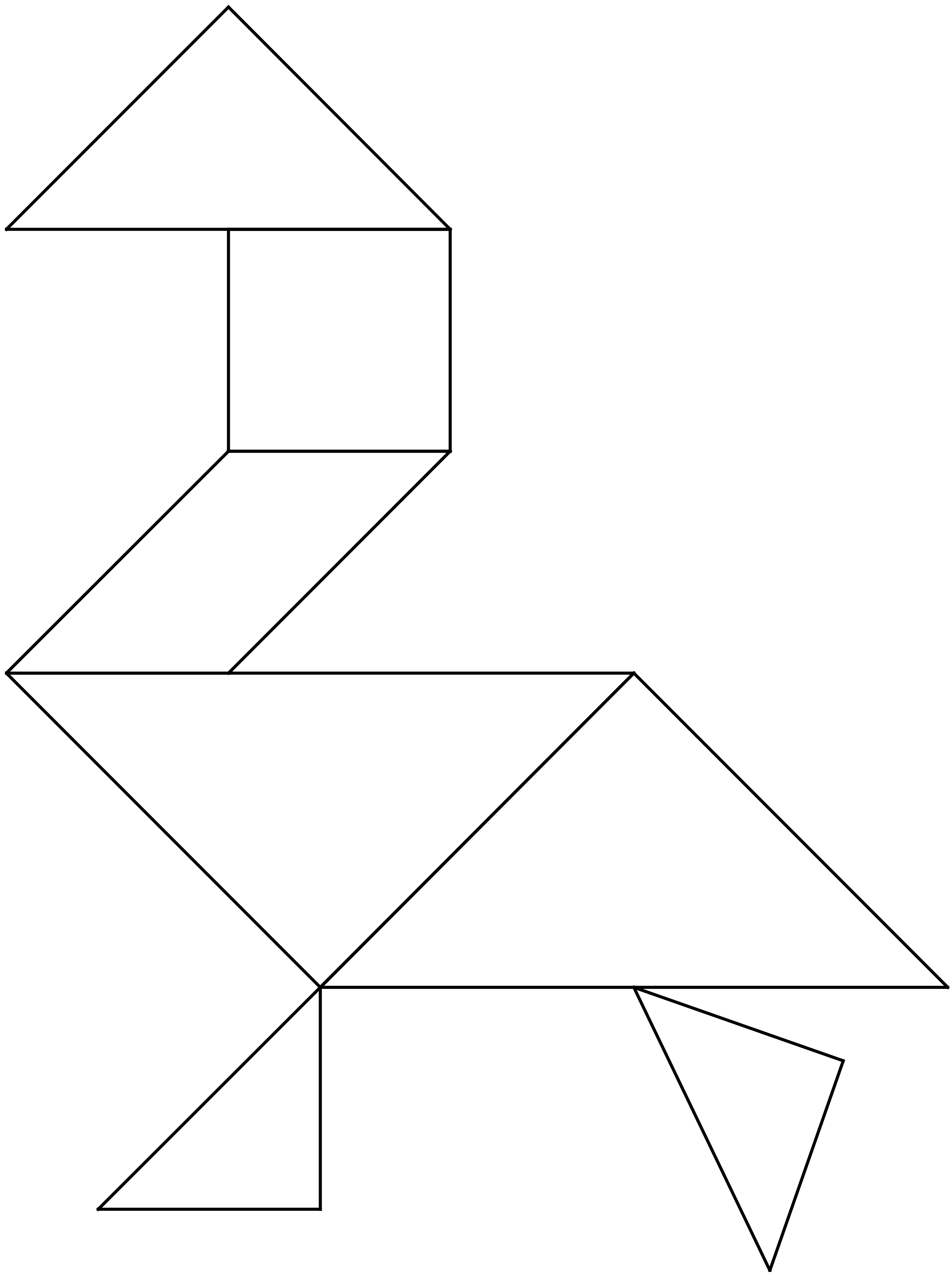

Hexagon

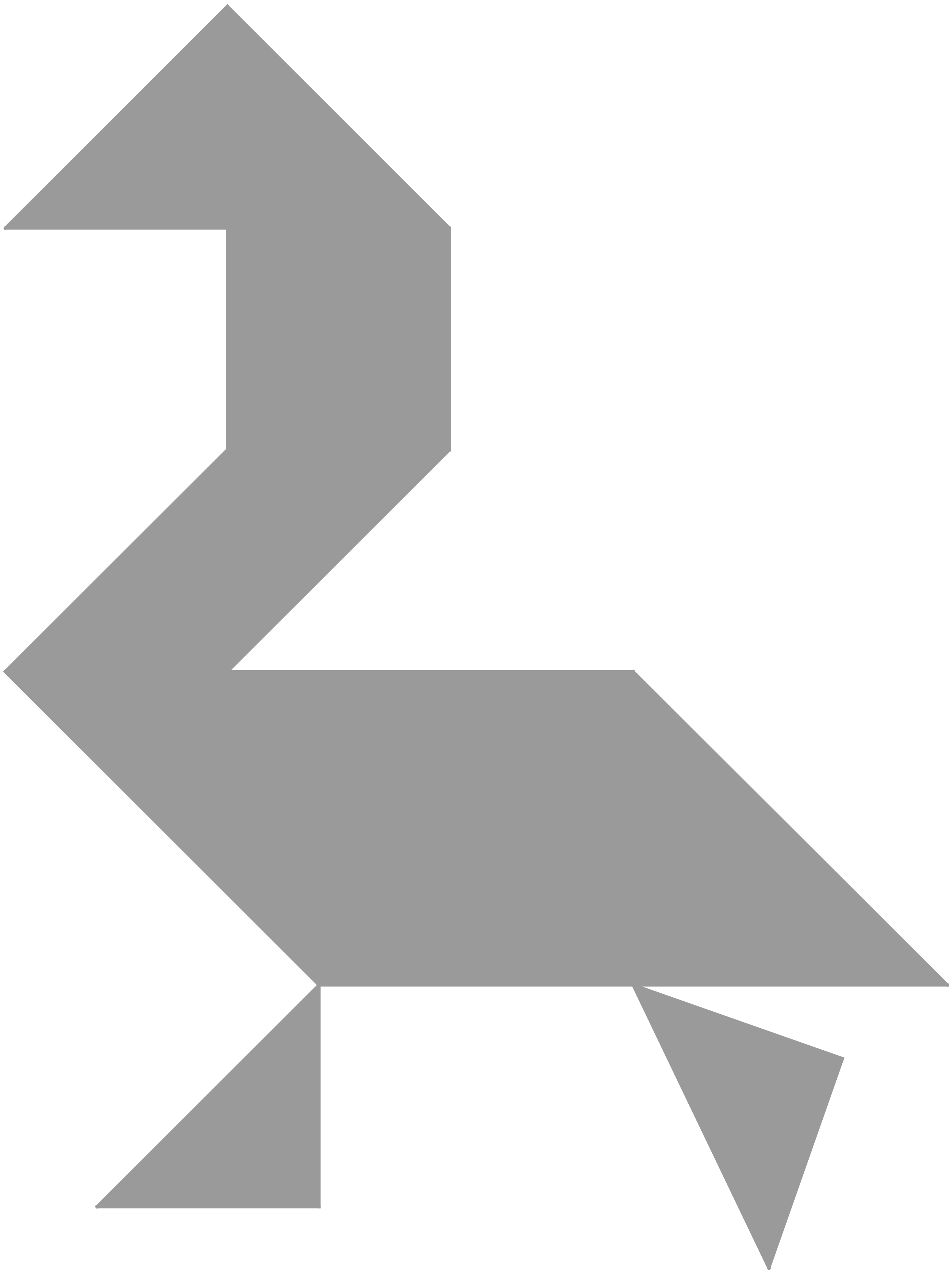

Right Triangle

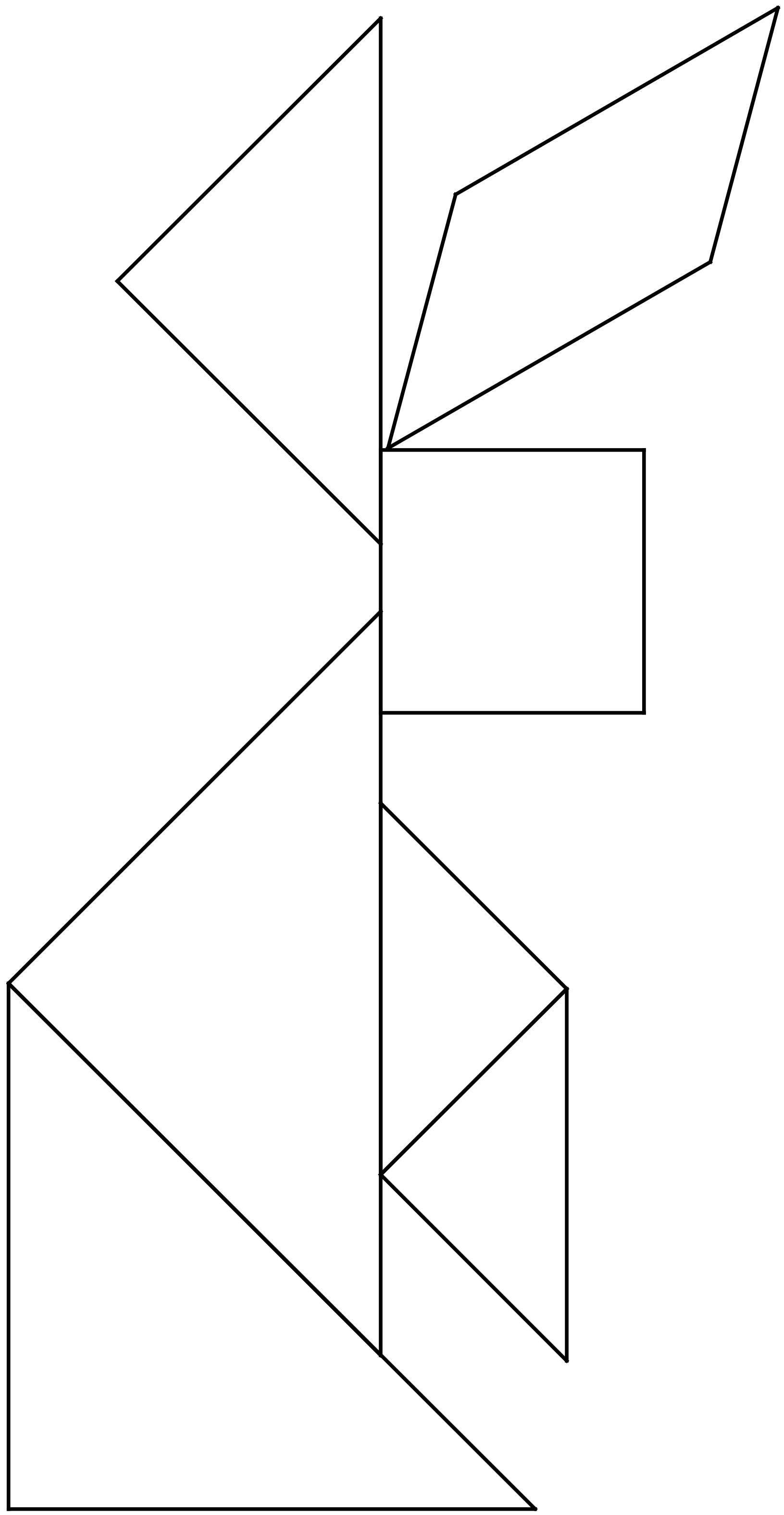

Candle

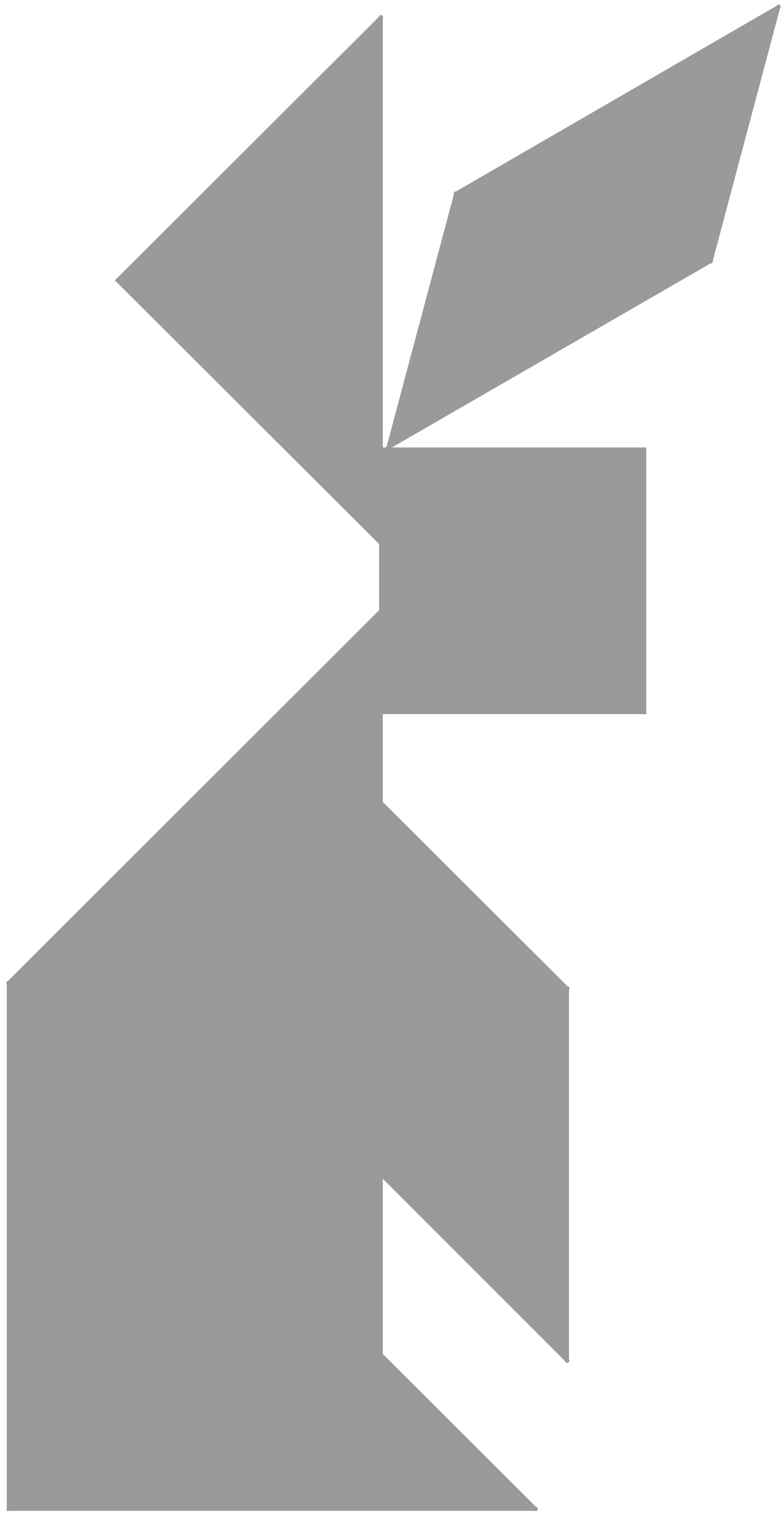

Sleepwalker

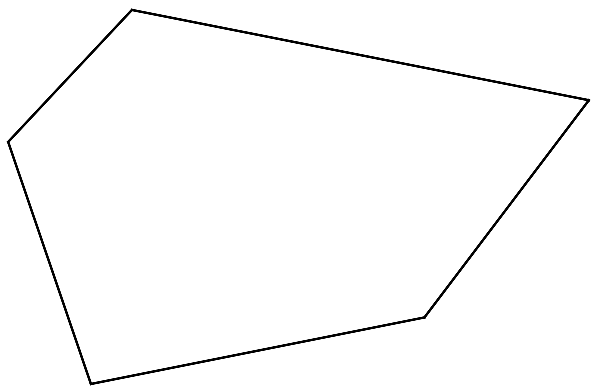

Mountain

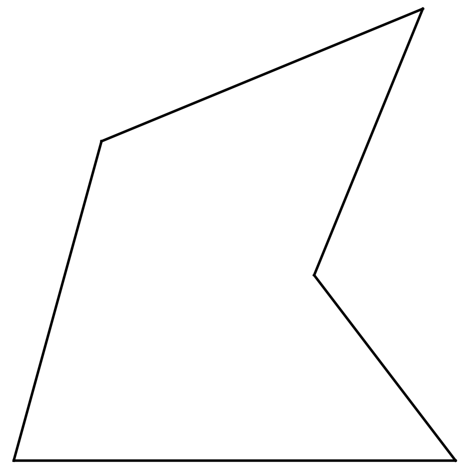


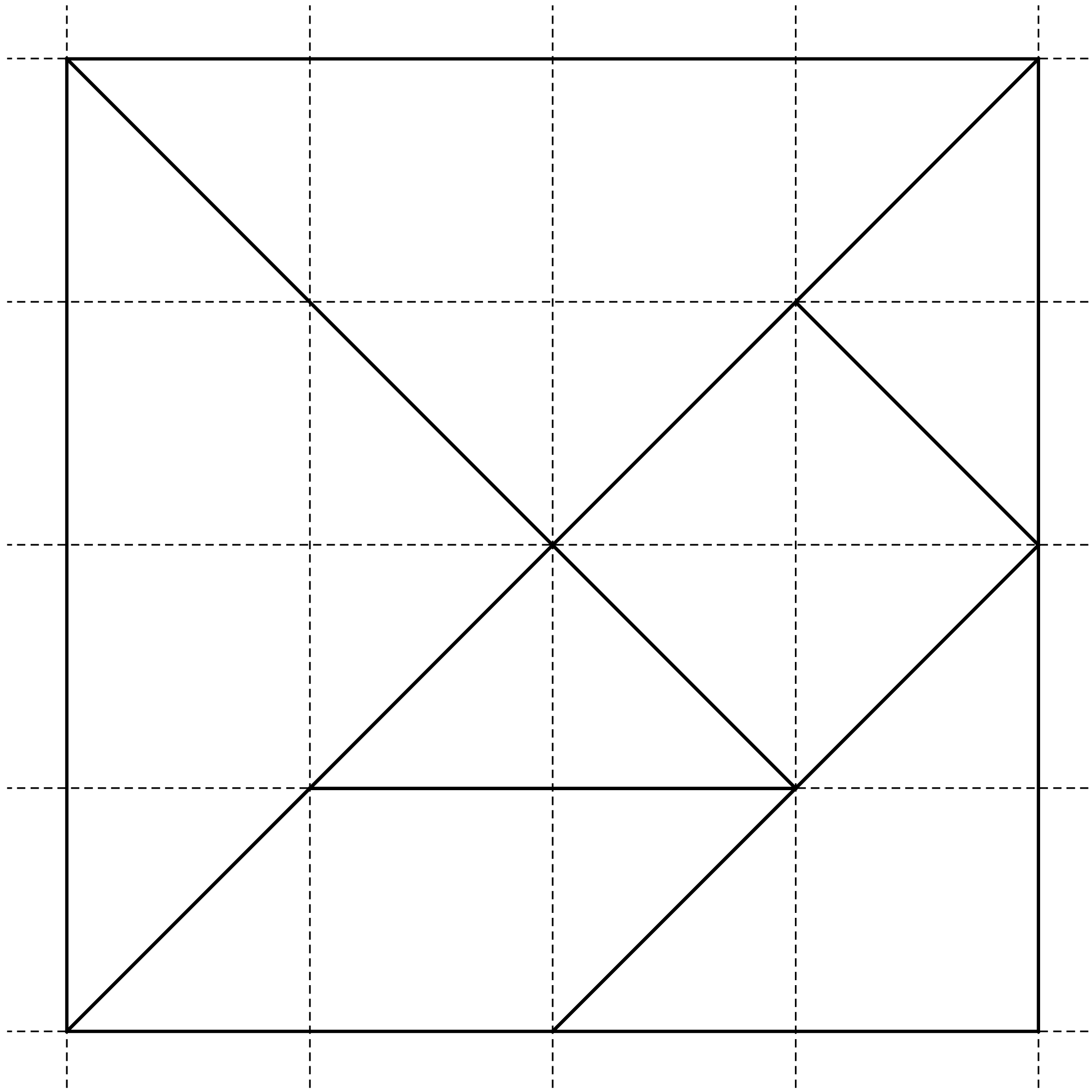


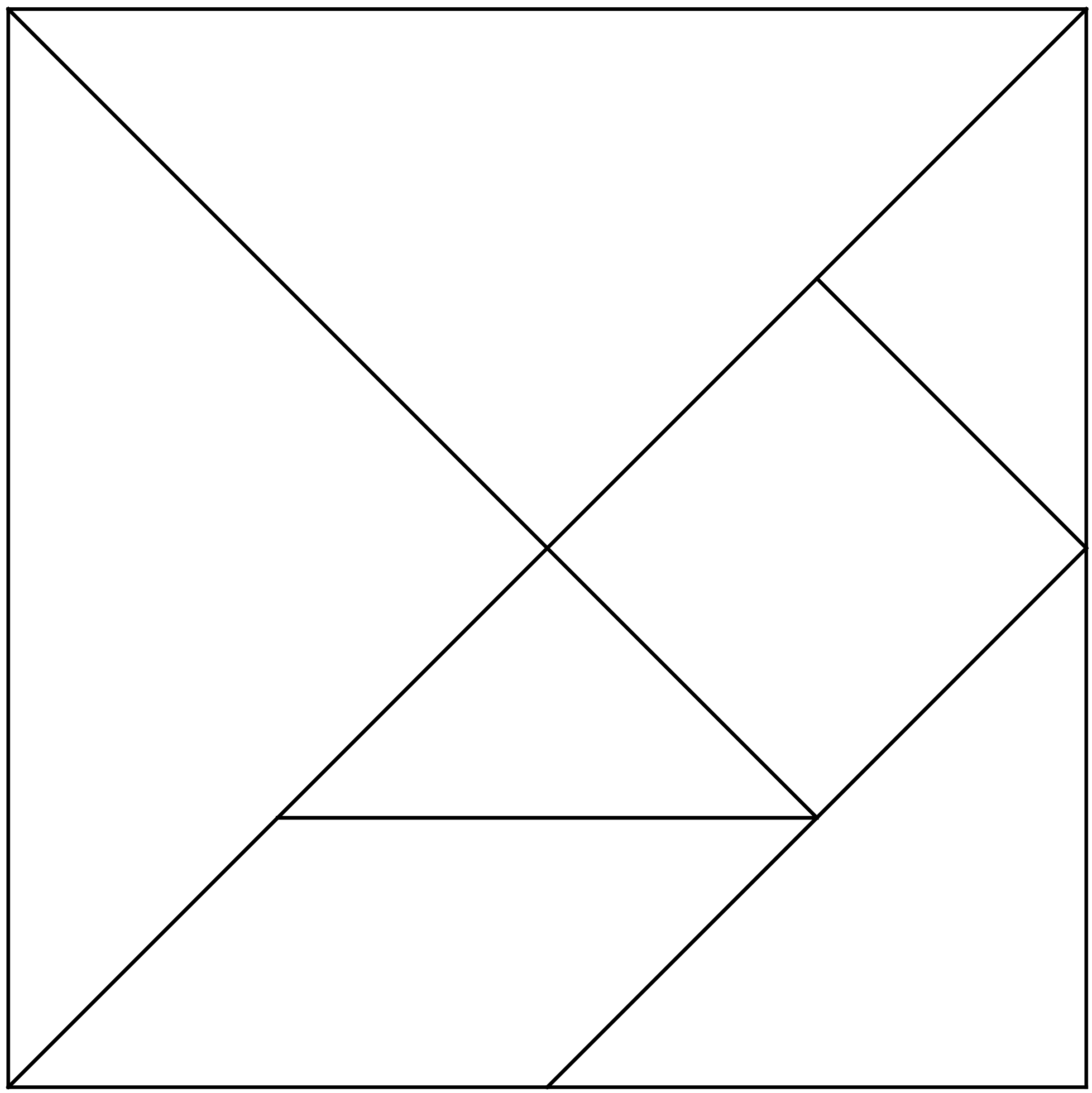




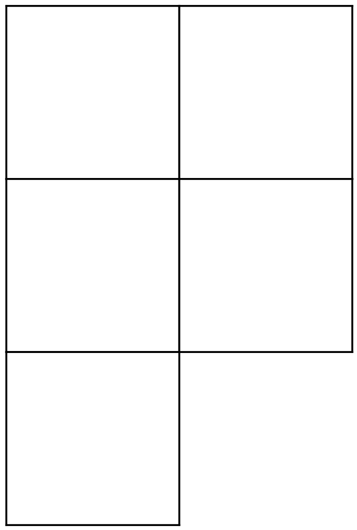


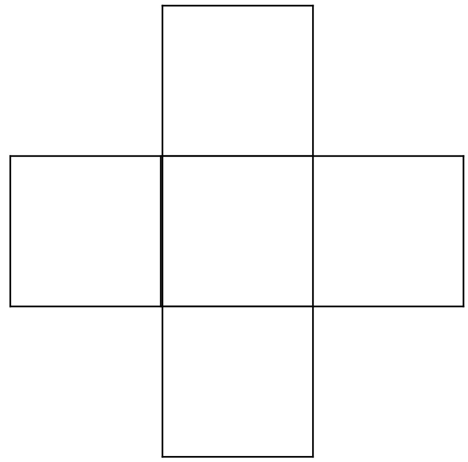


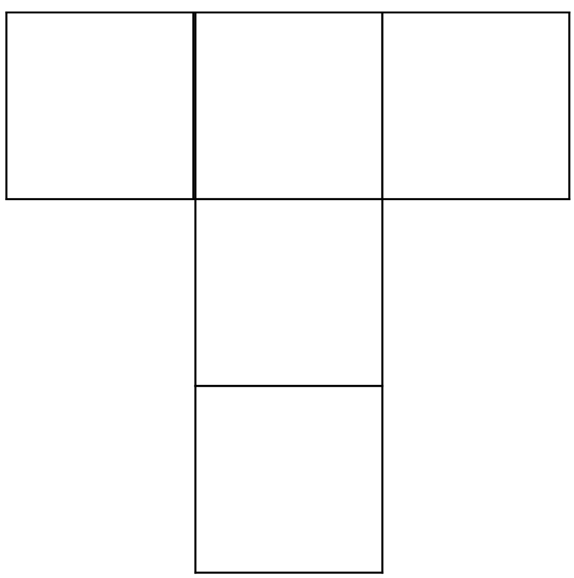


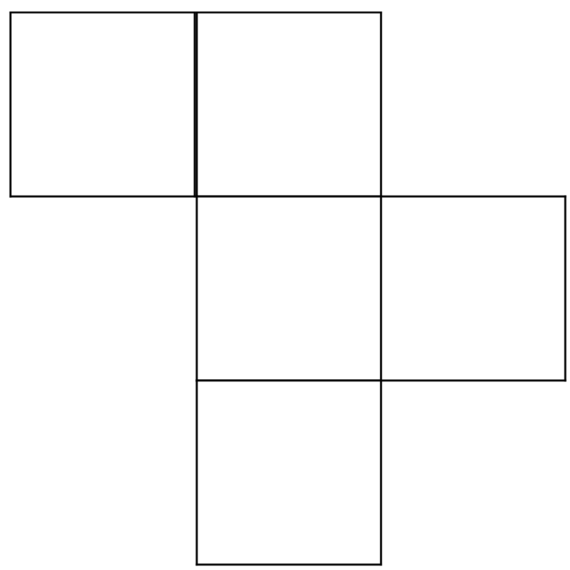


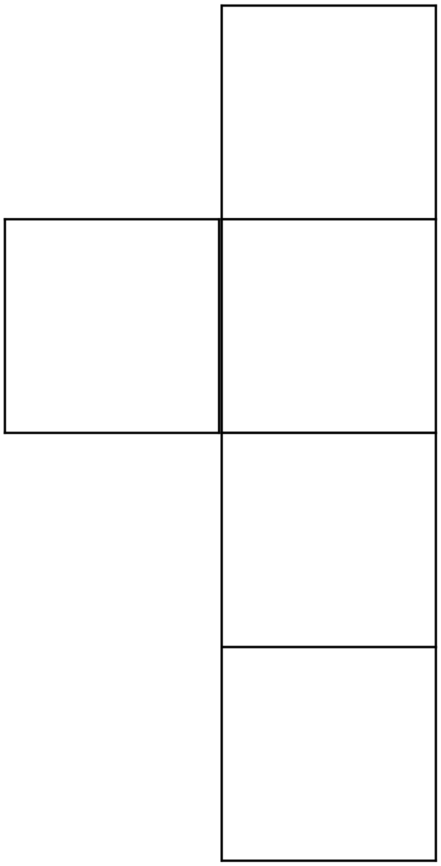
Pentominoes

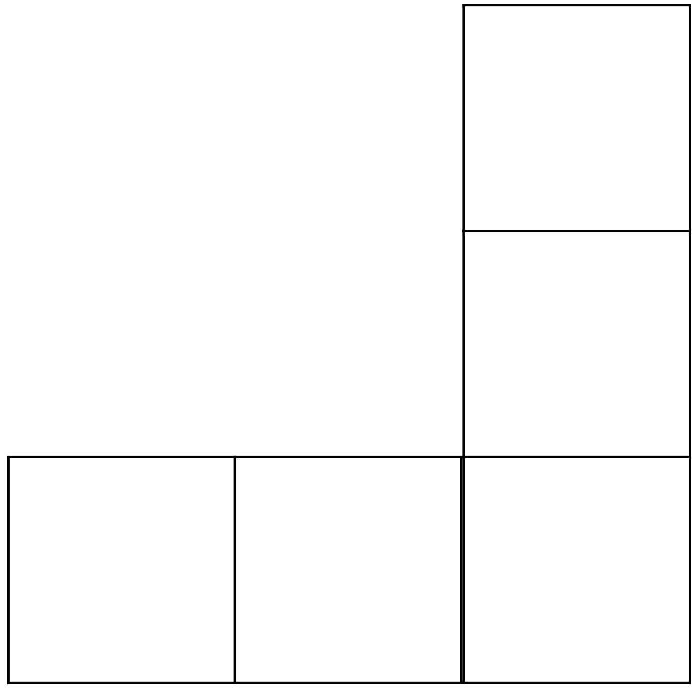


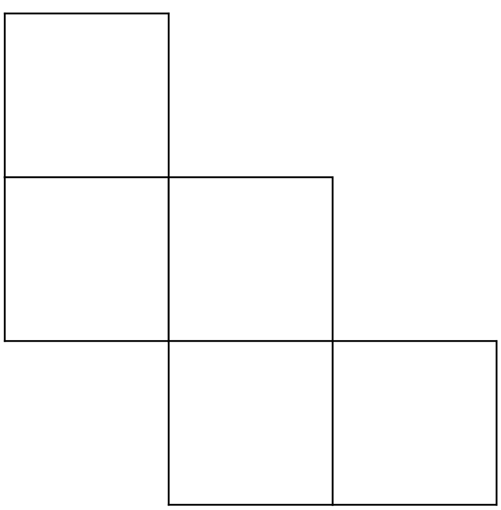


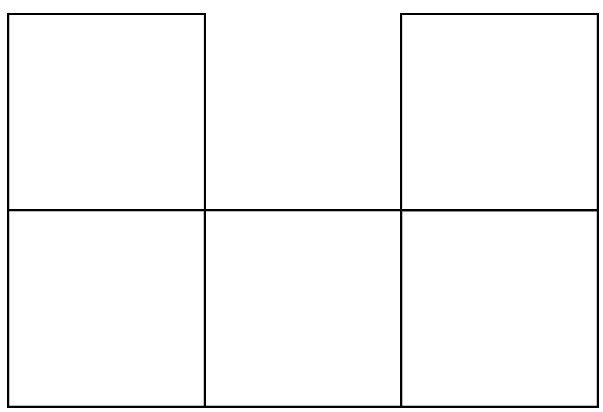


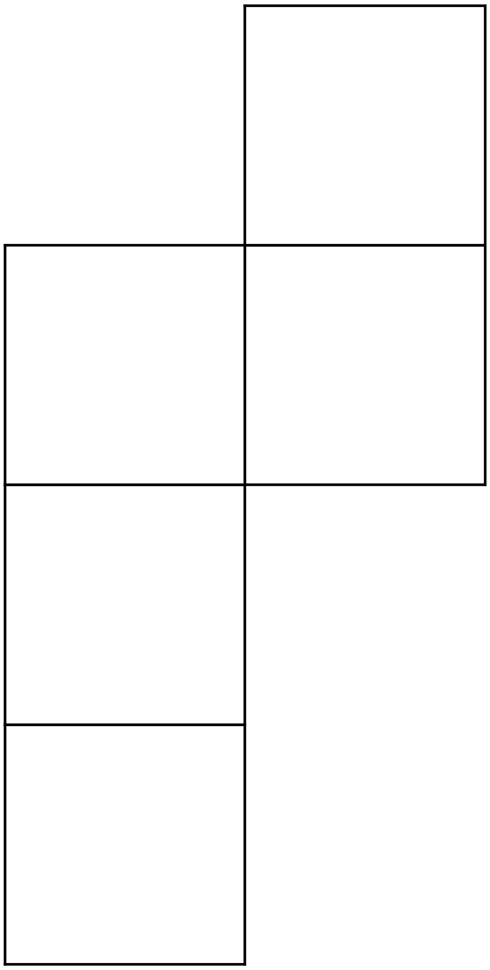


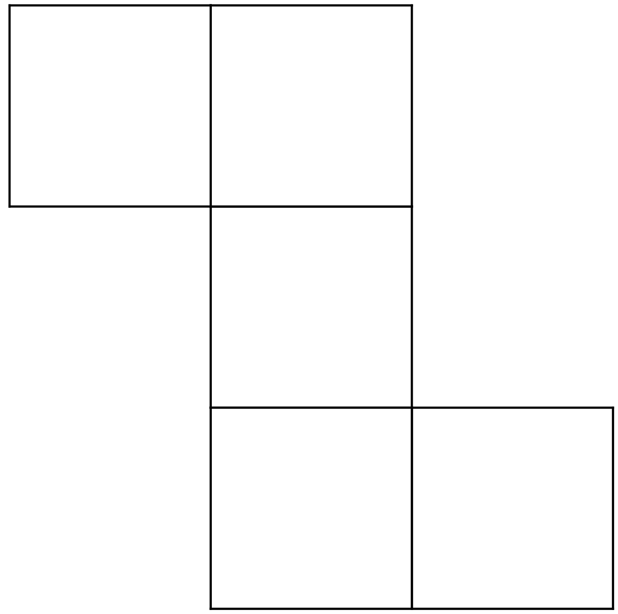


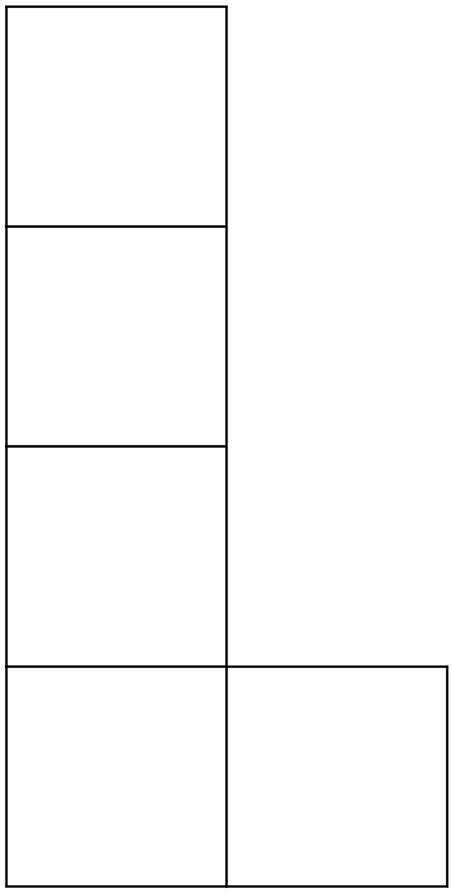


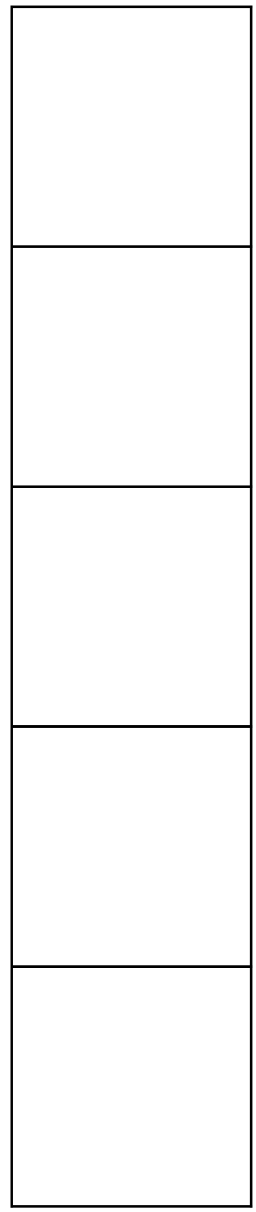












Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1