Mathematics 11

Guide

NOVA SCOTIA
NOUVELLE-ÉCOSSE

2014

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Mathematics 11

Prepared by the Department of Education and Early Childhood Development

This is the most recent version of the current curriculum materials as used by teachers in Nova Scotia.

Mathematics 11

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July 2014

CURRICULUM GUIDE

NOVA SCOTIA

Mathematics 11

Implementation Draft
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Contents

Introduction

Background and Rationale

Mathematics curriculum is shaped by a vision that fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society. It is essential that the mathematics curriculum reflect current research in mathematics instruction. To achieve this goal, Western and Northern Canadian Protocol's (WNCP) *The Common Curriculum Framework for Grades 10–12 Mathematics* (2008) has been adopted as the basis for the new mathematics curriculum in Nova Scotia.

The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan, and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and performance indicators agreed upon by the seven jurisdictions. The outcomes and performance indicators have been adapted for Nova Scotia. This document is based on both national and international research by the WNCP and the National Council of Teachers of Mathematics (NCTM).

There is an emphasis in the Nova Scotia curriculum on particular key concepts at each grade that will result in greater depth of understanding and, ultimately, stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

Purpose

This document provides sets of outcomes and performance indicators to be used as a mandated common base for defining mathematics curriculum expectations. This common base should result in consistent student outcomes in mathematics within the province of Nova Scotia. It should also enable easier transfer for students moving within the province or from any jurisdiction that has adopted the WNCP framework. This document is intended to clearly communicate to all education partners across the province the high expectations for students' mathematical learning.

Program Design and Components

Pathways

The *Common Curriculum Framework for Grades 10–12 Mathematics* (WNCP 2008), on which the Nova Scotia Mathematics 10–12 curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics* (WNCP 2006). In Nova Scotia, four pathways are available: Mathematics Essentials, Mathematics at Work, Mathematics, and Pre-calculus.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all four pathways are to provide prerequisite attitudes, knowledge, skills, and understandings for specific post-secondary programs or direct entry into the work force. All four pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents, and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour, and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of the Mathematics Essentials courses was designed in Nova Scotia to fill a specific need for Nova Scotia students. The content of each of the Mathematics at Work, Mathematics, and Pre-calculus pathways has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* (Alberta Education 2006) and on consultations with mathematics teachers.

Mathematics Essentials (Graduation)

This pathway is designed to provide students with the development of the skills and understandings required in the workplace, as well as those required for everyday life at home and in the community. Students will become better equipped to deal with mathematics in the real world and will become more confident in their mathematical abilities.

MATHEMATICS AT WORK (GRADUATION)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, and statistics and probability.

MATHEMATICS (ACADEMIC)

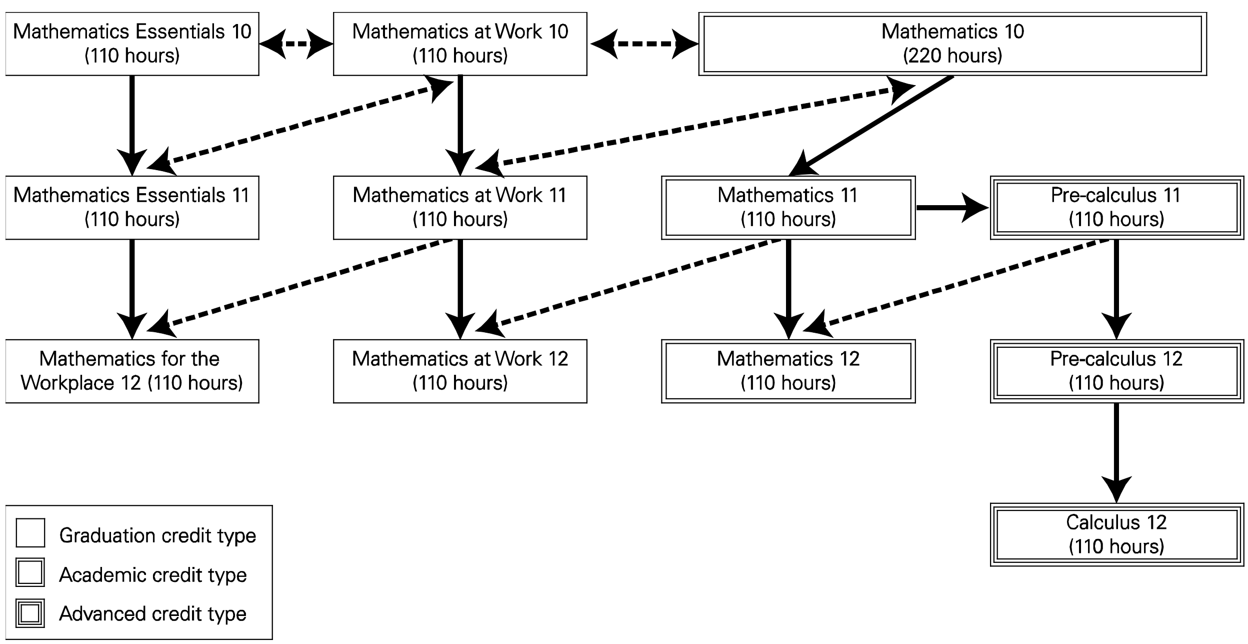
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that require an academic or pre-calculus mathematics credit. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, and statistics and probability. **Note:** After completion of Mathematics 11, students have the choice of an academic or pre-calculus pathway.

PRE-CALCULUS (ADVANCED)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Topics include algebra and number, measurement, relations and functions, trigonometry, and permutations, combinations, and binomial theorem.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.



Instructional Focus

Each pathway in senior high mathematics pathways is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful.

Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems, and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Assessment

Ongoing assessment for learning is essential to effective teaching and learning. Research has shown that assessment for learning (formative assessment) practices produce significant and often substantial learning gains, close achievement gaps, and build students' ability to learn new skills (Black & Wiliam 1998; OECD 2006). Student involvement in assessment promotes learning. Timely and effective teacher feedback and student self-assessment allow students to reflect on and articulate their understanding of mathematical concepts and ideas.

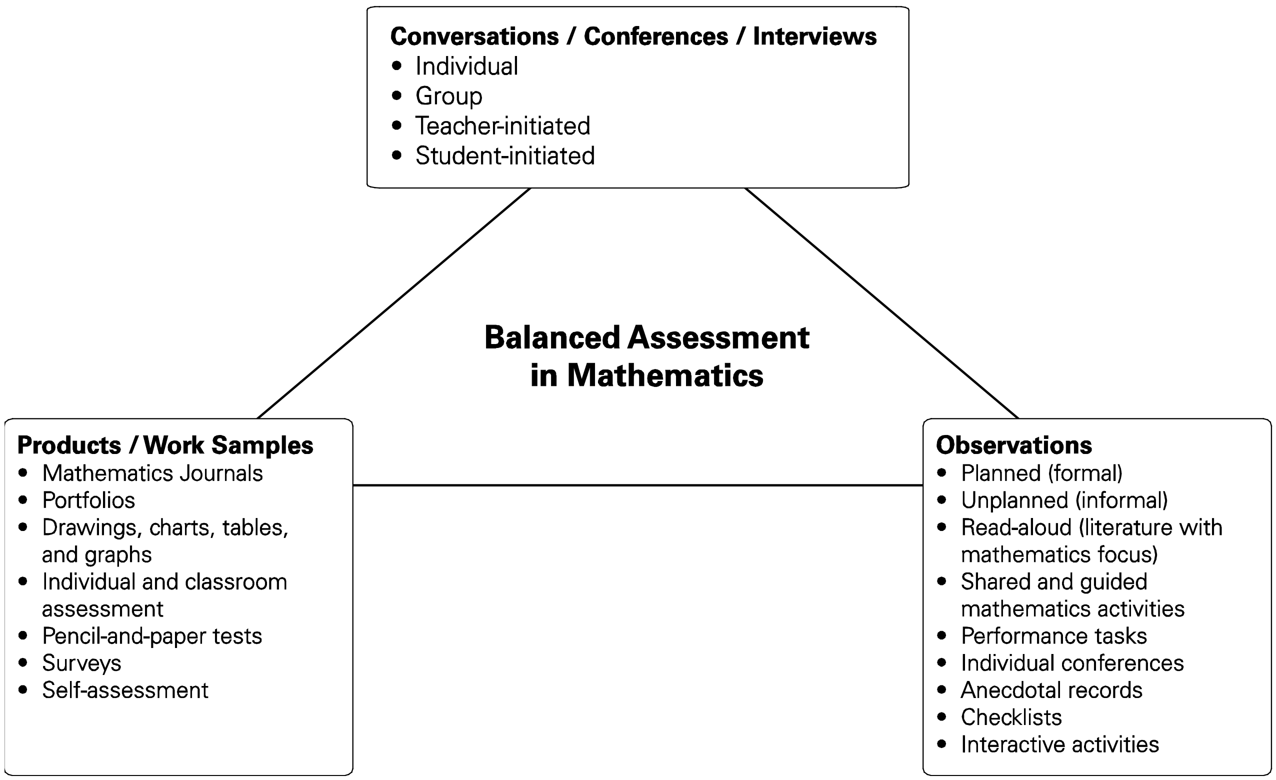
Assessment in the classroom includes

- providing clear goals, targets, and learning outcomes
- using exemplars, rubrics, and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning

(Davies 2000)

Assessment for learning practices act as the scaffolding for learning, which only then can be measured through assessment of learning (summative assessment). Assessment of learning tracks student progress, informs instructional programming, and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning, and produce achievement gains.

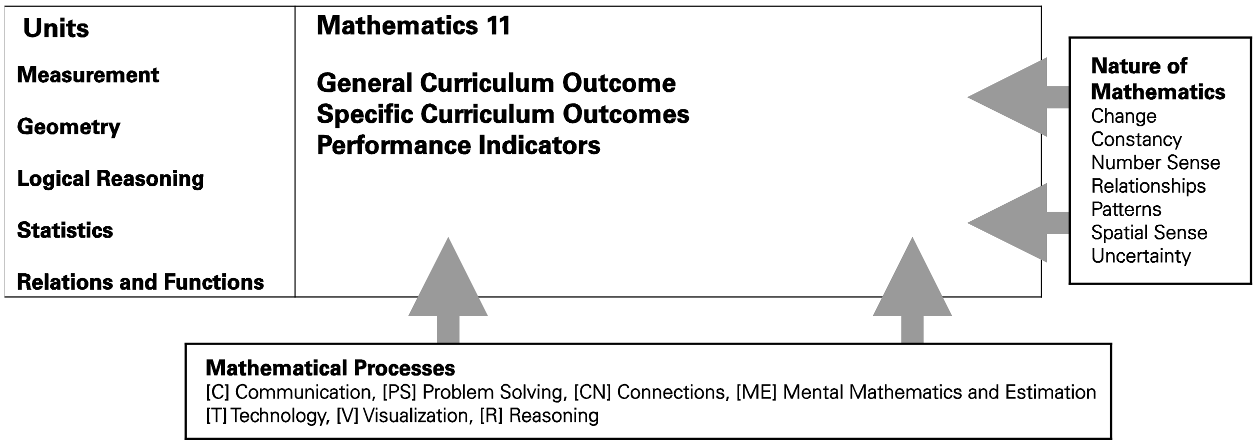
Assessment of student learning should
- align with curriculum outcomes
- clearly define criteria for success
- make explicit the expectations for students' performance
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction



Outcomes

Conceptual Framework for Mathematics 10–12

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



(Adapted with permission from Western and Northern Canadian Protocol, *The Common Curriculum Framework for K–9 Mathematics*, p. 5.)

Structure of the Mathematics 11 Curriculum

Units

Mathematics 11 comprises five units:
- Measurement (M) (15–20 hours)
- Geometry (G) (20–25 hours)
- Logical Reasoning (LR) (10 hours)
- Statistics (S) (20–25 hours)
- Relations and Functions (RF) (30–35 hours)

Outcomes and Performance Indicators

The Nova Scotia curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes, and performance indicators.

General Curriculum Outcomes (GCOs)

General curriculum outcomes are overarching statements about what students are expected to learn in each strand/sub-strand. The GCO for each strand/sub-strand is the same throughout the pathway.

Measurement (M)

Students will be expected to develop spatial sense and proportional reasoning.

Geometry (G)

Students will be expected to develop spatial sense.

Logical Reasoning (LR)

Students will be expected to develop logical reasoning.

Statistics (S)

Students will be expected to develop statistical reasoning.

Logical Reasoning (LR)

Students will be expected to develop algebraic and graphical reasoning through the study of relations.

Specific Curriculum Outcomes (SCOs) and Performance Indicators

Specific curriculum outcomes are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as expected for a given grade.

Performance indicators are samples of how students may demonstrate their performance of the goals of a specific curriculum outcome. The range of samples provided is meant to reflect the scope of the SCO. In the SCOs, the word **including** indicates that any ensuing items *must* be addressed to fully achieve the learning outcome. The phrase **such as** indicates that the ensuing items are provided for clarification only and are **not** requirements that must be addressed to fully achieve the learning outcome. The word **and** used in an outcome indicates that both ideas must be addressed to achieve the learning outcome, although not necessarily at the same time or in the same question.

MEASUREMENT (M)

M01 Students will be expected to solve problems that involve the application of rates.

Performance Indicators

M01.01 Interpret rates in a given context, such as the arts, commerce, the environment, medicine, or recreation.
M01.02 Solve a rate problem that requires the isolation of a variable.
M01.03 Determine and compare rates and unit rates.
M01.04 Make and justify a decision using rates.
M01.05 Represent a given rate pictorially.
M01.06 Draw a graph to represent a rate.
M01.07 Explain, using examples, the relationship between the slope of a graph and a rate.
M01.08 Describe a context for a given rate or unit rate.
M01.09 Identify and explain factors that influence a rate in a given context.
M01.10 Solve a contextual problem that involves rates or unit rates.

M02 Students will be expected to solve problems that involve scale diagrams, using proportional reasoning.

Performance Indicators

M02.01 Explain, using examples, how scale diagrams are used to model a 2-D shape or a 3-D object.
M02.02 Determine, using proportional reasoning, the scale factor, given one dimension of a 2-D shape or a 3-D object and its representation.
M02.03 Determine, using proportional reasoning, an unknown dimension of a 2-D shape or a 3-D object, given a scale diagram or a model.
M02.04 Draw, with or without technology, a scale diagram of a given 2-D shape according to a specified scale factor (enlargement or reduction).
M02.05 Solve a contextual problem that involves scale diagrams.

M03 Students will be expected to demonstrate an understanding of the relationships among scale factors, areas, surface areas, and volumes of similar 2-D shapes and 3-D objects.

Performance Indicators

M03.01 Determine the area of a 2-D shape, given the scale diagram, and justify the reasonableness of the result.
M03.02 Determine the surface area and volume of a 3-D object, given the scale diagram, and justify the reasonableness of the result.
M03.03 Explain, using examples, the effect of a change in the scale factor on the area of a 2-D shape.
M03.04 Explain, using examples, the effect of a change in the scale factor on the surface area of a 3-D object.
M03.05 Explain, using examples, the effect of a change in the scale factor on the volume of a 3-D object.
M03.06 Explain, using examples, the relationships among scale factor, area of a 2-D shape, surface area of a 3-D object and volume of a 3-D object.
M03.07 Solve a spatial problem that requires the manipulation of formulas.
M03.08 Solve a contextual problem that involves the relationships among scale factors, areas, and volumes.

GEOMETRY (G)

G01 Students will be expected to derive proofs that involve the properties of angles and triangles.

Performance Indicators

G01.01 Generalize, using inductive reasoning, the relationships between pairs of angles formed by transversals and parallel lines, with or without technology.
G01.02 Prove, using deductive reasoning, properties of angles formed by transversals and parallel lines, including the sum of the angles in a triangle.
G01.03 Generalize, using inductive reasoning, a rule for the relationship between the sum of the interior angles and the number of sides (n) in a polygon, with or without technology.
G01.04 Identify and correct errors in a given proof of a property involving angles.
G01.05 Verify, with examples, that if lines are not parallel the angle properties do not apply.
G01.06 Verify, through investigation, the minimum conditions that make a triangle unique.

G02 Students will be expected to solve problems that involve the properties of angles and triangles.

Performance Indicators

G02.01 Determine the measures of angles in a diagram that involves parallel lines, angles, and triangles and justify the reasoning.
G02.02 Identify and correct errors in a given solution to a problem that involves the measures of angles.
G02.03 Solve a contextual problem that involves angles or triangles.
G02.04 Construct parallel lines, using only a compass and straight edge or a protractor and straight edge, and explain the strategy used.
G02.05 Determine if lines are parallel, given the measure of an angle at each intersection formed by the lines and a transversal.

G03 Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case.

Performance Indicators

G03.01 Draw a diagram to represent a problem that involves the cosine law and/or sine law.
G03.02 Explain the steps in a given proof of the sine law and of the cosine law.
G03.03 Solve a problem involving the cosine law that requires the manipulation of a formula.
G03.04 Explain, concretely, pictorially or symbolically, whether zero, one or two triangles exist, given two sides and a non-included angle.
G03.05 Solve a problem involving the sine law that requires the manipulation of a formula.
G03.06 Solve a contextual problem that involves the cosine law and/or the sine law.

LOGICAL REASONING (LR)

LR01 Students will be expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems.

Performance Indicators

LR01.01 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
LR01.02 Explain why inductive reasoning may lead to a false conjecture.
LR01.03 Compare, using examples, inductive and deductive reasoning.
LR01.04 Provide and explain a counterexample to disprove a given conjecture.
LR01.05 Prove algebraic and number relationships, such as divisibility rules, number properties, mental mathematics strategies, or algebraic number puzzles.
LR01.06 Prove a conjecture, using deductive reasoning (not limited to two column proofs).
LR01.07 Determine if an argument is valid and justify the reasoning.
LR01.08 Identify errors in a given proof.
LR01.09 Solve a contextual problem involving inductive or deductive reasoning.

LR02 Students will be expected to analyze puzzles and games that involve spatial reasoning, using problem-solving strategies.

Performance Indicators

LR02.01 Determine, explain and verify a strategy to solve a puzzle or to win a game; for example,
- guess and check
- look for a pattern
- make a systematic list
- draw or model
- eliminate possibilities
- simplify the original problem
- work backward
- develop alternative approaches

LR02.02 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.
LR02.03 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

STATISTICS (S)

S01 Students will be expected to demonstrate an understanding of normal distribution, including standard deviation and *z*-scores.

Performance Indicators

S01.01 Explain, using examples, the meaning of standard deviation.
S01.02 Calculate, using technology, the population standard deviation of a data set.
S01.03 Explain, using examples, the properties of a normal curve, including the mean, median, mode, standard deviation, symmetry, and area under the curve.
S01.04 Determine if a data set approximates a normal distribution and explain the reasoning.
S01.05 Compare the properties of two or more normally distributed data sets.
S01.06 Explain, using examples that represent multiple perspectives, the application of standard deviation for making decisions in situations such as warranties, insurance, or opinion polls.
S01.07 Solve a contextual problem that involves the interpretation of standard deviation.
S01.08 Determine, with or without technology, and explain the *z*-score for a given value in a normally distributed data set.
S01.09 Solve a contextual problem that involves normal distribution.

S02 Students will be expected to interpret statistical data, using confidence intervals, confidence levels, and margin of error.

Performance Indicators

(It is intended that the focus of this outcome be on interpretation of data rather than on statistical calculations.)

S02.01 Explain, using examples, how confidence levels, margin of error, and confidence intervals may vary depending on the size of the random sample.
S02.02 Explain, using examples, the significance of a confidence interval, margin of error, or confidence level.
S02.03 Make inferences about a population from sample data, using given confidence intervals, and explain the reasoning.
S02.04 Provide examples from print or electronic media in which confidence intervals and confidence levels are used to support a particular position.
S02.05 Interpret and explain confidence intervals and margin of error, using examples found in print or electronic media.
S02.06 Support a position by analyzing statistical data presented in the media.

S03 Students will be expected to critically analyze society's use of inferential statistics.

Performance Indicators

S03.01 Investigate examples of the use of inferential statistics in society.
S03.02 Assess the accuracy, reliability, and relevance of statistical claims by
- identifying examples of bias and points of view
- identifying and describing the data collection methods
- determining if the data is relevant

S03.03 Identify, discuss, and present multiple sides of the issues with supporting data.

RELATIONS AND FUNCTION (RF)

RF01 Students will be expected to model and solve problems that involve systems of linear inequalities in two variables.

Performance Indicators

RF01.01 Model a problem, using a system of linear inequalities in two variables.
RF01.02 Graph the boundary line between two half planes for each inequality in a system of linear inequalities, and justify the choice of solid or broken lines.
RF01.03 Determine and explain the solution region that satisfies a linear inequality, using a test point when given a boundary line.
RF01.04 Determine, graphically, the solution region for a system of linear inequalities, and verify the solution.
RF01.05 Explain, using examples, the significance of the shaded region in the graphical solution of a system of linear inequalities.
RF01.06 Solve an optimization problem, using linear programming.

RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry.

Performance Indicators

(It is intended that completion of the square not be required.)
RF02.01 Determine, with or without technology, the intercepts of the graph of a quadratic function.
RF02.02 Determine, by factoring, the roots of a quadratic equation, and verify by substitution.
RF02.03 Determine, using the quadratic formula, the roots of a quadratic equation.
RF02.04 Explain the relationships among the roots of an equation, the zeros of the corresponding function, and the x-intercepts of the graph of the function.
RF02.05 Explain, using examples, why the graph of a quadratic function may have zero, one, or two *x*-intercepts.
RF02.06 Express a quadratic equation in factored form, using the zeros of a corresponding function or the *x*-intercepts of its graph.
RF02.07 Determine, with or without technology, the coordinates of the vertex of the graph of a quadratic function.
RF02.08 Determine the equation of the axis of symmetry of the graph of a quadratic function, given *x*-intercepts of the graph.
RF02.09 Determine the coordinates of the vertex of the graph of a quadratic function, given the equation of the function and the axis of symmetry, and determine if the *y*-coordinate of the vertex is a maximum or a minimum.
RF02.10 Determine the domain and range of a quadratic function.
RF02.11 Sketch the graph of a quadratic function.
RF02.12 Solve a contextual problem that involves the characteristics of a quadratic function.

Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.

Students are expected to
- communicate in order to learn and express their understanding of mathematics (Communication [C])
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines (Connections [CN])
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation [ME])
- develop and apply new mathematical knowledge through problem solving (Problem Solving [PS])
- develop mathematical reasoning (Reasoning [R])
- select and use technologies as tools for learning and solving problems (Technology [T])
- develop visualization skills to assist in processing information, making connections, and solving problems (Visualization [V])

The Nova Scotia curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning. The key to these process standards is presented in a box, as shown below, with each specific outcome within the units.

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics.

Students also need to communicate their learning using mathematical terminology. Communication can help students make connections between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, written and symbolic—of mathematical ideas. Students must communicate *daily* about their mathematics learning. This enables them to reflect, to validate, and to clarify their thinking and provides teachers with insight into students' interpretations of mathematical meanings and ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts.

When students encounter new situations and respond to questions of the type, How would you ... ? or How could you ... ?, the problem-solving approach is being modeled. Students develop their own problem-solving strategies by listening to, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families, or current events.

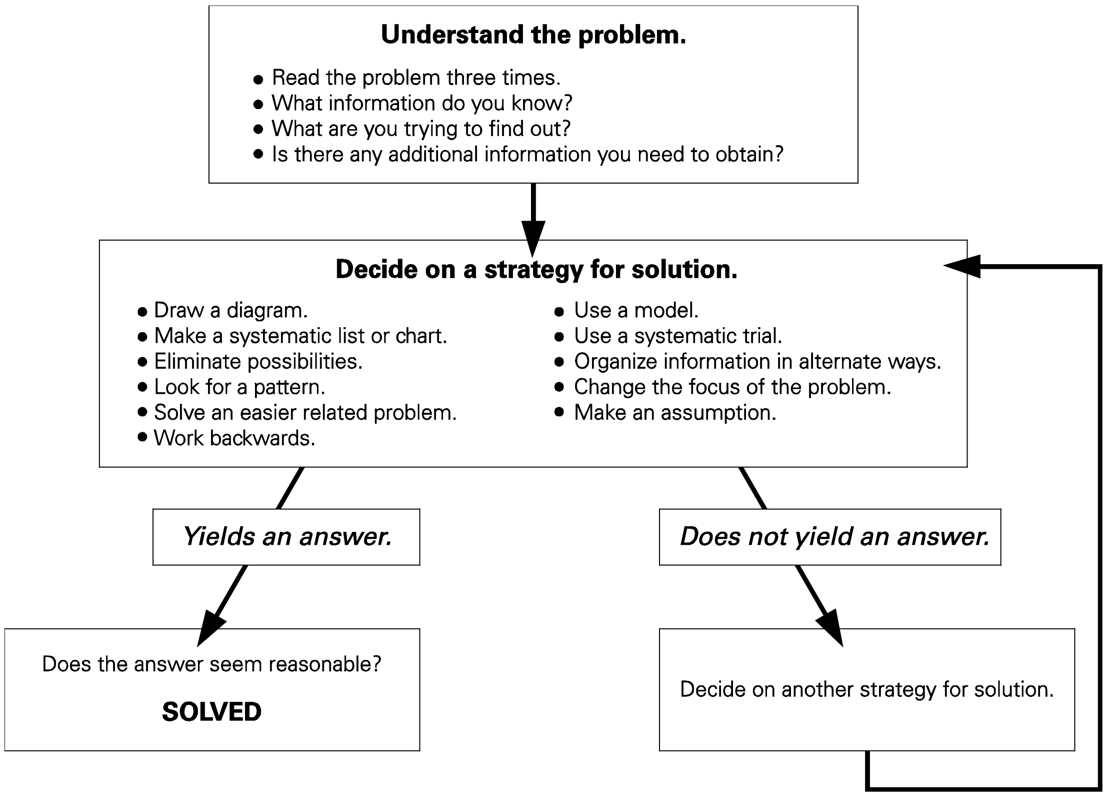
Both conceptual understanding and student engagement are fundamental in molding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill, or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive- and deductive-reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem, they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for, and engage in finding, a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

A possible flow chart to share with students is as follows:



Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to one another or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding. … *Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*." (Caine and Caine 1991, 5).

Mathematics should be viewed as an integrated whole rather than as the study of separate strands or units. Connections must also be made between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, and symbolic. The process of making connections, in turn, facilitates learning. Concepts and skills should also be connected to everyday situations and other curricular areas. For example, when developing literacy skills students learn to make text-to-world, text-to-text, and text-to-self connections. Students can also make connections to make mathematics come alive through math-to-world, math-to-math, and math-to-self connections.

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external aids.

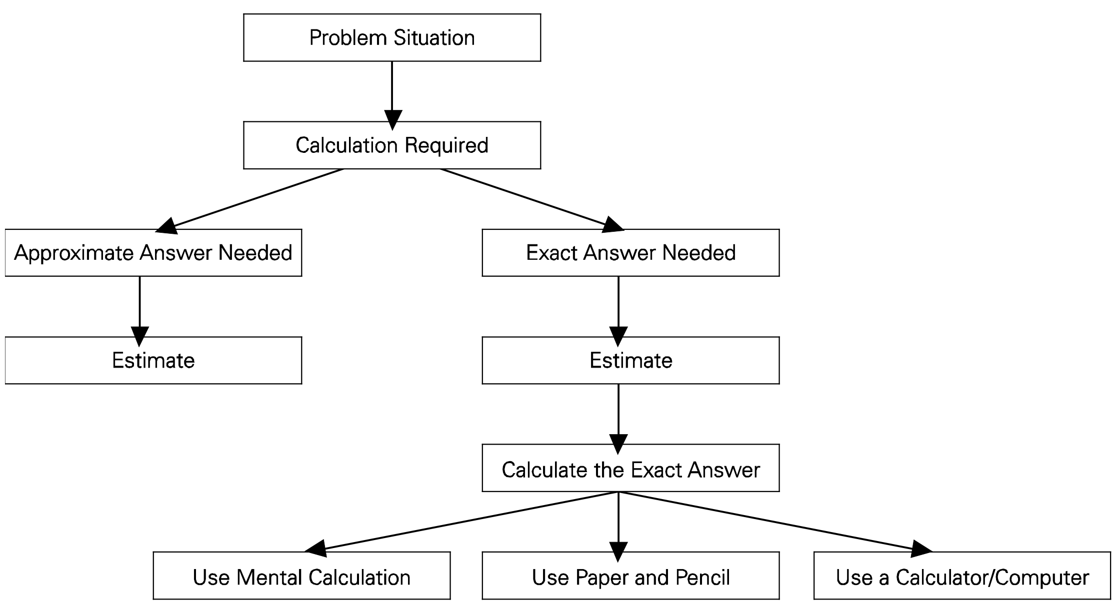
Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

“Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math.” (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers, and are more able to use multiple approaches to problem solving.” (Rubenstein 2001) Mental mathematics “provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers.” (Hope 1988, v)

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when, and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process as illustrated below.



The skill of estimation requires a sound knowledge of mental mathematics. Both are necessary to many everyday experiences, and students should be provided with frequent opportunities to practise these skills.

Technology [T]

Technology can be effectively used to contribute to and support the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Calculators, computers, and other technologies can be used to

- explore and represent mathematical relationships and patterns in a variety of ways
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of foundational concepts
- develop personal procedures for mathematical operations
- simulate situations
- develop number and spatial sense
- generate and test inductive conjectures

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization “involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world.” (Armstrong 1993, 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and to know several estimation strategies. (Shaw and Cliatt 1989, 150)

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations. It is through visualization that abstract concepts can be understood by the student. Visualization is a foundation to the development of abstract understanding, confidence, and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. Questions that challenge students to think, analyze, and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, Why do you believe that's true/correct? or What would happen if ...

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true.

Mathematics reasoning involves informal thinking, conjecturing, and validating—these help students understand that mathematics makes sense. Students are encouraged to justify, in a variety of ways, their solutions, thinking processes, and hypotheses. In fact, good reasoning is as important as finding correct answers. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics, and these are woven throughout this document. These components include change, constancy, number sense, relationships, patterns, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen 1990, 184).

Students need to learn that new concepts of mathematics as well as changes to previously learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers, and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms **stability**, **conservation**, **equilibrium**, **steady state** and **symmetry** (AAAS–Benchmarks 1993, 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.
- Lines with constant slope.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy. (British Columbia Ministry of Education, 2000, 146) Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities, and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and use benchmarks and referents. This results in students who are computationally fluent, flexible with numbers, and have intuition about numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables, and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory, or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create, and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

Curriculum Document Format

This guide presents the mathematics curriculum so that a teacher may readily view the scope of the outcomes that students are expected to achieve during that year. Teachers are encouraged, however, to examine what comes before and what follows, to better understand how students' learning at a particular grade level is part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes.

When a specific curriculum outcome is introduced, it is followed by the mathematical processes and performance indicators for that outcome. A scope and sequence is then provided, which relates the SCO to previous and next grade SCOs. Also for each SCO, there are background information, assessment strategies, suggested instructional strategies, and suggested models and manipulatives, mathematical vocabulary, and resource notes. For each section, the guiding questions should be used to help with unit and lesson preparation.

SCO

Mathematical Processes

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections
[ME] Mental Mathematics and Estimation		
[T] Technology	**[V]** Visualization	**[R]** Reasoning

Performance Indicators

Describes observable indicators of whether students have met the specific outcome.

Scope and Sequence

Previous grade or course SCOs	Current course SCO	Following grade or course SCOs

Background

Describes the "big ideas" to be learned and how they relate to work in previous grade and work in subsequent courses.

Assessment, Teaching, and Learning

Assessment Strategies

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Sample tasks that can be used to determine students' prior knowledge.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Some suggestions for specific activities and questions that can be used for both instruction and assessment.

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcome and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Suggestions for general approaches and strategies suggested for teaching this outcome.

SUGGESTED MODELS AND MANIPULATIVES

MATHEMATICAL VOCABULARY

Resources/Notes

Contexts for Learning and Teaching

Beliefs about Students and Mathematics Learning

"Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge." (National Council of Teachers of Mathematics 2000, 20).

- The Nova Scotia mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning that have grown out of research and practice. These beliefs include the following:
- Mathematics learning is an active and constructive process.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.
- Learners are individuals who bring a wide range of prior knowledge and experiences and who learn via various styles and at different rates.
- Leaning is most effective when standards of expectation are made clear with ongoing assessment and feedback.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best constructed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals, and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial, and symbolic representations of mathematics. The learning environment should value, respect, and address all students' experiences and ways of thinking so that students are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals of Mathematics Education

The main goals of mathematics education are to prepare students to
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- become mathematically literate adults, using mathematics to contribute to society
- commit themselves to lifelong learning

Students who have met these goals will

- gain understanding and appreciation of the contributions of mathematics as a science, a philosophy, and an art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding. Students should be encouraged to

- take risks
- think and reflect independently
- share and communicate mathematical understanding
- solve problems in individual and group projects
- pursue greater understanding of mathematics
- appreciate the value of mathematics throughout history

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for student success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations, and engage in reflective practices.

To experience success, students must be taught to set achievable goals and assess their progress as they work toward these goals. Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Engaging All Learners

*"*No matter how engagement is defined or which dimension is considered, research confirms this truism of education: *The more engaged you are, the more you will learn."* (Hume 2011, 6)

Student engagement is at the core of learning. Engagement in learning occurs when students are provided with opportunities to become more invested in their learning. This is critical for teachers to take into account when planning and implementing instruction. Effective instruction engages, embraces, and supports all learners through a range of learning experiences that are both age and developmentally appropriate.

This curriculum is designed to provide learning opportunities that are equitable, accessible, and inclusive of the many facets of diversity represented in today's classrooms. When teachers know their students as individual learners and as individual people, their students are more likely to be motivated to learn, persist in challenging situations, and apply reflective practices.

Supportive Learning Environments

A supportive and positive learning environment has a profound effect on students' learning. Students need to feel physically, socially, emotionally, and culturally safe in order to take risks with their learning. In classrooms where students feel a sense of belonging, see their teachers' passion for learning and teaching, are encouraged to actively participate, and are challenged appropriately, they are more likely to be successful.

Teachers recognize that not all students progress at the same pace nor are they equally positioned in terms of their prior knowledge of particular concepts, skills, and learning outcomes. Teachers are able to create more equitable access to learning when

- instruction and assessment are flexible and offer multiple means of representation
- students have options to engage in learning through multiple ways
- students can express their knowledge, skills, and understanding in multiple ways

(Hall, Meyer, and Rose 2012)

In a supportive learning environment, teachers plan learning experiences that support *each* student's ability to achieve curriculum outcomes. Teachers use a variety of effective instructional approaches that help students to succeed, such as

- providing a range of learning opportunities that build on individual strengths and prior knowledge
- providing all students with equitable access to appropriate learning strategies, resources, and technology
- involving students in the creation of criteria for assessment and evaluation
- engaging and challenging students through inquiry-based practices
- verbalizing their own thinking to model comprehension strategies and new learning
- balancing individual, small-group, and whole-class learning experiences
- scaffolding instruction and assignments as needed and giving frequent and meaningful descriptive feedback throughout the learning process
- integrating "blended learning" opportunities by including an online environment that extends learning beyond the physical classroom
- encouraging students to take time and to persevere, when appropriate, in order to achieve a particular learning outcome

Multiple Ways of Learning

"Advances in neuroscience and education research over the past 40 years have reshaped our understanding of the learning brain. One of the clearest and most important revelations stemming from brain research is that there is no such thing as a 'regular student.'" (Hall, Meyer, and Rose 2012, 2) Teachers who know their students well are aware of students' individual learning differences and use this understanding to inform instruction and assessment decisions.

The ways in which students make sense of and demonstrate learning vary widely. Individual students tend to have a natural inclination toward one or a few learning styles. Teachers are often able to detect learning strengths and styles through observation and through conversation with students. Teachers can also get a sense of learning styles through an awareness of students' personal interests and talents. Instruction and assessment practices that are designed to account for multiple learning styles create greater opportunities for all students to succeed.

While multiple learning styles are addressed in the classroom, the three most commonly identified are:

- auditory (such as listening to teacher-modelled think-aloud strategies or participating in peer discussion)
- kinesthetic (such as examining artifacts or problem-solving using tools or manipulatives)
- visual (such as reading print and visual texts or viewing video clips)

For additional information, refer to *Frames of Mind: The Theory of Multiple Intelligences* (Gardner 2007) and *How to Differentiate Instruction in Mixed-Ability Classrooms* (Tomlinson 2001).

A Gender-Inclusive Curriculum and Classroom

It is important that the curriculum and classroom climate respect the experiences and values of all students and that learning resources and instructional practices are not gender-biased. Teachers promote gender equity and inclusion in their classrooms when they

- articulate equally high expectations for all students
- provide equal opportunity for input and response from all students
- model gender-fair language, inclusive practices, and respectful listening in their interactions with students
- identify and openly address societal biases with respect to gender and sexual identity

Valuing Diversity: Teaching with Cultural Proficiency

"Instruction that is embedded in socially meaningful contexts, and tasks that are meaningful and relevant to the lives of students, will engage students in high-level problem-solving and reasoning and enhance students' engagement (Frankenstein 1995; Gutstein 2003; Ladson-Billings 1997; Tate 1995)." (Herzig 2005)

Teachers appreciate that students have diverse life and cultural experiences and that individual students bring different prior knowledge to their learning. Teachers can build upon their knowledge of their students as individuals, value their prior experiences, and respond by using a variety of culturally-proficient instruction and assessment practices in order to make learning more engaging, relevant, and accessible for all students. For additional information, refer to *Racial Equity Policy* (Nova Scotia Department of Education 2002) and *Racial Equity / Cultural Proficiency Framework* (Nova Scotia Department of Education 2011).

Students with Language, Communication, and Learning Challenges

Today's classrooms include students who have diverse language backgrounds, abilities, levels of development, and learning challenges. By observing and interacting with students and by conversing with students and/or their families, teachers gain deeper insights into the student as a learner. Teachers can use this awareness to identify and respond to areas where students may need additional support to achieve their learning goals. For students who are experiencing difficulties, it is important that teachers distinguish between those students for whom curriculum content is challenging and those for whom language-based factors are at the root of apparent academic difficulties. Students who are learning English as an additional language may require individual support, particularly in language-based subject areas, while they become more proficient in their English language skills. Teachers understand that many students who appear to be disengaged may be experiencing difficult life or family circumstances, mental health challenges, or low self-esteem, resulting in a loss of confidence that affects their engagement in learning. A caring, supportive teacher demonstrates belief in the students' abilities to learn and uses the students' strengths to create small successes that help nurture engagement in learning and provide a sense of hope.

STUDENTS WHO DEMONSTRATE EXCEPTIONAL TALENTS AND GIFTEDNESS

Modern conceptions of giftedness recognize diversity, multiple forms of giftedness, and inclusivity. Some talents are easily observable in the classroom because they are already well developed and students have opportunities to express them in the curricular and extracurricular activities commonly offered in schools. Other talents only develop if students are exposed to many and various domains and hands-on experiences. Twenty-first century learning supports the thinking that most students are more engaged when learning activities are problem-centred, inquiry-based, and open-ended. Talented and gifted students usually thrive when such learning activities are present. Learning experiences may be enriched by offering a range of activities and resources that require increased cognitive demand and higher-level thinking with different degrees of complexity and abstraction. Teachers can provide further challenges and enhance learning by adjusting the pace of instruction and the breadth and depth of concepts being explored. For additional information, refer to *Gifted Education and Talent Development* (Nova Scotia Department of Education 2010).

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in business education, career education, literacy, music, physical education, science, social studies, technology education, and visual arts.

Measurement
15–20 hours

GCO: Students will be expected to develop spatial sense and proportional reasoning.

SCO M01 Students will be expected to solve problems that involve the application of rates. [CN, PS, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M01.01 Interpret rates in a given context, such as the arts, commerce, the environment, medicine, or recreation.
M01.02 Solve a rate problem that requires the isolation of a variable.
M01.03 Determine and compare rates and unit rates.
M01.04 Make and justify a decision using rates.
M01.05 Represent a given rate pictorially.
M01.06 Draw a graph to represent a rate.
M01.07 Explain, using examples, the relationship between the slope of a graph and a rate.
M01.08 Describe a context for a given rate or unit rate.
M01.09 Identify and explain factors that influence a rate in a given context.
M01.10 Solve a contextual problem that involves rates or unit rates).

Scope and Sequence

Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
M02 Students will be expected to apply proportional reasoning to problems that involve conversions between SI and imperial units of measure. **FM01** Students will be expected to solve problems that involve unit pricing and currency exchange, using proportional reasoning.	**M01** Students will be expected to solve problems that involve the application of rates.	—

Background

In Mathematics 8, students explored the difference between a rate and a unit rate (8N05). In Mathematics 9, they used the concept of scale factor to create enlargements and reductions of 2-D shapes (9G03). In Mathematics 10, students explored the concept of slope as a measure of rate of change (RF03). In Mathematics 10 students also solved problems that involve unit pricing and currency exchange by using proportional reasoning (FM01).

In this unit, students will be expected to represent a rate in different ways. Students will also be expected to use rates to solve problems and make decisions.

Students will explore “rates of change” in a variety of practical situations such as trajectory, travel, and economic and population growth. They should begin with simple models represented by linear functions in which the average rate of change is constant; students should then progress to models represented by linear functions in which the rate of change is variable, which implies some sort of curve.

Students should explore some situations like heart rate, the rate at which runs are scored in a ball game, birth rate, population growth rate, and employment rate and learn how to calculate these rates.

For example, heart rate can be calculated by counting the number of beats in a time interval, divided by the time interval.

$$\text{heart rate} = \frac{\text{number of pulse beats in a time period}}{\text{time}}$$

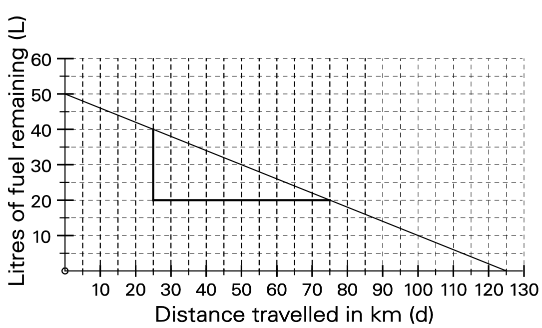
Rates are important because they tell students how one thing is changing in relation to another.

Students might read about an oil spill in the newspaper and how pollution control authorities monitor the spread by estimating the area covered by the oil at different times, then calculating the rate of spread in order to determine how fast the area of the oil slick is growing. Students can undergo a similar process to help them understand rates. Students might consider the rate of change in an oil spill's area of spread to be represented by

$$\frac{\Delta A}{\Delta t} = \frac{\text{change in area}}{\text{change in time}} = \frac{\text{ft.}^2}{\text{hour}}$$

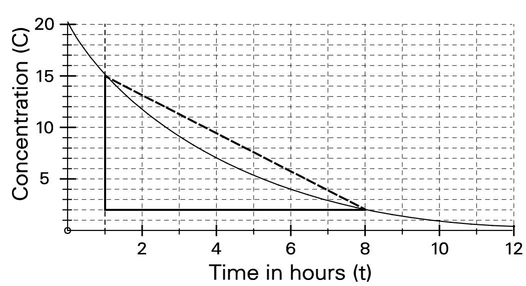
Specifically, if students were given the following information, $\frac{\Delta A}{\Delta t} = \frac{5\ \text{ft.}^2}{2\ \text{hour}} = 2.5\frac{\text{ft.}^2}{\text{hour}}$, it would mean that during the first 24-hour period after the spill, the oil slick's area was increasing at an average rate of 5 ft.^{2} every two hours or that it was increasing at an average rate of 2.5 ft.^{2} every hour.

Students should be able to discern different rates of change visually. Have students look at a graph and estimate how quickly the dependent value is changing with respect to the independent value. Students should also be able to determine if the dependent value is changing at the same rate all the time or if the rate of change varies over time. If a function is given by $y = f(x)$, then students should talk about "the average change in y with respect to x."



$$\frac{\Delta L}{\Delta d} = \frac{20-40}{75-25} = -\frac{20}{50} = -0.4\frac{L}{km}$$

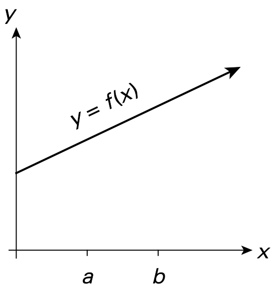
Constant change for all distances in domain.

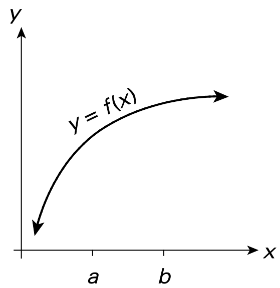


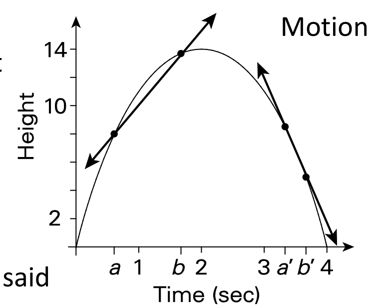
$$\frac{\Delta C}{\Delta t} = \frac{2-15}{8-1} = -\frac{13}{7} = -1.86\frac{\text{concentration}}{\text{hour}}$$

Non-constant change. Average change for time between 1 and 8 hours.

$\frac{\Delta y}{\Delta x}$ represents the change in *y* per unit change in *x*. If this change is constant, the relationship is linear. For non-linear functions, this average change will not be constant.







For non-linear functions, the rate of change is not the same everywhere. is often represented as a curve. If the graph slopes upward to the right, it means the rate of change is positive and that function is said to be increasing. For example, for each second that passes as a soccer ball is kicked into the air, the soccer ball's height continues to increase until the ball reaches its maximum height. At the maximum height, the rate of change is zero. As the ball begins to fall to the ground, the rate of change is negative. When a function's rate of change is negative, that function is said to be decreasing.

Students should recognize when they have been given a rate of change and what specific variables are being compared. For example, if students were told that air is being pumped into a spherical balloon at a rate of 20 in.^{3}/min., they should understand that the average rate of change is $\frac{\Delta V}{\Delta t}$ = 20 in.^{3}/min. and should be able to describe this change as an increase in volume of 20 in.^{3} every minute. If the function is linear, the rate is constant; otherwise the rate is changing over time and $\frac{\Delta V}{\Delta t}$ represents an average change rather than a constant change.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Serena used proportional reasoning to do the following conversion: 0.78 kg = ___ mg
  She wrote: $\frac{10\ 000 \text{ mg}}{1 \text{ kg}} = \frac{? \text{ mg}}{0.78 \text{ kg}}$
  Where did she make an error? Complete the conversion correctly.

- It takes Aisha 8 hours to wash 12 cars. How long will it take her to wash 21 cars?

- Vadim buys a package of 15 pencils for $4.50 at the corner store. Angela buys a box of 50 pencils at the grocery store for $14.00. Which is the better buy?

- Paper towels are sold in a 2-roll package for $2.49 and a 12-roll package for $12.99.
  (a) What package has the lower unit price?
  (b) If you need 12 rolls of paper towels, which is the better buy: one 12-roll package or six 2-roll packages?
  (c) When deciding which package size is the better buy for you, what should you consider in addition to unit price?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Each of your fingernails grows at about 0.05 cm/week. Each of your toenails grows at about 0.65 cm/year. Do your toenails or fingernails grow faster?

- The following table represents the average cost per litre of regular gasoline in Nova Scotia for the first six months of 2013.

Month	**January**	**February**	**March**	**April**	**May**	**June**
cost/volume ($/L)	121.2	123.6	131.4	139.2	133.1	131.9

  Using a graph, determine which two months had the smallest amount of decrease in the price of gas. Give reasons why you think this occurred.

- The average monthly temperature in °C for Truro, Nova Scotia is recorded in the chart below. The sampling period for this data covers 30 years.

Month	**Average Temperature (°C)**
January	– 6.9
February	– 6.5
March	– 1.8
April	3.9
May	9.8
June	14.7
July	18.4
August	17.8
September	13.4
October	7.7
November	2.8
December	– 3.5

  Using a graph, identify the two months with the greatest rate of change in temperature from one to the next. Which two months had the least rate of change?

- Betty earns $463.25 in five weeks. Assuming she does not receive any raises, how much will she earn in two years?

- A 12-bottle case of motor oil costs $41.88. A mechanic needs to order 268 bottles of motor oil. If she can only order by the case, how much money does she spend?

- A jet-ski rental operation charges a fixed insurance premium plus an hourly rate. The total cost for two hours is $50 and for five hours is $110. Determine the hourly rate to rent the jet-ski.

- As a custodian, John makes a cleaning solution by mixing 30 g of concentrated powdered cleanser into 2 L of water. At the same rate, how much powder will he need for 5 L of water?

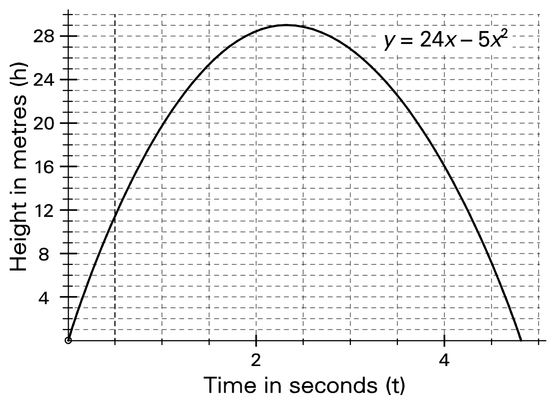
- An office has decided to track how much paper it uses to reduce waste. At the end of each month, the administrative assistant records the total number of sheets used and their weight. If paper weighs 10.8 lb. for every 500 sheets, how much will 700 sheets weigh?

- Saad jogs at a rate of 10 kmh. When he jogs at this rate for two hours, he burns 760 calories. Claudette jogs more slowly at 8 kmh, burning 150 calories in 30 minutes. If Saad jogs for three hours, how much longer will Claudette have to jog in order to burn the same number of calories?

- The low temperature in Summerside, Prince Edward Island, for a certain day was 9°C, at 4:30 a.m. The temperature then rose steadily at a constant rate until the high temperature of 25.5°C was recorded at 3:30 p.m. A weather forecaster predicted that the temperature would increase at the same rate for the next day, from a low of 11°C at 4:00 a.m. At that rate, what will be the temperature at 1:00 p.m. on the next day?

- When tested for diabetes, Sam was asked to consume a sweetened drink. After she finished drinking, her blood sugar level decreased slowly as time passed. Blood sugar concentration (*C*) is a function of time (*t*) in minutes.
  (a) Is the rate of change positive or negative over the first three hours?
  (b) Referring to the graph shown at the right, describe the rate of change of concentration in terms of time as positive or negative, and as constant or changing.
  (c) Estimate $\frac{\Delta C}{\Delta t}$ for the period from one hour to three hours after Sam finished drinking. Explain what it means.

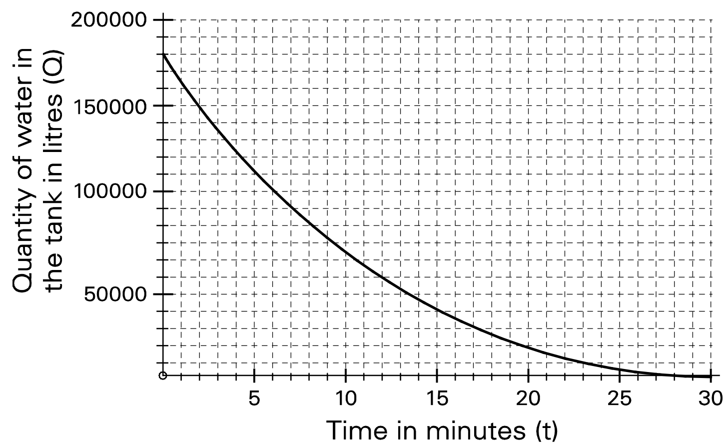
- A football is kicked. The height, *h*(*t*), of the football is depicted on the graph below. The height is in metres above the ground and the time is in seconds since the football was kicked.



  (a) What is the maximum height that the ball reaches?
  (b) When is the height of the ball increasing?
  (c) Referring to the graph shown above, describe the rate of change of height compared to time as positive or negative and as constant or changing.
  (d) What is the average rate of change of the ball's height for the interval [1, 4]?

- The number of litres of water in a tank (*Q*) can be described by the equation $Q(t) = 200(30 - t)^2$, where *t* is the number of minutes since the tank began to drain.
  (a) What is the average rate at which the water flows out during the first 10 minutes?
  (b) What is the average rate at which the water flows out during the 10-minute period from 10 minutes to 20 minutes?

(c) The graph of the function $Q(t)=200(30-t)^2$ is shown below. Explain how you could describe how the average rate at which the water flows out of the tank is changing as time progresses.



- A horse is running a five-furlong race. As the horse passes each furlong marker (*F*), a steward records the time elapsed (*t*) since the beginning of the race, as shown in the table below:

F	0	1	2	3	4	5
t (sec)	0	25	38	49	65	79

(a) How long does it take the horse to finish the race?
(b) What is the average speed of the horse during the last three furlongs of the race?
(c) During what part of the race is the horse running the fastest?

- Kim had a job mowing a lawn last Saturday. His employer paid him $50 to cut his lawn. Five hours later Kim was finished and walked home. On his way home, the mower sprung a leak and left a trail of gas. The tank was leaking 0.1 litres per minute, and the cost of gas is $1.339 per litre. If it took Kim one hour to get home, draw a graph that shows the relationship between time and net income for Kim's mowing.

- Kiyomi thinks that when finding the average rate of change, there is no difference whether the function is linear or non-linear. Would you agree with Kiyomi? Explain your thinking.

- The quantity (*Q*) of a chemical that is responsible for elasticity in human skin is given by $Q(t)=100-10\sqrt{t}$, where *t* is the age of a person.

(a) Find $Q(0)$, $Q(25)$, and $\dfrac{Q(25)-Q(0)}{25}$.
(b) Describe what each of the values in part (a) represent.
(c) Find $Q(49)$, $Q(81)$, and $\dfrac{Q(81)-Q(49)}{81-49}$.
(d) Describe what each of the values in part (c) represent.
(e) Describe how the average rate of change in elasticity in the human skin changes as a person ages.

- A 25-foot ladder is leaning against a vertical wall with its base two feet from the wall. The floor is slightly slippery and the base of the ladder slips farther away from the wall at the constant rate of 0.2 inches per second.
  (a) Draw a graph, with the axes labelled, that illustrates this rate of change.
  (b) Complete the following chart:

Time since ladder begins to slip (seconds)	Distance foot of ladder is from the wall (inches)	Size of angle that the ladder makes with the wall (degrees)
0	2	
1	2.2	
2		
3		
4		

  (c) For the time interval $0 \leq t \leq 3$, determine the average rate at which the angle that the ladder makes with the wall is changing. Include units.
  (d) For the time interval $0 \leq t \leq 4$, determine the average rate at which the angle that the ladder makes with the wall is changing. Include units.
  (e) Is the angle changing at a constant rate? Explain.

- A company's productivity (P) measured by the number of items produced is a function of the number of people (n) working on a job. For the interval [20, 50] the rate of change for P in terms of n is $\frac{\Delta P}{\Delta n} = 200$.
  (a) Draw a graph, with the axes labelled, that illustrates this rate of change.
  (b) Describe the meaning of the rate of change in terms of the number of people and the number of items produced.

- The number of four-litre containers of paint that a company sells (N) depends on the cost (c) of the container of paint. For prices between \$30 and \$45, the rate of change for the number of gallons sold in terms of the price charged per four-litre paint container can be described by $\frac{\Delta N}{\Delta c} = -\frac{500}{2}$.
  (a) Draw a graph, with the axes labelled, that illustrates this rate of change.
  (b) What does this rate of change mean in terms of the number of four-litre containers of paint the company sells and the cost of the container of paint?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Begin this lesson by giving students a graph, such as the one created by Pew Research Center about the use of social networking by age group, 2005–2013. This graph can be found on page 4 of their 2013 report, *72% of Online Adults Are Social Networking Site Users* (Brenner and Smith) at www.pewinternet.org/files/old-media/Files/Reports/2013/PIP_Social_networking_sites_update_PDF.pdf.
  Using the graph, discuss the change in use of social networking sites from 2005 to 2011.

  Pose the following questions to promote student discussion:
  - What does a positive slope represent? A negative slope?
  - What does a horizontal slope represent?
  - What does the steepness of the slope represent?
  - Locate the largest rate of change and explain what it represents.

- Students have previously been introduced to rate as a comparison between two things with different units. Invite students to talk about comparisons they make in their lives, such as fuel consumption, speed, prices in supermarkets, or monthly fees of a fitness centre.
  - Providing students the opportunity to work with rates in real-life scenarios should reinforce the students' understanding of rates, both their usefulness and their reasonableness. The focus here is to describe a situation in which a given rate might be used and is most useful. For example, ask students to answer questions such as the following if they were planning a road trip to Las Vegas:
    - Is it reasonable to discuss a road trip to Las Vegas in terms of m/s?
    - What rate(s) could be used to describe this road trip?
    - What factors might affect your rate of speed on this trip?
    - What factors might affect your fuel consumption?

- Students should be provided with several examples using various units and rates. Teachers should take this opportunity to work with problems involving rates that can be solved using equivalent ratios, proportional reasoning, and/or unit analysis. Examples may include the following:
  (a) During a Terry Fox Run, student volunteers distribute 250 mL cups of water to participants as they cross the finish line. Each volunteer has a cooler that can hold 64 L of water. How many cups of water can each volunteer dispense?
  (b) Loose-leaf paper costs $1.49 for 200 sheets or $3.49 for 500 sheets. What is the least you will pay for 100 sheets? 1600 sheets?

- In groups of two, ask one student to state the rate while the other student states it as a unit rate. Examples might include the following:

Rate	**Unit Rate**
Ten additional hats sell for every additional 120 people who attend the concert.	12 people to one hat
The amount of fuel in the car’s tank decreases by 4.2 litres every 100 km travelled.	0.042 litres per km or 23.8 km/litre

- In groups of two, ask one student to state the rate of change in words, while the other student states it using delta notation. Examples might include the following:

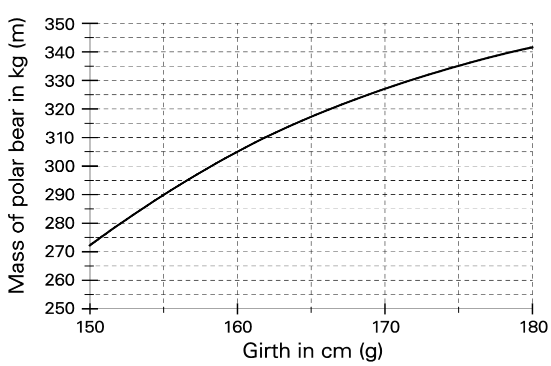
Rate (description)	**Rate (delta notation)**
The water flows into a conical water tank at a rate of 20 ft.^{3}/min.	$\frac{\Delta V}{\Delta t}$ = 20 ft.^{3}/min.
A man walks away from a lamp at a rate of 5 metres/min.	$\frac{\Delta d}{\Delta t}$ = 5 *m*/min.
A tutor is scheduling her time each week and considers a commitment of two hours per person she tutors.	$\frac{\Delta h}{\Delta p}$ = 2 hr./person
A child lets out the string of a kite at a rate of 2.5 ft./second.	$\frac{\Delta L}{\Delta t}$ = 2.5 ft./min.
Ten additional hats sell for every additional 120 people who attend the concert.	$\frac{\Delta h}{\Delta n} = \frac{1}{12}$ hats/person or $\frac{\Delta n}{\Delta h}$ = 12 persons/hat
The amount of fuel in the car’s tank decreases by 4.2 litres for every 100 km travelled.	$\frac{\Delta L}{\Delta km} = \frac{4.2}{100}$ litres/km or $\frac{\Delta km}{\Delta L}$ = 23.8 km/litre

- In groups of two, ask one student to state the rate using delta notation while the other student describes its meaning. Examples might include the following:

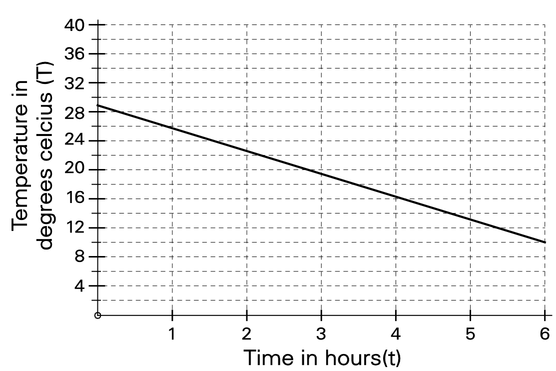
Rate (delta notation)	Rate (description)
$\frac{\Delta V}{\Delta t}$ = 20 ft.^{3}/min.	The volume is increasing by 20 ft.^{3} every minute, or there is an increase in volume of 20 ft.^{3} for every additional minute.
$\frac{\Delta d}{\Delta t}$ = 5 m/min.	The distance is changing by five metres every minute, or there is an increase of five metres in distance for every additional minute.
$\frac{\Delta h}{\Delta p}$ = 2 hr./person	The hours are changing by two for every person, or there is an additional two hours for every additional person.
$\frac{\Delta L}{\Delta t}$ = 2.5 ft./min.	The length is changing by 2.5 feet every minute, or there is an increase of 2.5 feet for every additional minute or the length increases by five feet every two minutes.
$\frac{\Delta h}{\Delta n} = \frac{1}{12}$ hats/person	There is one hat for every 12 people present, or one in 12 people wear a hat.
$\frac{\Delta n}{\Delta h}$ = 12 persons/hat	For every 12 people there is one hat, or 12 people share one hat.
$\frac{\Delta L}{\Delta km} = \frac{4.2}{100}$ litres/km	The car uses 4.2 litres of fuel for every 100 km driven, or the number of litres of fuel used increases by 4.2 every 100 km.
$\frac{\Delta km}{\Delta L}$ = 23.8 km/litre	The car drives 23.8 km using one litre of fuel.

- Present students with graphs such as the ones shown below and ask them to
  - describe the rate of change as positive or negative, and as constant or changing
  - calculate the rate of change for a specific interval
  - use delta notation to describe the average rate of change
  - state the meaning of the rate of change for a specific interval

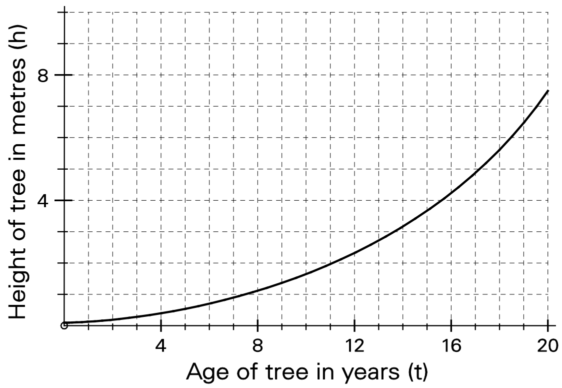
The mass of a polar bear in kg is shown in terms of its girth in cm.



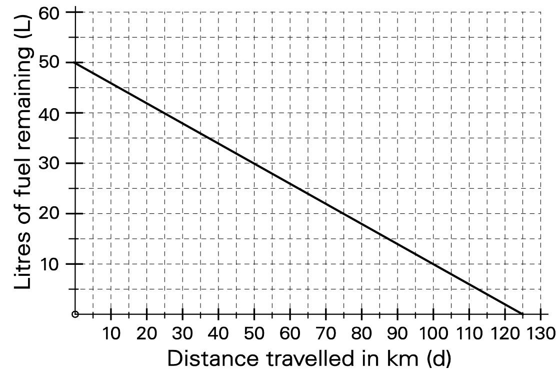
The temperature of food placed in cold storage.



Height of a particular type of tree in metres as a function of its age in years.



Number of litres of fuel remaining as a function of the number of km driven.



Suggested Models and Manipulatives

- graph paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- average rate of change
- constant change
- negative rate of change (decreasing)
- positive rate of change (increasing)
- unit rate

Resources/Notes

Internet

- *72% of Online Adults Are Social Networking Site Users* (Brenner and Smith 2013) www.pewinternet.org/files/old-media/Files/Reports/2013/PIP_Social_networking_sites_update_PDF.pdf.

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 8.1–8.2, pp. 452–469

Notes

SCO M02 Students will be expected to solve problems that involve scale diagrams, using proportional reasoning. [CN, PS, R, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M02.01 Explain, using examples, how scale diagrams are used to model a 2-D shape or a 3-D object.
M02.02 Determine, using proportional reasoning, the scale factor, given one dimension of a 2-D shape or a 3-D object and its representation.
M02.03 Determine, using proportional reasoning, an unknown dimension of a 2-D shape or a 3-D object, given a scale diagram or a model.
M02.04 Draw, with or without technology, a scale diagram of a given 2-D shape according to a specified scale factor (enlargement or reduction).
M02.05 Solve a contextual problem that involves scale diagrams

Scope and Sequence

Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
M02 Students will be expected to apply proportional reasoning to problems that involve conversions between SI and imperial units of measure.	**M02** Students will be expected to solve problems that involve scale diagrams, using proportional reasoning.	–

Background

In Mathematics 9, students were introduced to scale factors and scale diagrams (9SS03). Students explored the concepts of enlargements and reductions. They also determined the scale factor given the scaled diagram of two-dimensional images, and used a scale factor to create an image from its original figure.

In this unit, students will use scale diagrams involving 2-D shapes before moving on to 3-D objects. Although students were exposed to scale factors and scale diagrams of 2-D shapes in Mathematics 9, a review of these concepts is important. This is the first time that a variable, such as *k*, will be used to represent the scale factor, $k = \frac{\text{diagram measurement}}{\text{actual measurement}}$.
In this Measurement unit, using scale factors and measurements, students should be able to determine the dimensions of a reduced or enlarged object.

Consider the following example:

- A dinosaur model has a scale of 1:12. If the head of the dinosaur model is 8 cm in length, how long was the head of the real dinosaur?

Use questions such as the following to promote student discussion:

- What is the scale factor?
- What does this value mean?
- Will the value of the scale factor result in an enlargement or reduction? Explain.
- How did you determine the length of the head of the dinosaur?

Students should recognize that for each 1 cm the model measures, the corresponding part of the dinosaur measures 12 cm. They may set up and solve a proportion equation or they may simply recognize the dinosaur is 12 times the length of the model dinosaur.

Ask students to determine scale factors from a variety of sources, such as models and scale diagrams, by using corresponding lengths. As students work through different examples, they should recognize that the scale factor for an enlargement is greater than one and the scale factor of a reduction is between 0 and 1. When working with problems, students should be exposed to scale factors in a variety of forms including decimals, fractions, and percentages. If available, students should be encouraged to explore scale diagrams with the use of technology such as Geometer's Sketchpad (Key Curriculum Press 2013), Google SketchUp (SketchUp 2013) or GeoGebra (International GeoGebra Institute 2013).

Review of the conversion of different units in both the metric and imperial systems may be necessary. It is also important to emphasize the meaning of units so students better understand the concept of scale and can gain a visual appreciation of the object that is being reduced or enlarged.

Students will extend their work with scale factors and scale diagrams of 2-D shapes to scale factors and scale diagrams of 3-D objects. This will be their first exposure to relating scale factors to a 3-D object.

Students will use a scale factor to determine unknown measurements of similar 3-D objects. This would be a good opportunity to bring in a regular-size cereal box and a similar jumbo-size cereal box. Students can create nets of the boxes and record the measurements. Ask if the boxes are similar and why. Are the dimensions related by a scale factor? Students should recognize that corresponding measurements of the boxes (length, width, and height) are proportional.

Students should also be able to use a given scale factor to determine the unknown dimensions of a 3-D object. If the dimensions of a scale drawing of a patio chair are 2 cm × 1.5 cm × 4 cm, for example, and a scale factor of 1:30 is applied, ask them to determine the actual dimensions of the patio chair.

It is important to use various everyday objects and apply a given scale factor when asking students to determine the dimensions of the actual object or its image.

Note: While scale factors may sometimes be written as 1 cm = 2 km for example, this does not mean that these are equivalent, and it could be more accurately written as 1 cm represents 2 km or 1 cm: 2 km.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

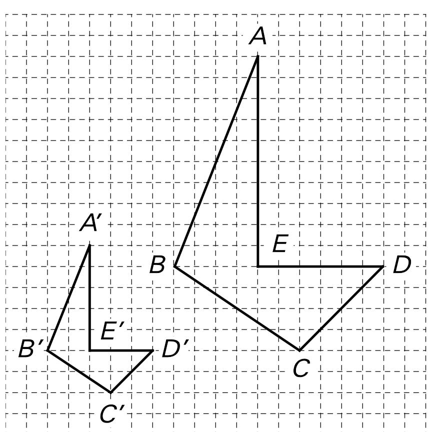
- Filipo bought a second-hand treadmill online. It would only register in miles. Describe a conversion factor that could be used to estimate a conversion from miles to kilometres or vice versa.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

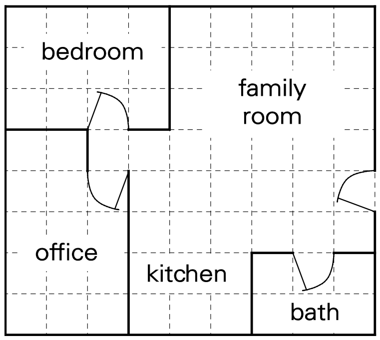
- A circle has been transformed so that its image radius is 14 cm. If the scale factor is 0.4, what is the radius of the original circle?

- Determine the scale factor from the diagram below.

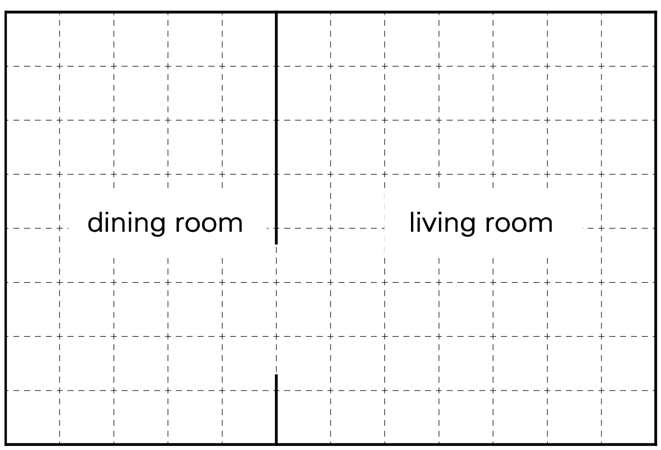


- ΔABC is similar to ΔSTU. If the sides of ΔABC measure 5.2, 3.8, and 6.4 respectively, and the scale factor is 2:3, what is the perimeter of ΔSTU?

- In the following blueprint, each grid mark represents 4 feet.
  (a) Ceramic tile costs $5 per square foot. How much would it cost, before tax, to purchase tiles for the bathroom?
  (b) Carpet costs $18 per square yard. The bedroom and living room are to be carpeted. How much would the carpet cost, before tax?
  (c) Which has a higher unit cost, the tile or the carpet? Explain.



- Louis intends to have the floor of the dining room in his rental property redone with ceramic tiles that come in one-foot wide squares. He has a scale diagram of the dining room and living room, shown below, where one unit represents two feet. A 12-pack of squares sells for $38.40 and single squares can be purchased for $3.70 each. Determine the cost, in dollars, for Louis to buy just enough squares to cover his dining room floor?



- During an art class, students are projecting the image of a can of evaporated milk on the wall. The projector applies a scale factor of 250%. If the can has a diameter of 10 cm and a height of 12.5 cm, what are the dimensions of the image on the wall?

- Tony drew a scale diagram of his new skateboard to show a friend. He used a scale factor of 0.4. The scaled diagram has dimensions 3.2 in. × 1.8 in. × 10.8 in. What are the dimensions of the skateboard?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Examples of real-world applications, such as maps, sewing patterns, car models, doll house furniture, and construction blueprints, would be an effective way to introduce this concept and capture students' interest. Students could be given a road map, the price of gasoline per litre, and the miles-per-gallon value for a vehicle and asked to determine an estimate of the cost for a trip between two towns.

- Refer to the emergency exit plan that is posted in each classroom in your building.
  - Ask students how they could determine if the schematic is drawn to scale. You could ask students to make this determination and justify their answer.
  - If the schematic is drawn to scale, ask them to explain how they could determine what scale was used and then to find the scale used.

- An appropriate task would involve students designing a floor plan of their dream bedroom, kitchen, or a room of their choice. They will have to consider what to include in their drawing, the scale they will use, and the measurements needed. Students must also realize that it is important for the measurements to be realistic. If a blueprint is using a scale of 1 in. representing 1 ft., for example, the doorway should be at least 3 in. wide on the drawing so that in real-life it would be 3 ft. wide.

Consider asking the following questions to guide students with their design:

- How are the dimensions recorded on the diagram?
- How are the doors, windows, closets, and walls represented?
- Is the scale indicated?
- Should a key be included to identify the symbols used in the drawing?
- Will furniture be included?

Completed floor plans can be posted around the classroom for students to see other examples.

- Have students bring in some models, such as small versions of cars, trucks, airplanes, dolls, figurines, or action figures. Ask them to do some research to determine the scale factor that such models represent.

SUGGESTED MODELS AND MANIPULATIVES

- cereal boxes and other objects of various sizes
- toy cars, trucks, airplanes, dolls, doll furniture, or action figures

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- scale factor

Resources/Notes

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 8.3–8.6, pp. 474–510

Notes

SCO M03 Students will be expected to demonstrate an understanding of the relationships among scale factors, areas, surface areas, and volumes of similar 2-D shapes and 3-D objects. [C, CN, PS, R, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M03.01 Determine the area of a 2-D shape, given the scale diagram, and justify the reasonableness of the result.
M03.02 Determine the surface area and volume of a 3-D object, given the scale diagram, and justify the reasonableness of the result.
M03.03 Explain, using examples, the effect of a change in the scale factor on the area of a 2-D shape.
M03.04 Explain, using examples, the effect of a change in the scale factor on the surface area of a 3-D object.
M03.05 Explain, using examples, the effect of a change in the scale factor on the volume of a 3-D object.
M03.06 Explain, using examples, the relationships among scale factor, area of a 2-D shape, surface area of a 3-D object and volume of a 3-D object.
M03.07 Solve a spatial problem that requires the manipulation of formulas.
M03.08 Solve a contextual problem that involves the relationships among scale factors, areas, and volumes.

Scope and Sequence

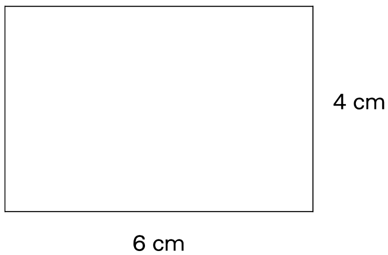
Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
M03 Students will be expected to solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including right cones, right cylinders, right prisms, right pyramids, and spheres.	**M03** Students will be expected to demonstrate an understanding of the relationships among scale factors, areas, surface areas, and volumes of similar 2-D shapes and 3-D objects.	–

Background

In Mathematics 10, students solved problems related to surface area and volume of 3-D objects (M03). It would be beneficial to review the surface area and volume formulas of 3-D objects such as a rectangular prism, right cylinder, right cone, right pyramid, and sphere.

In this Measurement unit, students will focus on the relationship between scale factor and area of similar 2-D shapes. They will solve problems that involve scale factor, surface area, and volume of 3-D objects.

Students will analyze how area is affected when the lengths of shapes are enlarged or reduced by a particular scale factor. This may be a good opportunity to review area formulas for shapes such as a parallelogram, triangle, rectangle, and circle. The following rectangle could be used as an illustration.



Students will explore the relationship between scale factor and area. They should use different scale factors such as 2, 3 or 0.5. Consider the following questions for discussion:

- What is the area of the original rectangle?
- What are the dimensions and area of the resulting similar rectangles?
- What do you notice?

Students should observe that the resulting areas are not directly proportional to the lengths. When students double the sides of a rectangle, for example, the area does not just double, it quadruples.

It is important for students to recognize that the scale factor is applied to each dimension of the 2-D shape. As a result, the area will change by a factor of k^2. (The new area will be k^2 times the original area.)

Therefore, $k^2 = \dfrac{\text{area of similar 2D shape}}{\text{area of original shape}}$.

Using manipulatives such as linking cubes, students will investigate the relationship between the scale factor and the surface area of two similar 3-D objects, in addition to investigating the relationship between the scale factor and the volume of two similar 3-D objects.

Students are expected to use the dimensions of a scale diagram of a 3-D object as well as the scale factor to determine the surface area and volume of the enlarged/reduced object.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Calculate the volume of a cylinder with a height of 10 cm and a diameter of 20 cm.

- How does the volume of a sphere with a radius of four inches compare to a sphere with a radius of eight inches?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- A 4 in. × 6 in. picture has dimensions that have been tripled. What is the area of the new picture?

- Chad and Sheina painted a mural on the wall, measuring 12 ft. × 8 ft. using an overhead projector. If the original sketch had an area of 216 in.^{2}, what is the scale factor?

- Find at least three examples of enlargements or reductions in real-world objects. Estimate their scale factors.

- Dashiell claims that when you enlarge every side of a cube *n* times, its volume also increases *n* times. Jane says that the volume of a cube increases 3*n* times, and Élaine is convinced that the volume increases n^3 times. Who do you agree with and why?

- The surface area of a cone is 36 ft.^{2}. What is the surface area of its image if a scale factor of 1:4 is applied?

- Find the volume of a cylinder if its image has a volume of 450 cm^3 and a scale factor of 2:3. Round your answer to the nearest cubic centimetre.

- What is the scale factor of the following pairs of similar spheres?
  - Volume of the original is 450 mm^3 and its image is 1518.75 mm^3.
  - Surface area of the original is 248 in.^{2} and its image is 126.5306 in.^{2}.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

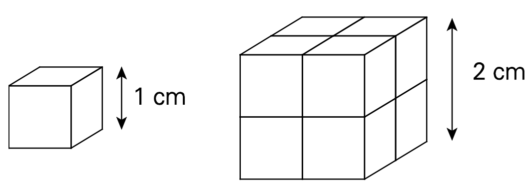
Consider the following sample instructional strategies when planning lessons.

- Begin by giving examples of careers that use scale models. Professionals such as architects, engineers, filmmakers, and sales associates use scale-size models to do their jobs. Show pictures of these examples or bring in actual examples for the students to observe.

- Videos can be used to stimulate student interest (such as the DVD extras showing the making of films where scale models were used—*Star Wars*, *Lord of the Rings*, etc.). Scale factor is normally represented in science in terms of a fraction equal to a number less than one, reduced so that the numerator is a one (i.e., $\frac{1}{20}$ scale). However, it can also be expressed as an integer multiplier (i.e., a scale factor of 20). It may be simpler to work with the latter idea, but students and teachers should be comfortable working with either conceptualization.

- The use of manipulatives is very important as students discover the impact on the surface area and volume when the dimensions of an object are multiplied by a scale factor.

  – Ask students to measure the dimensions of one linking cube. The volume of such a cube is 1 cm^3 while its surface area is 6 cm^2.

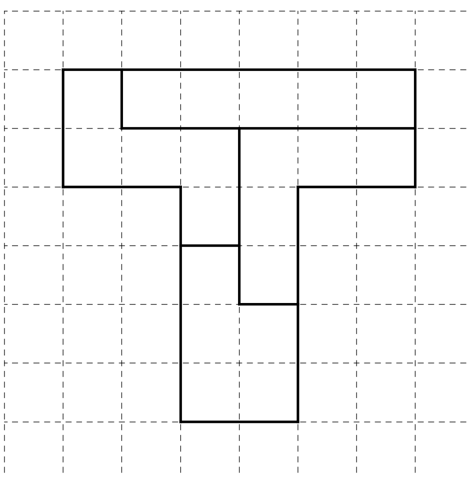


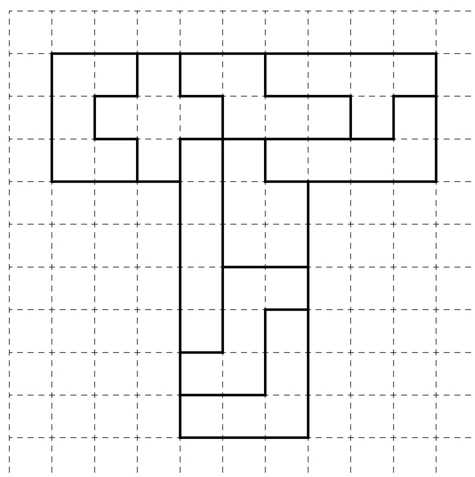
– Encourage students to work in groups, build various cubes using pre-determined scale factors, find the surface area and volume, and make the connection to its scale factor. Information could be organized in a chart.

Scale Factor	**Length**	**Width**	**Height**	**Surface Area**	**Volume**
1	1	1	1	6	1
2	2	2	2	24 (changes by a factor of 4)	8 (changes by a factor of 8)
3	3	3	3	54 (changes by a factor of 9)	27 (changes by a factor of 27)
4	4	4	4	96	64

– Ask students what will happen if the length of each edge of the cube is doubled. [They should observe that the volume of the cube changes by a factor of 8 or 2^3, while the surface area only changes by a factor of 4 or 2^2.]
– Ask students what will happen to the surface area and the volume of the original cube if the length of every edge of the original cube triples.
– Students should recognize that when the dimensions of similar 3-D objects are related by a scale factor k, their surface areas are related by k^2 and their volumes are related by k^3.

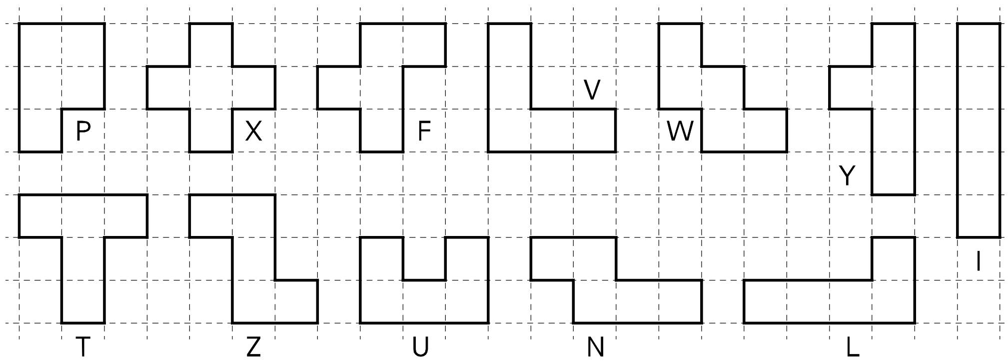
- Ask students to build the scale models of the T pentomino pieces, shown below.





Ask them to describe the scale factor for each and then compare their areas.

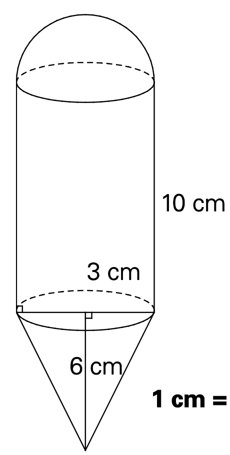
Note: The 12 Pentominoes are shown below.



Use some of the following challenges to integrate LR02 (*Analyze puzzles and games that involve spatial reasoning, using problem-solving strategies.*) into this outcome.

- Using the four pentominoes named L, N, P, and V, create a shape that is double the dimensions of the Z pentomino.
- Using the four pentominoes named L, N, P, and V, create a shape that is double the dimensions of the U pentomino.
- Using the four pentominoes named L, N, P, and V, create a shape that is double the dimensions of the L pentomino in two different ways.
- Using the four pentominoes named L, N, P, and Y, create a shape that is double the dimensions of the N pentomino.
- Using the four pentominoes named I, L, N, and P, create a shape that is double the dimensions of the L pentomino.
- Not every pentomino can be doubled. For example, the pentominoes named V and X cannot be doubled, although the rest can. Find solutions for doubling as many of the remaining pentominoes as you can?
- Using all the pentominoes except those named U, X, and Y, create a shape that is triple the dimensions of the L pentomino.
- Using all the pentominoes except those named T, W, and Z, create a shape that is triple the dimensions of the W pentomino.
- Using all the pentominoes except those named L, W, and Y, create a shape that is triple the dimensions of the Z pentomino.
- Using all the pentominoes except those named F, I, and X, create a shape that is triple the dimensions of the X pentomino.
- The dimensions of all the pentominoes can be tripled. Find solutions for tripling as many of the remaining pentominoes as you can.

- Once students have discovered the impact of scale factor on surface area and volume, post a question such as the following:
  - Consider the scaled-down diagram of the storage tank to the right.
  - Guide students through the process using the following instructions and questions.
    (a) Find the total surface area of the original tank.
    (b) Find the total volume of the original tank.
    (c) Was it necessary to determine the dimensions of the original drawing to answer the above questions?
    (d) How would you determine the surface area and volume of the original if the scale diagram was not given?



1 cm = 1.5 m

SUGGESTED MODELS AND MANIPULATIVES

- linking cubes
- pentominoes

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- scale
- scale diagram
- scale factor

Resources/Notes

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Section 8.4, pp. 483–490
  - Section 8.6, pp. 502–510

Software

- Geometer's Sketchpad (Key Curriculum Press 2013) (NSSBB #: 50474, 50475, 51453)
- SketchUp 2013 (Google.com 2013)
  www.sketchup.com
- GeoGebra (International GeoGebra Institute 2013)
  www.geogebra.org/cms/en

Notes

Geometry
20–25 hours

GCO: Students will be expected to develop spatial sense.

SCO G01 Students will be expected to derive proofs that involve the properties of angles and triangles. [CN, R, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(*It is intended that deductive reasoning be limited to direct proof.*)

G01.01 Generalize, using inductive reasoning, the relationships between pairs of angles formed by transversals and parallel lines, with or without technology.

G01.02 Prove, using deductive reasoning, properties of angles formed by transversals and parallel lines, including the sum of the angles in a triangle.

G01.03 Generalize, using inductive reasoning, a rule for the relationship between the sum of the interior angles and the number of sides (n) in a polygon, with or without technology.

G01.04 Identify and correct errors in a given proof of a property involving angles.

G01.05 Verify, with examples, that if lines are not parallel the angle properties do not apply.

G01.06 Verify, through investigation, the minimum conditions that make a triangle unique.

Scope and Sequence

Mathematics 10	Mathematics 11	Mathematics 12
RF08 Students will be expected to solve problems that involve the distance between two points and the midpoint of a line segment.	**G01** Students will be expected to derive proofs that involve the properties of angles and triangles.	**LR03** Students will be expected to solve problems that involve conditional statements.

Background

Students have had no formal instruction with proof in previous mathematics courses. If students have completed the Logical Reasoning (SCO LR01) outcome prior to beginning this unit, they will have had experience with making, testing, and proving conjectures.

Students form **conjectures** through the use of **inductive reasoning** and then prove their conjectures through the use of **deductive reasoning**.

A **conjecture** is a testable expression that is based on available evidence but is not yet proven.

Inductive reasoning is a form of reasoning in which a conclusion is reached based on a pattern present in numerous observations. The premise makes the conclusion likely, but does not guarantee it to be true.

Deductive reasoning is the process of coming up with a conclusion based on information that is already known to be true. The facts that can be used to prove a conclusion deductively may come from accepted definitions, properties, laws, or rules. The truth of the premises guarantees the truth of the conclusion.

Students are introduced to inductive reasoning through the investigation of geometric situations and are introduced to conjectures through the observation of patterns. They will be required to justify the reasoning used for all conjectures.

Students will be presented with situations to explore and will be encouraged to develop conjectures based on these situations. They will also explore the role that **counterexamples** play in disproving conjectures.

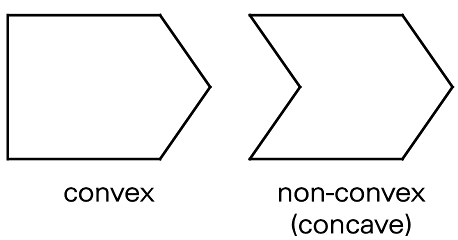
In the Geometry and Logical Reasoning units students are introduced to the notion of formal **proof** including the **two-column proof** format and other formats.

When constructing proofs, students must use proper mathematical terminology. Students have been exposed to the Pythagorean theorem, the number system, and divisibility rules in previous grades. There are some concepts, however, that will need to be introduced, such as **supplementary angles**, **complementary angles**, and the **transitive property**.

In Grade 6, students learned that the sum of the angles in a triangle totals 180° and that the sum of the angles in a quadrilateral is 360°. They will now use deductive reasoning to prove that this is true for all triangles and for all quadrilaterals. They will also use deductive reasoning to determine a formula for finding the sum of the angles in a polygon with *n* sides.

Students should distinguish between convex and non-convex (concave) polygons. The focus, however, will be on convex polygons.

Convex polygon: A polygon in which each interior angle measures less than 180°.



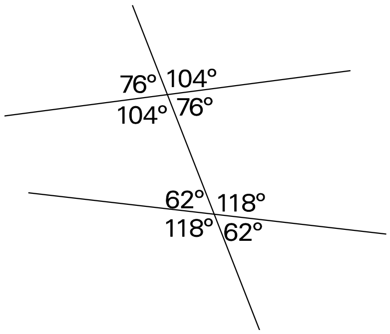
In Grade 7, students identified parallel and perpendicular lines. They also used various strategies to draw a line segment that was perpendicular (or parallel) to a given line segment.

Students are not familiar with the terms **complementary angles**, **supplementary angles**, or **vertically opposite angles**.

Students are expected to discover relationships both with and without technology. Software programs such as FX Draw (Efofex Software 2013) and Geometer's Sketchpad (Key Curriculum 2013) or an interactive whiteboard could also be used to develop an understanding of the angle relationships. The construction of a geometric situation, either electronically or by hand, is as powerful for learning as the analysis of the situation.

Students will be introduced to the term transversal before developing the various angle properties. They will discover that vertically opposite angles are congruent and that when two parallel lines are intersected by a transversal, the following are true: corresponding angles are congruent; alternate interior angles are congruent; alternate exterior angles are congruent; and interior angles on the same side of the transversal are supplementary.

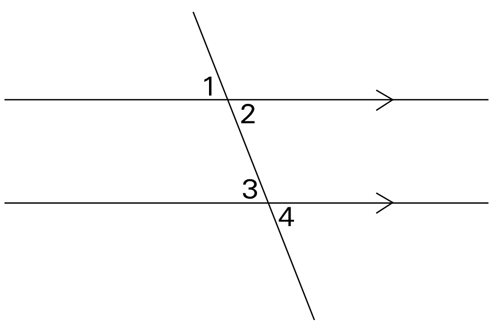
Students should recognize that when a transversal intersects a pair of non-parallel lines, the angle properties do not apply except for vertically opposite angles.



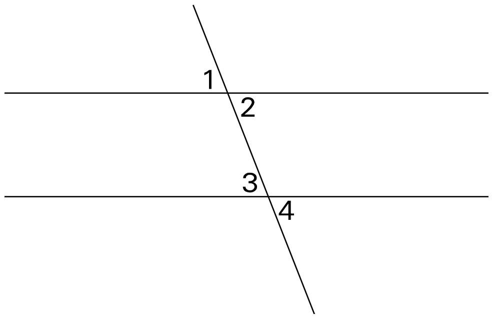
Once students have explored the angle relationships when two parallel lines are cut by a transversal, they will use their knowledge of corresponding angles, vertically opposite angles, and supplementary angles to formally prove the other relationships, such as alternate interior angles. A **two-column proof** and a **paragraph proof** are two of the most common strategies used to construct proofs involving properties of angles formed by transversals and parallel lines.

Teachers could ask students to make a conjecture that involves alternate exterior angles formed by parallel lines and a transversal. Students may conjecture that alternate exterior angles are equal and use the following to prove their conjecture.

Statement	**Justification**
$\angle 1 = \angle 2$	vertically opposite angles
$\angle 2 = \angle 4$	corresponding angles
$\angle 1 = \angle 4$	transitive property



In both the Geometry and Logical Reasoning units, students are expected to identify errors in reasoning and proofs (both formal and informal) such as the one shown below.



Statement	Reason
$\angle 1 = \angle 2$	Vertically opposite angles are congruent.
$\angle 2 = \angle 3$	Alternate interior angles are congruent.
$\angle 1 = \angle 3$	Both equal $\angle 2$.
$\angle 3 = \angle 4$	Vertically opposite angles are congruent.
$\angle 1 = \angle 4$	Both equal $\angle 3$.

In Grades 8 and 9, students were exposed to the properties of congruent and similar polygons.

The intent of this Geometry unit is for students to investigate what makes a triangle unique. It is not the aim of the unit to have students prove triangles congruent.

Using the investigative approach, students should recognize that there are four sets of conditions that guarantee the uniqueness of a triangle:

- side-side-side (SSS)
- side-angle-side (SAS)
- angle-side-angle (ASA)
- side-angle-angle (SAA)

Teachers should not present these as theorems without students having discovered the requirements for a triangle to be unique first. If the lengths of three sides of a triangle are given, for example, only one unique triangle can be constructed. As a result, any triangles constructed with these side lengths will be a replica of the given triangle and will therefore be congruent to the given triangle.

Students should also be given time to investigate other relationships. Students may determine the side-side-angle (SSA) only works for a right triangle, known as the hypotenuse-leg (HL) theorem. Although this theorem is not part of this outcome, discussion may be warranted here as to why it works for a right triangle. They will also determine that the angle-angle-angle (AAA) property only indicates similarity not congruency.

The SSA non-unique case is also called the ambiguous case, and this will be considered in detail when the sine law is studied (SCO G03).

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

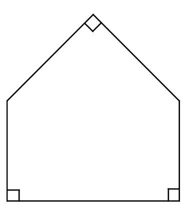
- How can you determine if two lines are parallel?
- Draw a quadrilateral with two parallel sides and use a protractor to measure the angles in the quadrilateral.
- In what real-life situations do you see parallel lines?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Find an object that models lines cut by a transversal. Sketch the object and highlight the various pairs of angles that are formed. State which angles are congruent and why you can be certain that they are without measuring them.

- In baseball, the home plate is shaped like the one shown. It has 3 right angles and 2 other congruent angles (A and B). Find the measures of $\angle A$ and $\angle B$.



- Using at least three examples, verify inductively that the conjectures regarding the sum of interior angles of a polygon are valid for convex polygons.

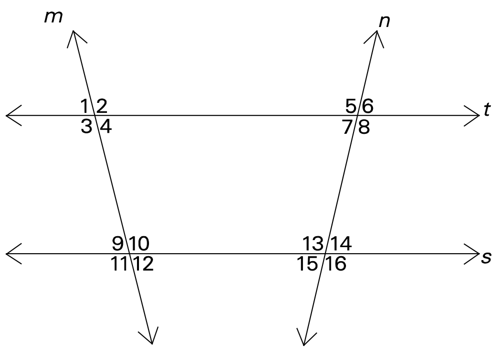
- Draw ΔDEF with DF = 8 cm, $\angle D$ = 35° and $\angle F$ = 95°. Is your triangle unique? Explain. Compare your triangle with several classmates to check.

- When asked to construct ΔPQR with PQ = 12", QR = 15", and $\angle D$ = 27°, Marie and Jackie drew congruent triangles. Are there any other possible triangles with these measurements that are not congruent to the ones that were drawn? Explain.

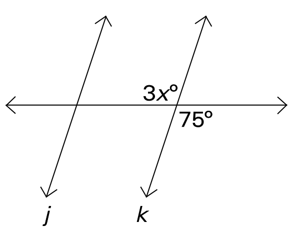
- A daycare is building a triangular sandbox. In the plan, two of the sides of this sandbox are 20 feet and the third side is 16 feet. Is there more than one sandbox that will meet these specifications?

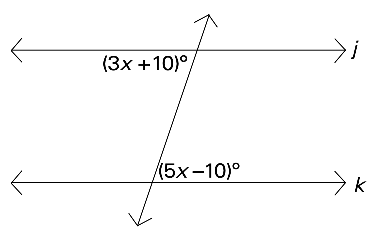
- Two neighbours have built triangular fire pits with the following specifications. Two of their three sides are 64 inches and 30 inches in length. The angle opposite the 30 inch side is 60°. Are the fire pits guaranteed to be the same shape and size?

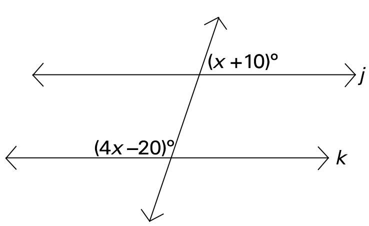
- For the following diagram $\angle 2$ = 95° and $\angle 6$ = 80°, determine the other angle measures given that lines t and s are parallel.

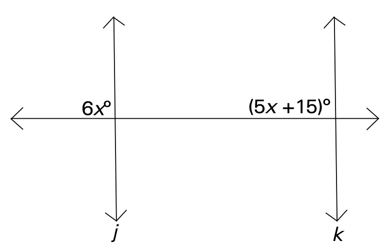


- Find the value of the variable, given that line j and line k are parallel.

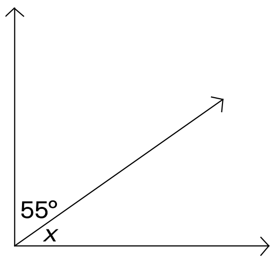


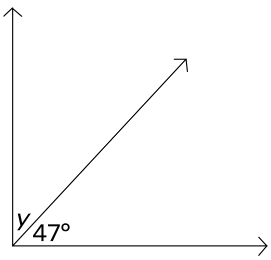




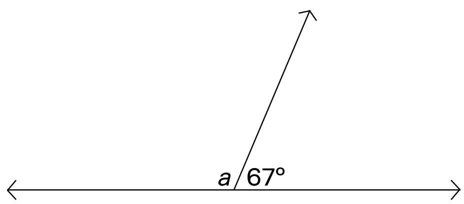


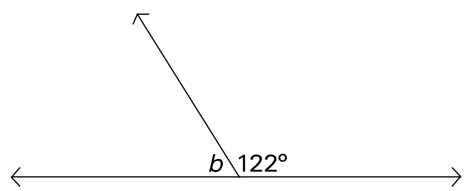
- Given that the following are right angles, find the value of the variables.





- Given that the following are straight angles, find the value of the variables.





FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

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- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
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Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
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- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

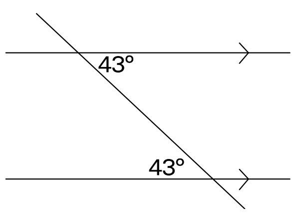
- Students need to be introduced to the term **transversal** before developing the various angle properties. It is important that students realize that a transversal does not have to cut parallel lines.

- Ask students, in groups of three or four, to draw five sets of lines cut by a transversal.
  - Draw the first four sets with non-parallel lines crossed by a transversal, with each set getting closer to being parallel. The fifth set should be parallel lines crossed by a transversal.
  - Number the sets from 1 to 5.
  - Measure the same pair of alternate interior angles. Also have them find the sum of the same-side exterior angles in each set of lines and to record their data in an organized chart.
  - After reviewing and discussing the data in their chart, students should draw conclusions about the measures of alternate interior angles and the sum of same-side exterior angles for lines that are not parallel and lines that are parallel.
  - Ask students to measure all of the angles and record the patterns they observe. Which angles are equal? Which angles are supplementary? Is this an example of inductive reasoning?

- Patty paper (tracing paper) or geo-strips can be used as an effective tool to investigate which angles are congruent when two lines, parallel and non-parallel, are intersected by a transversal.

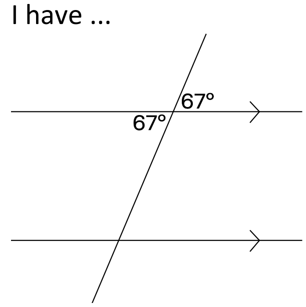
- Students can work in groups of six to play the game “I Have ... , Who Has ... ”. Each student is given a card such as the following:

I have ...



Who has vertically opposite angles?

I have ...



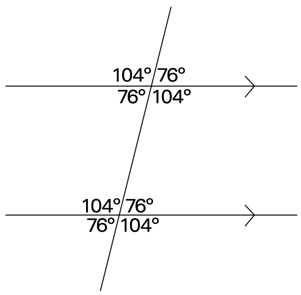
Who has corresponding angles?

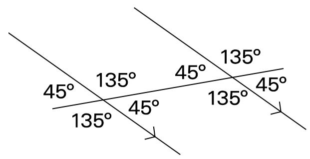
Student #1 begins by identifying the pair of angles illustrated in their diagram and reading the “Who has ...” statement at the bottom of their card. All other students must then look at their pictures to see who has the card illustrating that particular pair of angles. Play continues until the game comes back to the original card.

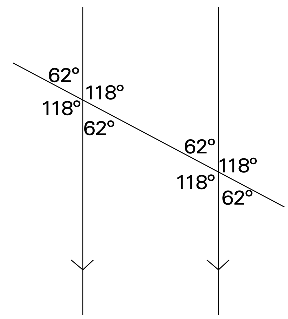
- Use tape to create two parallel lines with a transversal on the floor. Students should work in pairs. Student A stands on the angle. Student B picks a card (from a deck with “vertically opposite,” “alternate interior,” “alternate exterior,” “corresponding,” “same side interior,” and “congruent”) to direct Student A to move to the described angle. Students switch roles after six moves.

- Ask students to create a graphic organizer highlighting the relationships between
  - vertically opposite angles
  - alternate interior angles
  - alternate exterior angles
  - corresponding angles
  - same side interior angles

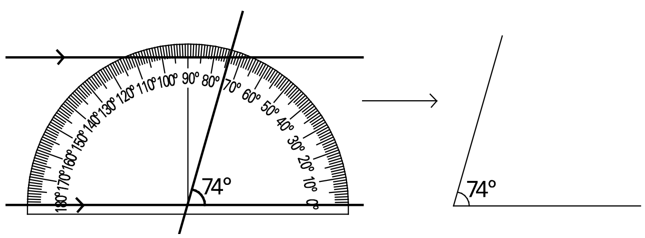
After exploring multiple examples, students make their generalizations about the angle relationships that exist.





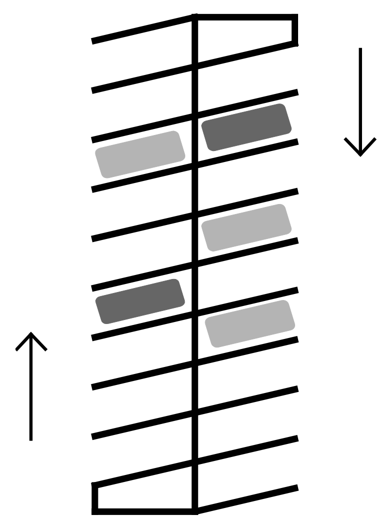


- Software programs such as FX Draw (Efofex Software 2013) and Geometer's Sketchpad (Key Curriculum 2013) could also be used to develop the angle relationships. This alternative to paper and pencil creates a dynamic learning environment where students can interactively investigate and generalize the relationship between pairs of angles formed by transversals and parallel lines. An interactive whiteboard could also be used to develop these relationships. Students could be invited to use the interactive protractor to measure all the angles.



- Ask students to look for examples of parallel lines cut by a transversal around their school. Some examples might include court lines on the gymnasium floor or floor tile patterns in the classroom. Students could trace the lines onto a piece of paper and then measure the angles using a protractor to determine relationships amongst the angles. Have students look to see if they can find parallel lines with a non-right transversal.

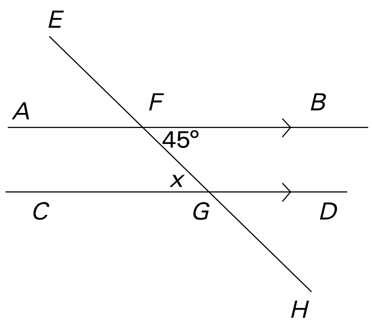
- As an alternative task, students could investigate these angles using a parking lot. It would be beneficial if students were exposed to a parking lot where all the angles are not right angles. When workers paint lines for a parking lot, they aim to paint lines that are parallel to each other. The lines in a parking lot, therefore, provide an ideal illustration of the relationship between angles created by parallel lines and a transversal. For example, students can discuss and mark the different types of angles in the school's parking lot with chalk. This task could also be set up in the gymnasium using tape to represent parking spaces. Students can then measure the angles to determine which angles are equal and which are supplementary.

  This would be a good opportunity to promote discussion by asking students the following questions:
  - Why would a parking lot have parallel lines that intersect at non-right angles?
  - Why would a parking lot have one-way traffic?

- It is beneficial to have students analyze solutions that contain errors, explain why errors might have occurred, and explain how errors can be corrected. This reinforces the angle relationships that have been developed throughout this unit. Have students identify and correct errors such as those present in the following example:

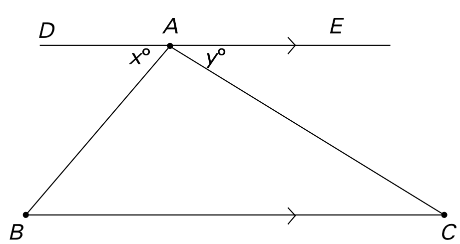
  Determine the measure of x.



Statement	Justification
$\angle BFG = 45°$	given
$\angle BFG = \angle FGD$	interior angles on the same side of the transversal are equal
$\angle FGD + \angle FGC = 180°$ $\angle FGC = 180° - \angle FGD$ $\angle FGC = 180° - 45°$ $\angle FGC = 135°$ $x = 135°$	supplementary angles (angles forming a straight line)

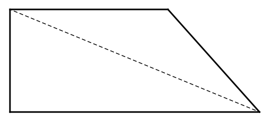
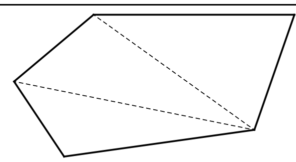
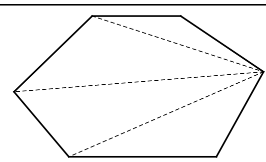
  Students should recognize that interior angles on the same side of the transversal are not equal. They are supplementary angles.

- Students will prove, using deductive reasoning, that the sum of the interior angles of any triangle is 180°. Give students direction by asking them to draw a triangle and to draw a line that is parallel to one of the sides of the triangle and tangent at one of the vertices.



  Students should be able to complete the proof using the properties of angles formed by transversals and parallel lines.

- Encourage students to discover the relationship between the sum of the interior angles and the number of sides in a convex polygon using the angle sum property. They are aware that the sum of the angles in a triangle is 180°. Students can separate each polygon into triangles by drawing diagonals. They can then use the following table to help them with their investigation. Each vertex of a triangle must be a vertex of the original polygon.

Number of Sides	**Diagram**	**Number of Triangles Formed**	**Sum of Angles**
4		2	360°
5		3	540°
6		4	720°

The objective of this investigation is for students to recognize that the sum of the interior angles increases by 180° as the number of sides increase by one. They should also observe that the number of triangles formed is always two less than the number of sides in the polygon. Using this information, encourage students to develop a formula for the sum of the measures of the interior angles of a polygon, $S = 180°(n - 2)$ where S represents the sum of the interior angles and n is the number of sides of the polygon.

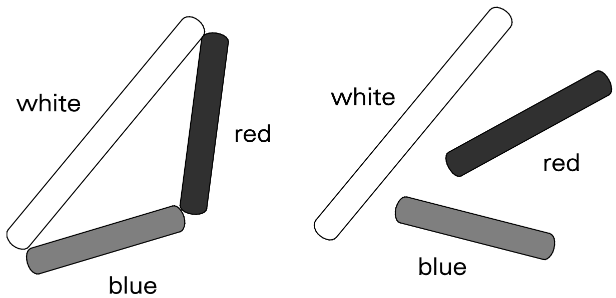
If the polygon is regular, students can use their knowledge of the sum of the measures of the angles in a polygon to determine the measure of each interior angle.

- The following task, which uses either pipe cleaners or geo-strips, will help students conceptualize the idea of the necessary conditions for unique triangles.

Note: After giving students three different lengths of pipe cleaners or geo-strips, have them build a reference triangle that will be used for all of the following investigations.

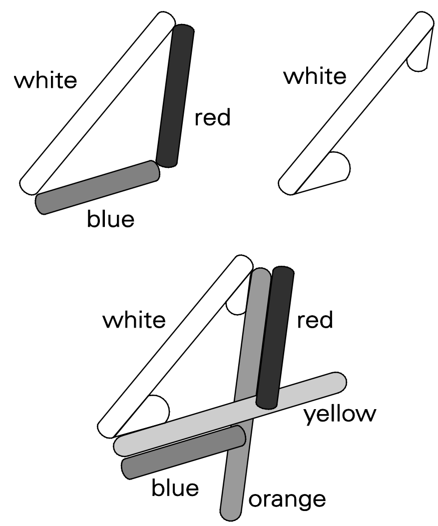
Side-Side-Side (SSS)

- Provide students with a set of pipe cleaners or geo-strips that are identical to those they used to form their reference triangle.
- Ask students to build a triangle using the pipe cleaners or geo-strips.



- Ask students what they noticed about both triangles.
- Students should notice that no matter where they put the sides, they will build a replica of the reference triangle. Therefore knowing the three sides of a triangle is one condition that will guarantee a unique triangle because any other triangle built with the same conditions will be a replica of the given triangle. This will lead into a discussion around the side-side-side relationship (SSS) of congruent triangles because what makes a triangle unique will also make triangles congruent.

Angle-Side-Angle (ASA)



- Provide students with additional pipe cleaners or geo-strips, a protractor, scissors, and construction paper.
- Ask students to use one pipe cleaner or geo-strip identical to one side in the reference triangle. They should then measure the angles formed by that pipe cleaner or geo-strip and the other sides of the reference triangle and cut an angle out of construction paper to represent each of these measured angles.
- Ask students to attach an angle to each end of the selected pipe cleaner or a geo-strip and then use the extra pipe cleaners to construct the other two sides of the triangle.
- Ask students to compare the side lengths of their construction to the side lengths of the base or reference triangle.
- Encourage discussion using questions such as the following:
  - > Are these triangles unique? Why
  - > What did you notice about the side in relation to the two angles?
  - > What are the conditions that will create a unique triangle?
  - > Will these conditions produce congruent triangles?

Side-Angle-Side (SAS)

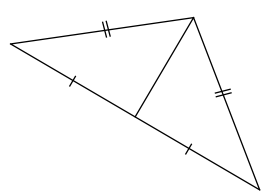
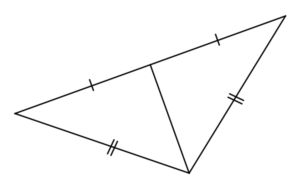
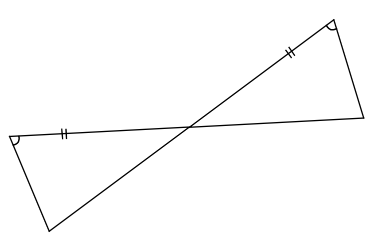
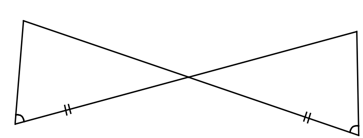
- Provide students with additional pipe cleaners or geo-strips, a protractor, scissors, and construction paper.
- Ask students to use two pipe cleaners or geo-strips identical to those in the base triangle. They should then measure the angle formed between the two selected pipe cleaners or geo-strips and cut an angle out of construction paper to represent this measured angle.

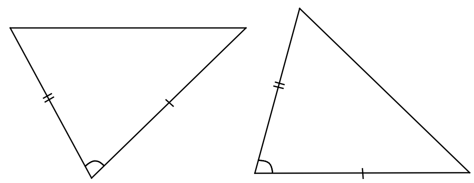
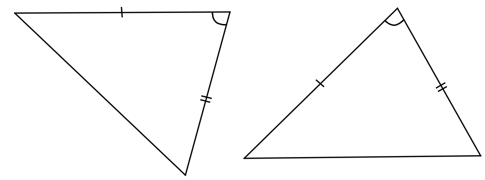
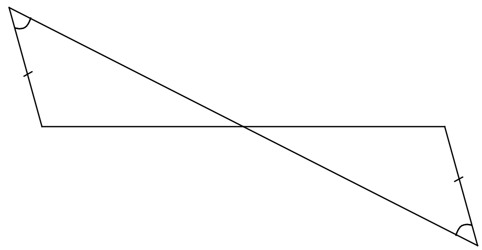
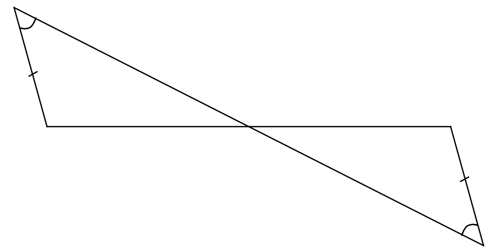
- Ask students to attach the two pipe cleaners so the construction paper angle fits at their joining point. Then use the extra pipe cleaner to construct the other side of the triangle.
- Ask students to compare the side lengths of their construction to the side lengths of the base or reference triangle. Ask the same questions that you asked for the angle-side-angle relationship.

Side-Angle-Angle (SAA)

- Provide students with additional pipe cleaners or geo-strips, a protractor, scissors, and construction paper.
- Ask students to use one pipe cleaner or geo-strip identical to that in the reference triangle. They should then measure two of the angles formed in the base triangles. One of these angles must be across from the selected pipe cleaner or geo-strip. Students should then cut angles out of construction paper to represent each of these measured angles.
- Ask students to use their pipe cleaner or geo-strip and the constructed angles along with additional pipe cleaners or geo-strips to form a triangle.
- Ask students to compare the side lengths of their construction to the side lengths of the base or reference triangle.
- Use questions such as the following to help students work through the task.
  > Can congruent triangles be constructed using these properties?
  > What three pieces of information are required to prove congruent triangles?

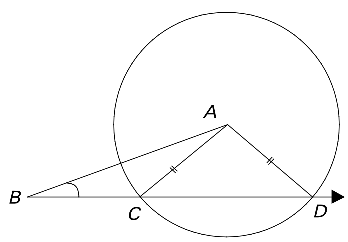
▪ To reinforce the conditions that will ensure that you have congruent triangles, have students play the following memory game.
  - Students have to find three matching cards. One card will state a condition for congruent triangles (SSS, SAS, ASA) and the other two cards show sample triangles that satisfy the postulate. In groups of two, students will take turns trying to find the three matching cards for each postulate, and the student at the end of the game with the most cards wins.

SSS		
ASA		

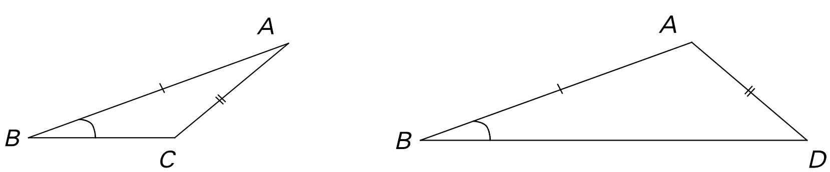
SAS		
AAS		

- Using a compass, you may wish to have students explore why SSA does not work for non-right triangles
  - Draw a diagram with a fixed angle *B* and fixed side *AB*. Create a circle with centre at *A*.

  - Make a triangle by drawing two lines that are radii of the circle.

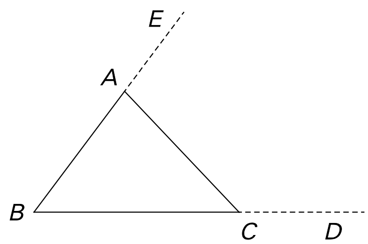


  - Two triangles are constructed. However, ΔABC is not congruent to ΔABD.

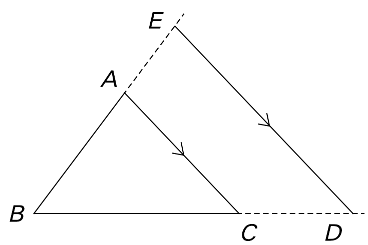


  - Using this construction, students should recognize that knowing only side-side-angle (SSA) does not work because the unknown side could be located in two different places.

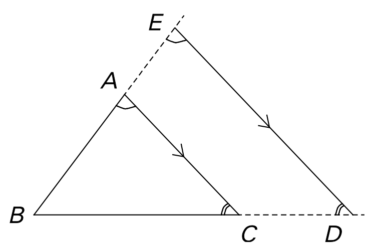
- A construction can also clarify the angle-angle-angle (AAA) case.
  - Draw a triangle *ABC*. Extend the lines on two sides of the triangle.



  - Draw a line parallel to *AC*.



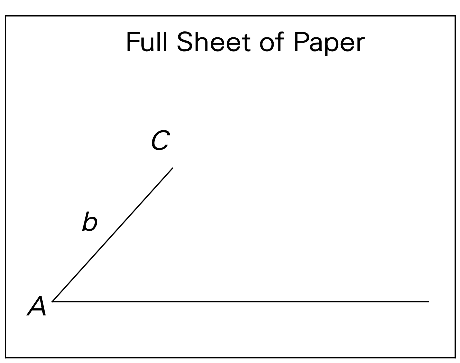
  - Use the fact that the lines are parallel to mark the corresponding angles.



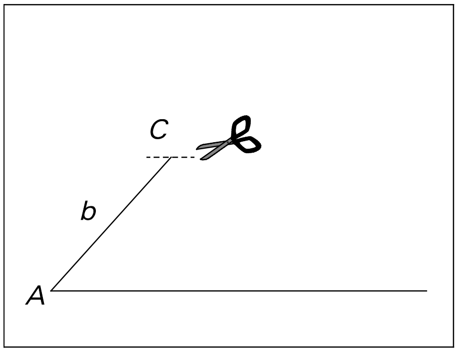
  - Two triangles are constructed. However, ΔABC is clearly not congruent to ΔEBD.
  - From this, students should recognize that knowing only angle-angle-angle can produce similar triangles but cannot guarantee congruent triangles.

- The following task can be used with students to discover that, under certain conditions, there may be no possible triangles, exactly one unique triangle, or two different triangles. This sets the background that will be explored further as part of the ambiguous case for the law of sines.

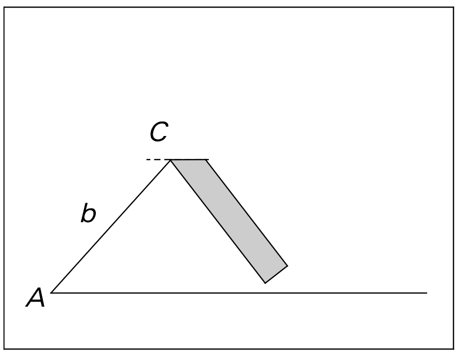
  Step 1: Provide or have students construct a diagram similar to the one below, where side *b* and angle *a* are fixed.



Step 2: Cut a slit in the paper at point *C* large enough to fit a small strip of paper. This strip will represent side *a* of the triangle.



Step 3: Insert the strip into the slot at *C* as shown.



Step 4: Have students explore various lengths for side *a* by pulling the strip out from *C* and rotating it left and right.

As teachers observe, consider the following questions.

> How many triangles can be formed when the strip is too short? (no triangle)
> How many triangles can be formed when the strip is perpendicular to side *C*? (one right triangle)
> Create a triangle using a longer strip. Can another triangle be created using the same strip length? (two unique triangles)

SUGGESTED MODELS AND MANIPULATIVES

- compasses
- dynamic geometry software
- foam or cardboard angles
- patty paper or tracing paper
- pipe cleaners or geo-strips
- protractors
- rulers
- scissors

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- adjacent interior angles
- alternate exterior angles
- alternate interior angles
- converse
- convex polygon
- corresponding angles
- exterior angles
- interior angles
- non-adjacent interior angles
- regular polygon
- transversal

Resources/Notes

Internet

- *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2013)
  http://illuminations.nctm.org
  This interactive game allows students to explore the conditions to create congruent triangles. (Keyword: congruence theorems)

- Math Warehouse (Morris 2013)
  www.mathwarehouse.com
  An interactive game where students can explore and discover the rules for angles of parallel lines cut by a transversal. (Keyword: parallel line and angle)

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 2.1–2.4, pp. 70–99

Notes

SCO G02 Students will be expected to solve problems that involve the properties of angles and triangles. [CN, PS, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

G02.01 Determine the measures of angles in a diagram that involves parallel lines, angles, and triangles and justify the reasoning.
G02.02 Identify and correct errors in a given solution to a problem that involves the measures of angles.
G02.03 Solve a contextual problem that involves angles or triangles.
G02.04 Construct parallel lines, using only a compass and straight edge or a protractor and straight edge, and explain the strategy used.
G02.05 Determine if lines are parallel, given the measure of an angle at each intersection formed by the lines and a transversal.

Scope and Sequence

Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
RF10 Students will be expected to solve problems that involve systems of linear equations in two variables, graphically and algebraically.	**G02** Students will be expected to solve problems that involve the properties of angles and triangles.	—

Background

Students have worked with parallel lines in previous grades. In G01 they determined various angle relationships when two parallel lines were cut by a transversal. They are now asked to verify that the lines are parallel by measuring the angles they have created (corresponding, vertically opposite, alternate interior, alternate exterior, interior angles on the same side of the transversal). They should understand how many angle relationships are necessary to measure in order to prove the lines are parallel. A common student error occurs when students identify lines as being parallel based only on vertically opposite angles that are equal.

Students will be expected to construct two parallel lines using both a compass-straight edge combination and a protractor-straight edge combination.

Students will use angle properties to calculate identified angles in diagrams involving parallel lines, transversals, and triangles, as well as using these angle relationships to solve contextual problems.

Students are expected to analyze solutions that contain errors, explain why errors might have occurred, and explain how they can be corrected. This process reinforces the angle relationships that have been developed throughout this unit. Students should be able to identify and correct errors.

Students could be exposed to examples where variables represent the angles requiring them to solve a linear equation. In ΔABC , for example, $\angle A = 8x - 15$, $\angle B = -x + 42$, $\angle C = 2x$. Ask students to determine the measure of $\angle A$.

Students should be exposed to problems where they can make a connection between mathematics and their environment.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

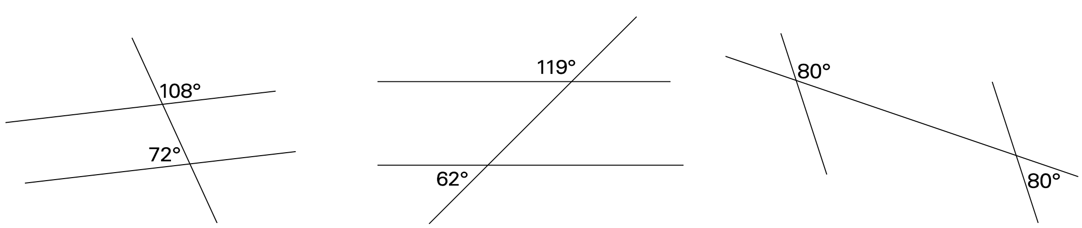
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve for x given that $2x + 15 = 5x + 3$
- If one of the acute angles of a right triangle is 42°, what is the measure of the other angle?
- Provide an example of a situation where it is important for lines to be parallel. What would happen if they were not parallel?

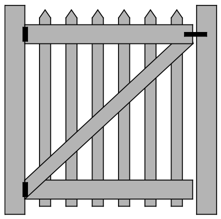
Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

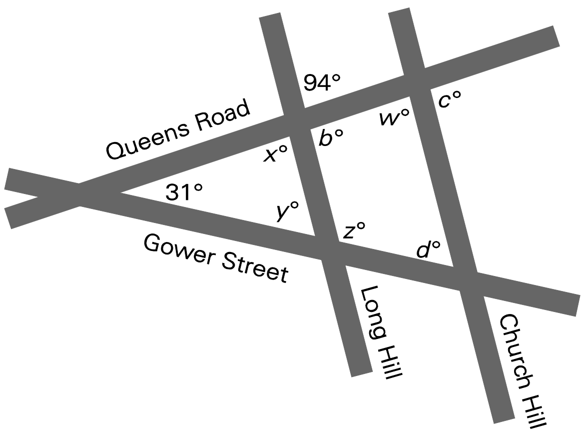
- Determine if the lines shown below are parallel. Explain your reasoning.



- What measurements would be necessary to determine if the top and bottom rails of the gate are parallel?

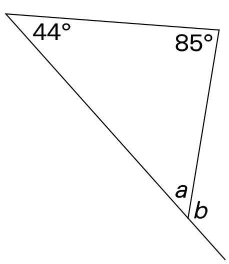


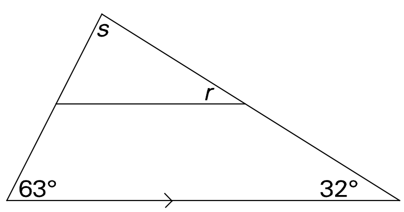
- Long Hill and Church Hill are parallel to each other. Determine the missing measures on the map shown below:

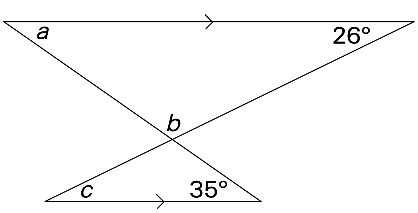


- Design a map involving parallel lines cut by a transversal. Provide at least one angle measurement, and create a question to identify unknown angles.

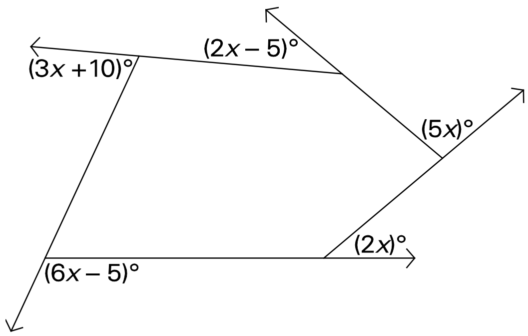
- Determine the measures of unknown angles indicated by the variables in the diagrams.



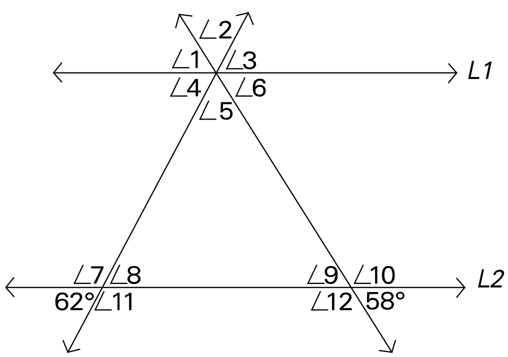




- Determine the value of x in the diagram.

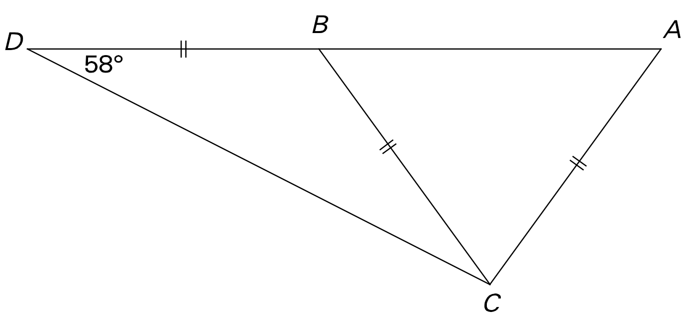


- The measure of which of the following angles can be determined to ensure that the lines L1 and L2 are parallel?



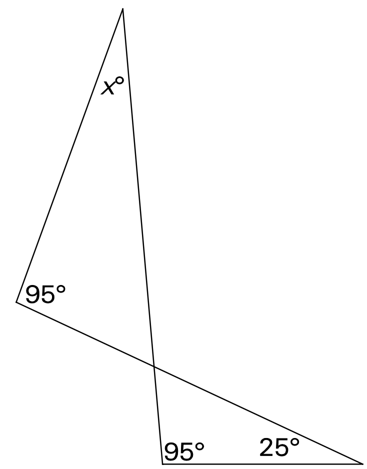
- In quadrilateral $KLMN$, $\angle K = 6x - 30$, $\angle L = -2x + 50$, $\angle M = 5x + 10$, and $\angle N = 3\angle M$. Determine if the quadrilateral has a right angle.

- Determine the measure of $\angle A$ and explain your reasoning.

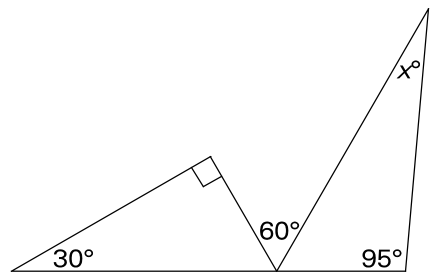


- Determine the value of x.

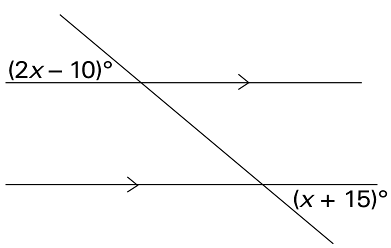
(a)



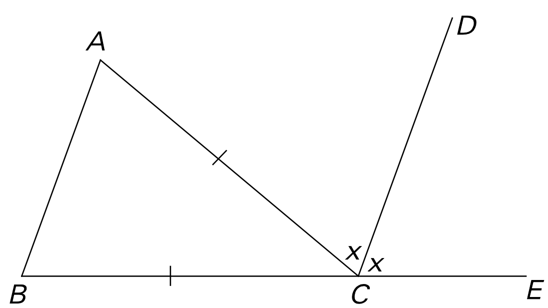
(b)



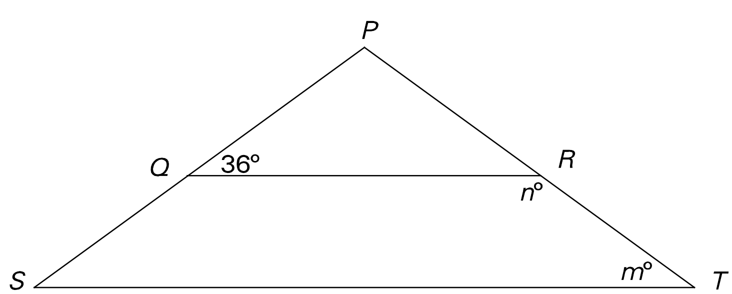
(c)



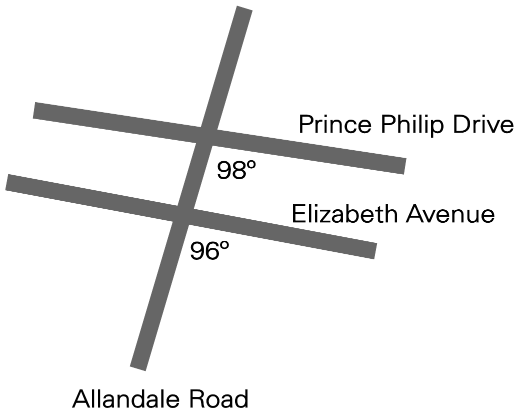
- Given the information in the diagram below, prove that $AB \parallel DC$.



- Calculate the values of m and n, if $PQ = PR$ and $QR \parallel ST$.

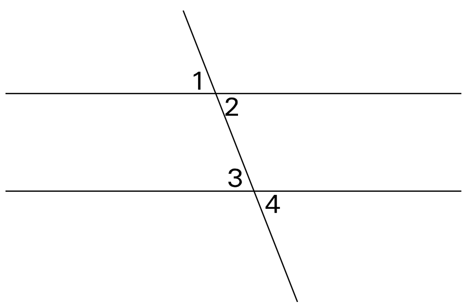


- Suppose Prince Philip Drive and Elizabeth Avenue each follow a straight line path and intersect at Allandale Road at angles of 98° and 96°, as shown in the map below. If the streets were to continue in a straight line, would their paths ever cross? Explain your reasoning.



- McKenzie said that if a triangle is obtuse, two of the angles of the triangle are acute. Ask students if they agree with McKenzie. Explain your reasoning.

- Identify and correct errors found in the following example:



Statement	**Reason**
$\angle 1 = \angle 2$	Vertically opposite angles are congruent.
$\angle 2 = \angle 3$	Alternate interior angles are congruent.
$\angle 1 = \angle 3$	Both are congruent $\angle 2$.
$\angle 3 = \angle 4$	Vertically opposite angles are congruent.
$\angle 1 = \angle 4$	Both are congruent $\angle 3$.
Lines are parallel	Alternate exterior angles are congruent.

- Prove that the line formed by joining the midpoints of two sides of a triangle is parallel to the third side of the triangle.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

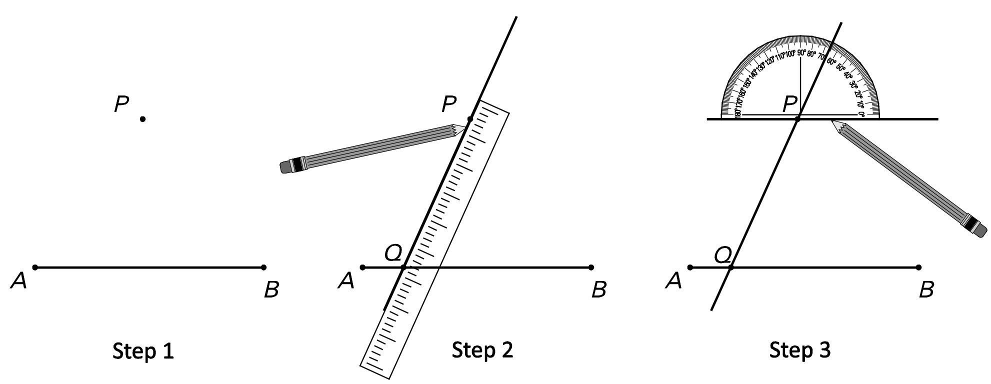
SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Have the students construct parallel lines and verify the angle relationships that were developed earlier. Use a variety of methods—a compass and straight edge, a protractor and ruler, or paper folding and protractor—to ensure that the lines are parallel, and then consider the following:

  Using a protractor and a straight edge

  Step 1: Draw a segment *AB*. Place a point *P* above the line.
  Step 2: Draw a line through *P* and intersecting *AB* at *Q*.
  Step 3: Using a protractor, measure $\angle QPB$. Use your protractor to then form an angle having the same measure at *P*. One arm of the angle is parallel to *AB*.

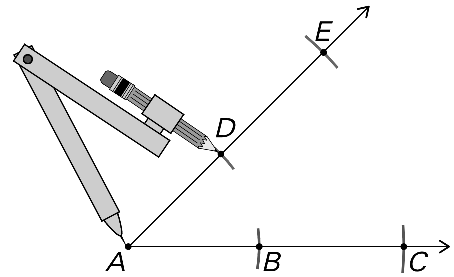


Using paper folding and a protractor

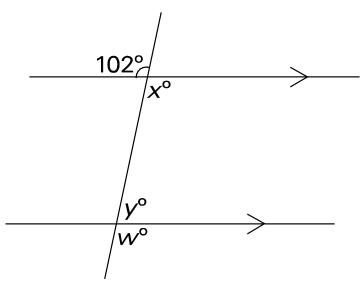
Step 1: Take a blank sheet of paper and fold it in half.
Step 2: Fold it in half again.
Step 3: Unfold the paper and trace any two of the fold lines, using a ruler and protractor.
Step 4: Construct a transversal.

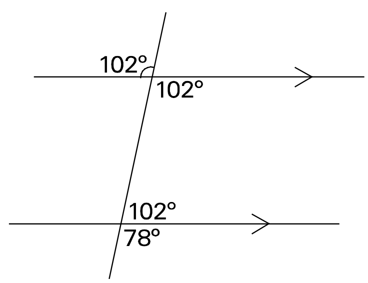
Using a compass and straight edge

Step 1: Draw a straight line.
Step 2: Use your compass to mark off segments of equal length along this line. Label these points *A*, *B*, and *C*.
Step 3: Draw a second straight line that intersects the first one at point *A*.
Step 4: Use your compass to mark off segments, beginning at point *A*, of equal length along this second line. Label these points *D* and *E*.
Step 5: Join *B* and *D*; join *C* and *E*.

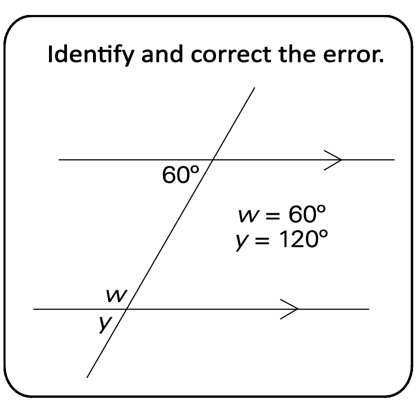


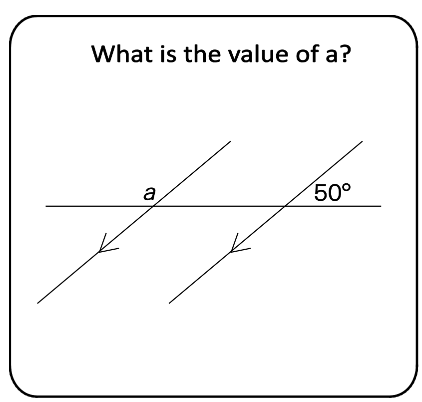
- A common error occurs when students incorrectly identify pairs of angles, leading to an incorrect measurement. For example, students identify same side interior angles as congruent rather than supplementary. Students should be able to identify and correct errors such as those present in the following example:



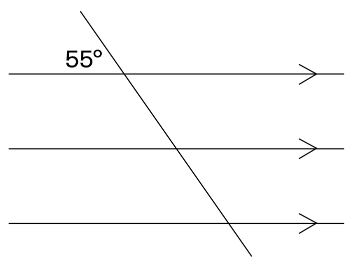


- Quiz-Quiz-Trade Game:
  - Each student is given a card with a problem. The answer is written on the back of the card.
  - In groups of two, partner *A* asks the question and partner *B* answers.
  - They switch roles and repeat.
  - Partners trade the cards and then find a new partner.
  - Sample cards are shown below.





How many angles in the diagram measure 55º?



- Create centres in the classroom containing solutions to problems that involve the measurement of angles. Students will participate in a carousel activity in which they will be asked to move throughout the centres to identify and correct the errors.

Suggested Models and Manipulatives

- compasses
- patty paper or tracing paper
- protractor
- straight edges

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- acute angle
- adjacent interior angles
- alternate exterior angles
- alternate interior angles
- converse
- convex polygon
- corresponding angles
- exterior angles
- interior angles
- non-adjacent interior angles
- obtuse angle
- regular polygon
- right angle
- transversal

Resources/Notes

Internet

- *Math Is Fun*, Parallel Lines, and Pairs of Angles (MathIsFun.com 2011)
  www.mathsisfun.com/geometry/parallel-lines.html
  Basic interactive site that can be used to review the terminology.

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 2.2–2.4, pp. 73–99

Notes

SCO G03 Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case. [CN, PS, R]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

G03.01 Draw a diagram to represent a problem that involves the cosine law and/or sine law.
G03.02 Explain the steps in a given proof of the sine law and of the cosine law.
G03.03 Solve a problem involving the cosine law that requires the manipulation of a formula.
G03.04 Explain, concretely, pictorially or symbolically, whether zero, one or two triangles exist, given two sides and a non-included angle.
G03.05 Solve a problem involving the sine law that requires the manipulation of a formula.
G03.06 Solve a contextual problem that involves the cosine law and/or the sine law.

Scope and Sequence

Mathematics 10	Mathematics 11	Pre-calculus 12
M04 Students will be expected to develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles.	**G03** Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case.	**T06** Students will be expected o prove trigonometric identities, using ▪ reciprocal identities ▪ quotient identities ▪ Pythagorean identities ▪ sum or difference identities (restricted to sine, cosine, and tangent) ▪ double-angle identities (restricted to sine, cosine, and tangent)

Background

In Mathematics 10 students used the primary trigonometric ratios to solve problems involving right triangles. They will now use the sine law and cosine law to solve problems that can be modelled with acute triangles, including the ambiguous case.

In Mathematics 10 students used the terminology angle of elevation, angle of depression, and angle of inclination.

Students will explain the steps in the derivation of both the sine law (also commonly referred to as the law of sines), and the cosine law (also commonly referred to as the law of cosines), and then use the relationships to calculate unknown sides and angles in triangles. Students apply the laws to calculate unknown side lengths and angle measures, explaining their reasoning.

Students need to be able to explain their thinking about whether they will use the law of sines, the law of cosines, the trigonometric ratios, or the Pythagorean theorem.

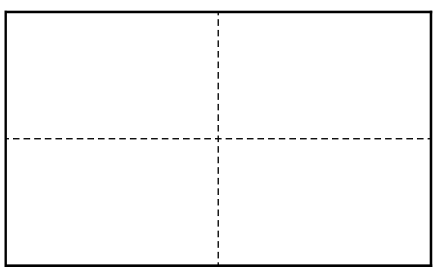
Students examine the conditions that lead to the ambiguous case of the sine law—two given sides and the non-included angle, SSA. Based on given information, students decide whether zero, one, or two triangles exist in non-contextual situations and explain their reasoning in a variety of ways. For contextual problems in which the given information leads to two possible triangles, students decide whether the acute or obtuse situation applies and justify their decision.

Students will solve problems represented by more than one triangle. They will use a combination of strategies such as the primary trigonometric ratios, the Pythagorean theorem, the sum of the angles in a triangle, the sine law, and the cosine law to solve problems.

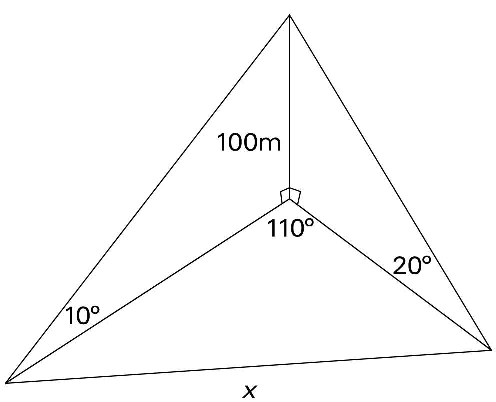
In Mathematics 10 students solved 3-D problems in which the planes were at right angles to each other. In this unit, students will extend these 3-D situations to include instances in which the planes are not perpendicular. Models will be essential for many students when visualizing these 3-D problems.

This can be done simply by using a piece of paper, card stock, or a cue card.

- Fold along the dotted lines, as shown. Next, cut along one of the dotted lines from the edge to the centre.



- Fold to create the corner of a box. The lengths and angles can then be placed in the appropriate locations on the card as shown below.



Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

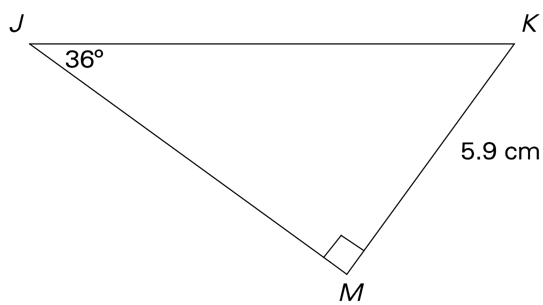
Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

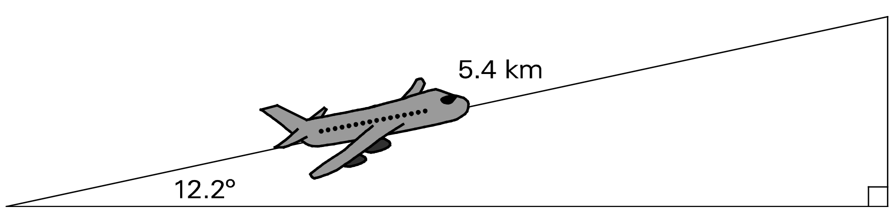
ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Determine the length of the hypotenuse.



- A pilot starts her takeoff and climbs steadily at an angle of 12.2°. Determine the horizontal distance the plane has travelled when it has climbed 5.4 km along its flight path. Express your answer to the nearest tenth of a kilometre.

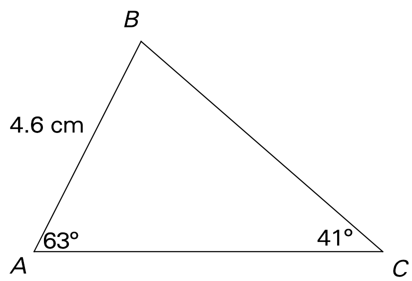


- The angle between the shorter side of a rectangle and its diagonal is 56°. The shorter side of the rectangle is 2.3 cm. How long is the diagonal?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

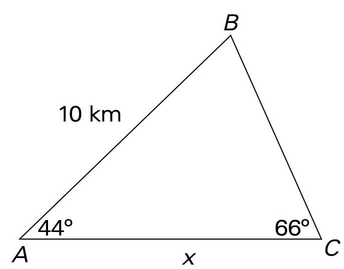
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the length of *AC* in the following diagram using two different methods.



- Answer the following questions:
  (a) Why does the sine law have three ratios in its equation?
  (b) Do you use all three ratios at once? How can you tell which ratios to use?
  (c) How many pieces of information about a triangle's side lengths and angles are needed in order to solve it using the sine law? What possible combinations will work?
  (d) Would you use the sine law to solve right triangles? Explain.

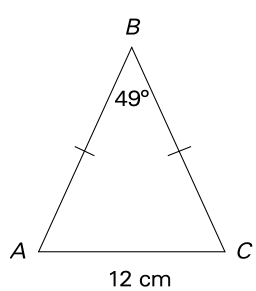
- Two observers sight a plane at angles of elevation of 44° and 66° respectively. If one observer is 10 km away from the plane, how far apart are the two observers from each another?



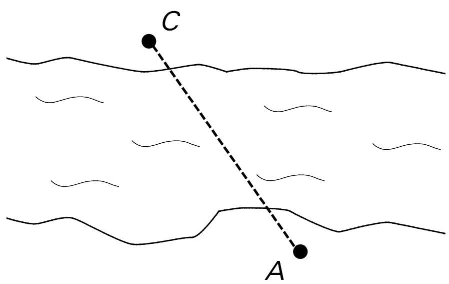
- Make up a problem in which a side length can be determined using law of sines. Show the solution. Make up a problem in which an angle measure can be determined using law of sines.

- Make up a problem in which a side length can be determined using law of cosines. Show the solution. Make up a problem in which an angle measure can be determined using law of cosines.

- Find the missing side lengths in the isosceles triangle below.

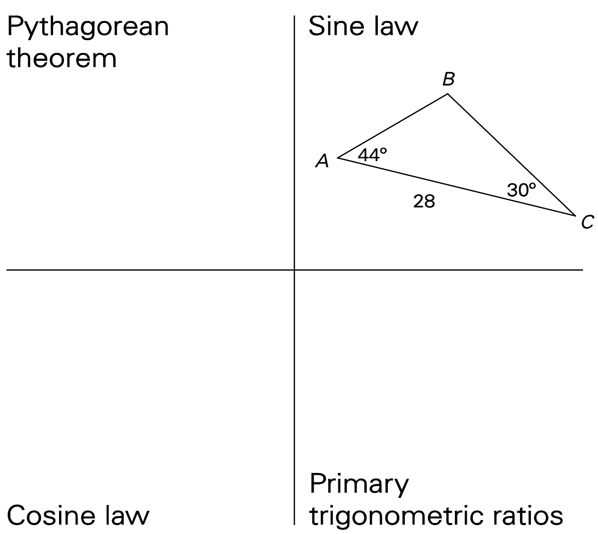


- A surveyor is located on one side of a river that is impossible to cross and only has a 100 m measuring tape and a sextant (used to measure angles) in his possession. Explain how the surveyor could use only these two tools and the law of sines to find the distance from point *A* to point *C*.



- Explain how the law of cosines validates the Pythagorean theorem if the included angle is 90°.

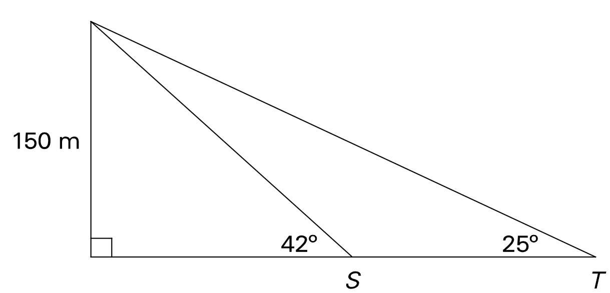
- Obtain a set of cards/questions from your teacher. Each card or question will contain a triangle with information regarding the lengths of the sides and angles. Sort these cards/questions according to a specific strategy you would use to solve the question. [Pythagorean theorem, sine law, cosine law, primary trigonometric ratios].



- Research a real-world problem that makes use of the sine law, the cosine law, or both the sine and the cosine laws. Share the problem with the class and discuss the method for solving.

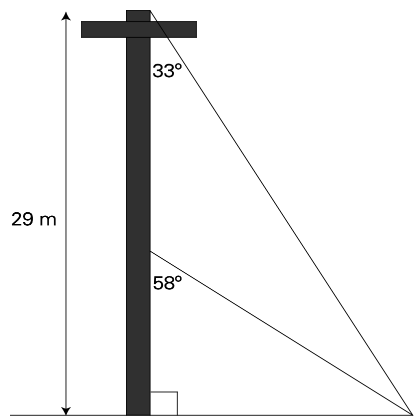
- What quantities must be known in a triangle before the sine law can be used? The cosine law?

- A tower is 150 metres high. Two wires located at positions *S* and *T*, are fastened at the top of the tower and make angles of elevation of 42° and 25° with the ground. How far apart are the two wires?



- Yujin enjoys swimming in the ocean. One day, Yujin decides to swim 9.2 km from Island *A* to Island *B*; then, after resting a few moments, she swims 8.6 km to Island *C*. If Island *C* to Island *A* to Island *B* forms a 52° angle, determine how much further Yujin has to swim by swimming to Island *B* first rather than simply swimming straight from Island *A* to Island *C*.

- A 29-foot-high pole on a farm is supported by two guy wires as shown below. Find the length of the two guy wires.



- An engineer is asked to build a triangular support frame for an airplane wing with $\angle A = 42°$ and with sides $b = 13.2$ cm and $a = 10.1$ cm respectively. Can the engineer determine the length of the third side? Explain.

- For ΔABC, $a = 8$, $b = 9$, and $c = 7$. What is the measure of $\angle C$?

- A forest ranger is standing at the top of a 100-foot-high tower. She observes a fire at an angle of depression of 25°, turns 120°, and sees, at a 40° angle of depression, a group of campers. How far are the campers from the fire?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
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Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

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- Does the lesson fit into my yearly/unit plan?
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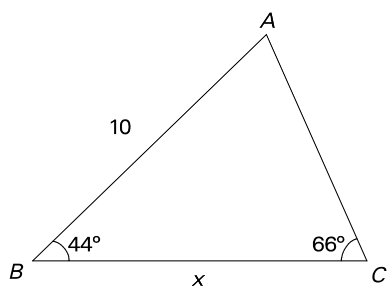
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

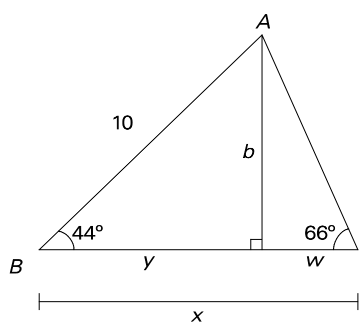
SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Students have been exposed to right-triangle trigonometry to solve problems involving right triangles. Before introducing the sine law, it may be beneficial for students to solve, using the primary trigonometric ratios, a triangle that is not a right triangle. Encourage students to draw a diagram to represent the problem to help them gain a visual understanding of the problem. Consider the following example:



- Ask students if this triangle can be divided into two right triangles and what strategies can be applied to find the indicated length. They should recognize that this requires a multi-step solution. A strategy must be developed before a solution is attempted. Drawing an altitude from vertex *A*, students can use the primary trigonometric ratios and the Pythagorean theorem to solve for the unknown value.



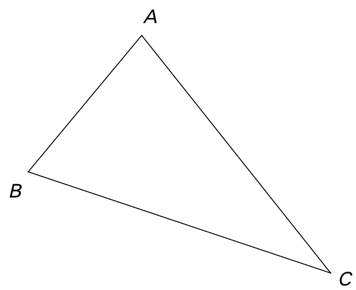
$\cos 44° = \dfrac{y}{10}$
$(10)\cos 44° = y$
$7.19 = y$

$\sin 44° = \dfrac{b}{10}$
$(10)\sin 44° = b$
$6.95 = b$

$\tan 66° = \dfrac{6.95}{w}$
$(w)\tan 66° = 6.95$
$w = \dfrac{6.95}{\tan 66°}$
$w = 3.09$

$x = y + w$
$x = 7.19 + 3.09$
$x = 10.28$

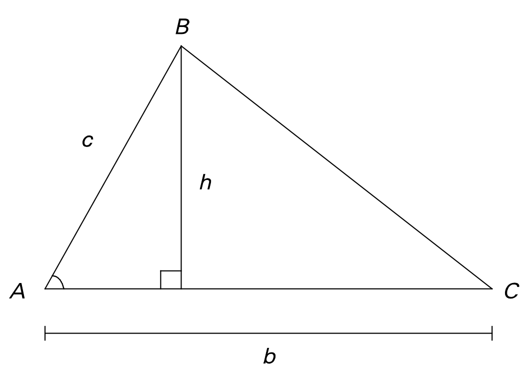
- Provide students with a triangle and have them measure the side lengths and angles using a ruler and protractor. Students can then use inductive reasoning to make a conjecture about the law of sines prior to proving it deductively.



- Ask students to answer the following questions:
  (a) What conjecture can you make regarding the ratios calculated below?
  (b) Would your conjecture be valid if you were to use the reciprocal of the ratios?

	Measure		**Measure**		**Calculate**
$\angle A$		Side a		$\frac{\sin A}{a}$	
$\angle B$		Side b		$\frac{\sin B}{b}$	
$\angle C$		Side c		$\frac{\sin C}{c}$	

- The law of sines can be derived using the area formula of a right triangle. Consider the following diagram:



Triangle *ABC* is not a right triangle. Therefore, students will draw an altitude from vertex *B*.

The area formula for a right triangle is Area = $\frac{1}{2}$(base)(height).

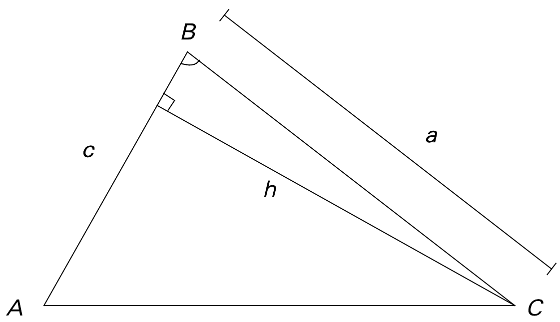
Ask students to write an expression for the height (h) using the sine ratio.

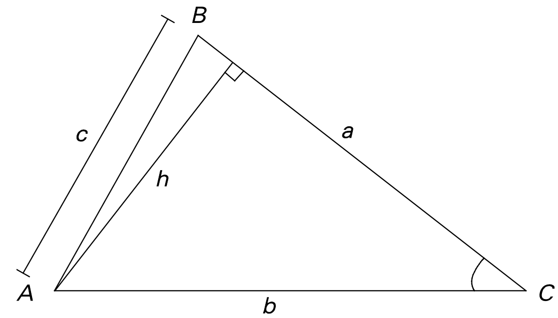
Since, $\sin A = \frac{h}{c}$

then (c) $\sin A = h$

Therefore, Area = $\frac{1}{2}(b)(h) = \frac{1}{2}(b)(c \sin A)$

- Ask students to repeat this procedure by drawing an altitude from the other vertices and writing an equation for the area of triangle *ABC*.





They should conclude that Area = $\frac{1}{2}(b)(c \sin A) = \frac{1}{2}(a)(c \sin B) = \frac{1}{2}(a)(b \sin C)$

- Promote student discussion as to why they can set the three area expressions equal to each other.

$\frac{1}{2}(b)(c \sin A) = \frac{1}{2}(a)(c \sin B) = \frac{1}{2}(a)(b \sin C)$
$(b)(c \sin A) = (a)(c \sin B) = (a)(b \sin C)$

Dividing each expression by *abc*:

$$\frac{(b)(c \sin A)}{abc} = \frac{(a)(c \sin B)}{abc} = \frac{(a)(b \sin C)}{abc}$$

Simplifying this yields the law of sines:

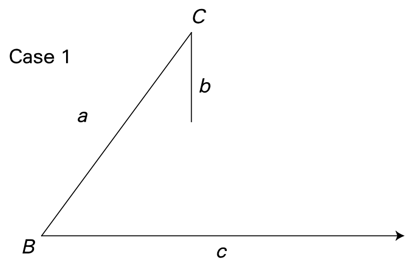
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

In other words, the law of sines is a proportion that compares the ratio of each side of a triangle to the sine of its opposite angle.

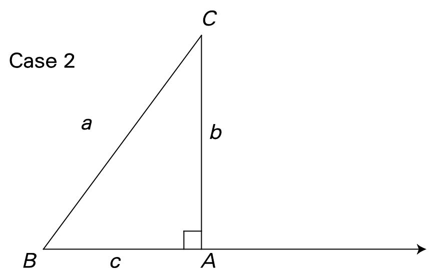
- Encourage students to draw diagrams with both the given and unknown information marked. Students should recognize that when they have ASA there is a unique triangle that can be found using the law of sines.

- Students should also recognize that when they have SSA, there *may* be more than one solution. When this happens it is called the ambiguous case. There are eight cases to consider (five when the angle is acute and three when the angle is obtuse).
  - *If the angle is acute*, then there are five cases possible, only one of these situations yields the ambiguous case.

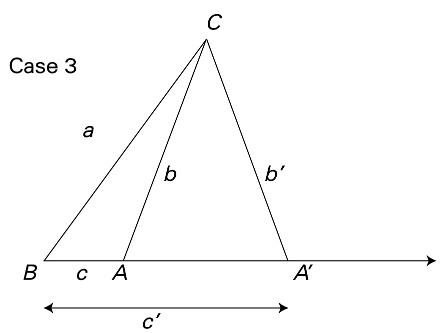
Case 1: The third side is too short, and there is no solution.



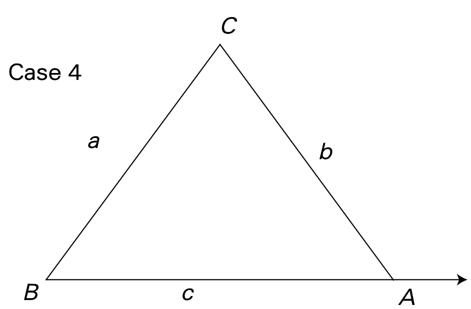
Case 2: The third side forms a right angle triangle, and there is a single solution.



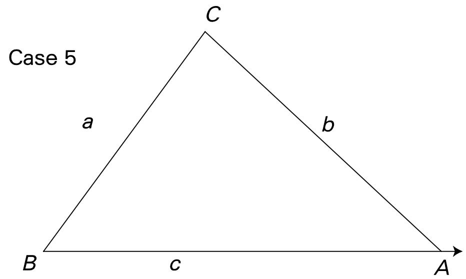
Case 3: The third side is longer than the perpendicular would be, but shorter than the fixed side *a*. In this case, there would be two solutions.



Case 4: The third side is the same length as the fixed side *a*. In this case there is only one solution (an isosceles triangle).

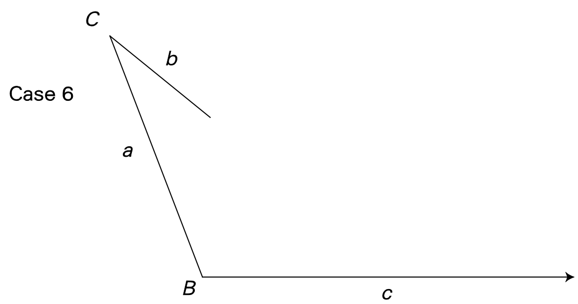


Case 5: The third side is longer than the fixed side *a*. In this case there is only one solution.

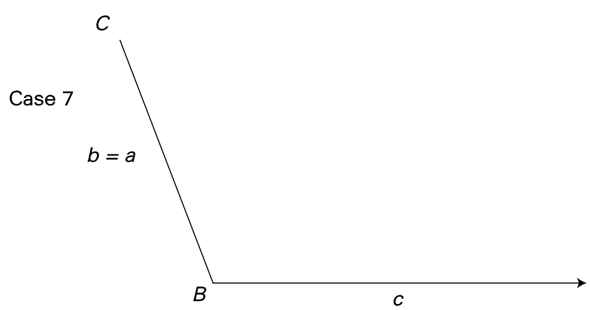


– *If the angle is obtuse*, there are three situations possible; none of these situations yields the ambiguous case.

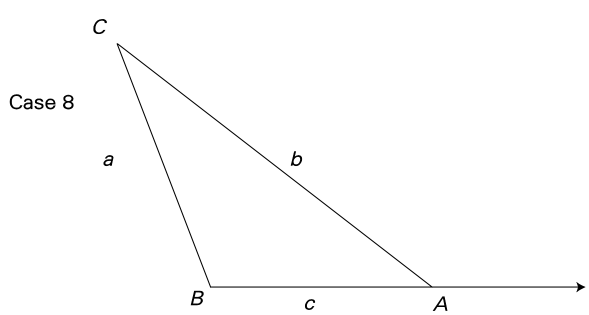
Case 6: The third side is shorter than the fixed side. There will be no possible solution.



Case 7: The third side is the same length as the fixed side. There will be no possible solution.

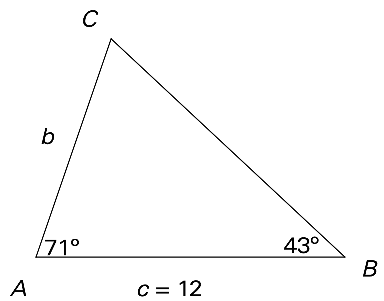


Case 8: The third side is longer than the fixed side. There will be one solution.



– Students should recognize when they have information that would create the ambiguous case.

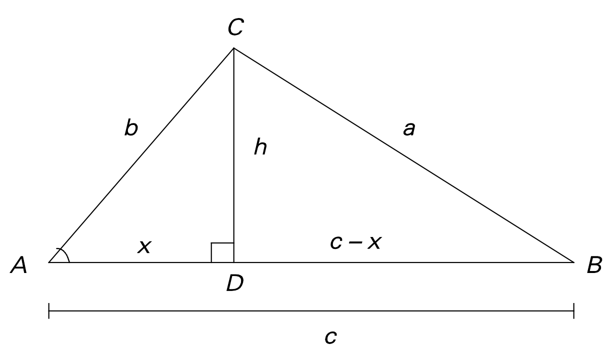
- When working with the sine law, students sometimes incorrectly identify side and opposite angle pairs. To avoid this error, encourage them to use arrows on the diagram when identifying the angle and its opposite side. They could also encounter problems when they multiply to solve the ratios. When solving $\frac{\sin A}{12} = \frac{\sin 30^\circ}{4}$, for example, students may incorrectly write 4 sin A = sin 360° instead of 4 sin A = 12 sin 30°. The use of brackets may clarify the situation for many students and thus teachers should encourage students to write 4(sin A) = 12(sin 30°). Another common student error occurs when students try to solve a triangle given two angles and an included side, mistakenly thinking that there is not enough information to use the sine law. Consider an example such as the following:



Students can use the property that the sum of the angles in a triangle is 180°. The measure of $\angle C$, therefore, is 66°. They can then proceed to use the sine law to find the length of side AC.

- Encourage students to check the reasonableness of their answer. For example, since $\angle C$ is a little smaller than $\angle A$, we expect the length of side AB to be a little shorter than the length of side CB. Students should also consider asking questions such as the following: Is the shortest side opposite the smallest angle? Is the longest side opposite the largest angle?

- Students will prove and use the cosine law to solve triangles. Provide students with a triangle ABC with side lengths a, b and c. Ask them to draw an altitude, h, from vertex C and let D be the intersection of AB and the altitude, as shown in the figure below. If x is the length of AD, they should recognize $BD = c - x$. Guide students through the following process:



  - Use the Pythagorean theorem in ΔBCD: $a^2 = h^2 + (c - x)^2$
  - Expand the binomial: $a^2 = h^2 + (c^2 - 2cx + x^2)$
  - Rearrange to obtain: $a^2 = x^2 + h^2 + c^2 - 2cx$
  - For ΔACD: $x^2 + h^2 = b^2$
  - substitute to obtain: $a^2 = b^2 + c^2 - 2cx$
  - In ΔACD: $\cos A = \frac{x}{b}$

- resulting in: $b \cos A = x$
- Substitute for x into $a^2 = b^2 + c^2 - 2cx$: $a^2 = b^2 + c^2 - 2c(b \cos A)$ or $c^2 = a^2 + b^2 - 2a(b \cos C)$

- Students can then express the formula in different forms to find the lengths of the other sides of the triangles. It is important that students be able to apply the cosine formula flexibly. (as $a^2 = b^2 + c^2 - 2bc \cos A$ or $b^2 = a^2 + c^2 - 2ac \cos B$ or $c^2 = a^2 + b^2 - 2ab \cos C$)

- The cosine law can be used to determine an unknown side or angle measure in a triangle. Continue to encourage students to draw diagrams with both the given and unknown information marked when solving problems. Students should consider why the cosine law is used to find the unknown angle if three sides are known (SSS) or if two sides and the included angle are known (SAS).

  Have students consider the following possibilities:

Included angle is acute	**Included angle is right**	**Included angle is obtuse**
$c^2 = a^2 + b^2 - 2ab \cos 60°$ $c^2 = a^2 + b^2 - 2ab(0.5)$ $c^2 = a^2 + b^2 - ab$ (Side opposite an acute angle is smaller than if it had been opposite a right angle.)	$c^2 = a^2 + b^2 - 2ab \cos 90°$ $c^2 = a^2 + b^2 - 2ab(0)$ $c^2 = a^2 + b^2$ (Pythagorean theorem)	$c^2 = a^2 + b^2 - 2ab \cos 150°$ $c^2 = a^2 + b^2 - 2ab(-0.5)$ $c^2 = a^2 + b^2 + ab$ (Side opposite an obtuse angle is larger than if it had been opposite a right angle.)

- When three sides of a triangle are known, students will use the cosine law to find one of the angles. Some students may rearrange the equation to solve for a particular angle. Others may substitute the unknown values into the cosine law and then rearrange the equation to find the angle. It is important for students to recognize that they have a choice when trying to find the second angle—they can either use the cosine law or the sine law. Students should notice the third angle can then be determined using the sum of the angles in a triangle.

- When solving triangles, encourage students to consider the following questions:
  - What is the given information?
  - What am I trying to solve for?
  - With the given information, should I use the sine law or the cosine law? Is there a choice?

- If students know two sides and a non-included angle (SSA) of a triangle, they can use the cosine law in conjunction with the sine law to find the other side. As an alternative, they could apply the sine law twice if they are aware of the possible ambiguous case. Students need to be exposed to numerous examples in order to find the method that works best for them.

Example

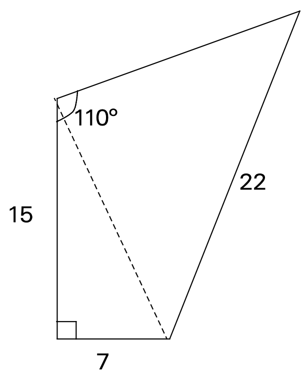
Two sides of a triangle are 5 " and 3 " in length. The angle opposite the 5 " side is 70°. Determine the length of the other side.

Law of cosines and law of sines	**Law of sines done twice**
$\frac{\sin 70^\circ}{5} = \frac{\sin A}{3}$ $\frac{3\sin 70^\circ}{5} = \sin A$ $0.5638 = \sin A$ $A = 34.32^\circ$ $\angle x = 180^\circ - (34.32^\circ + 70^\circ) = 75.68^\circ$ $X^2 = 3^2 + 5^2 - 2(3)(5)\cos 75.68^\circ$ $X^2 = 9 + 25 - 7.42$ $X^2 = 26.58$ $X = 5.16$	$\frac{\sin 70^\circ}{5} = \frac{\sin A}{3}$ $\frac{3 \sin 70^\circ}{5} = \sin A$ $0.5638 = \sin A$ $A = 34.32^\circ$ $\angle x = 180^\circ - (34.32^\circ + 70^\circ) = 75.68^\circ$ $\frac{x}{\sin 75.68^\circ} = \frac{5}{\sin 70^\circ}$ $x = \frac{5 \sin 75.68^\circ}{\sin 70^\circ}$ $x = 5.16$
Law of cosines (using quadratics)—This could be considered after students have learned the quadratic formula.	
$5^2 = 3^2 + x^2 - 2(3)(x)\cos 70^\circ$ $25 = 9 + x^2 - 2.052(x)$ $0 = x^2 - 2.052(x) - 16$ Using the quadratic formula $x = \frac{-(-2.052) \pm \sqrt{(-2.052)^2 - 4(1)(-16)}}{2(1)}$	$x = \frac{2.052 \pm \sqrt{(68.211)}}{2}$ $x = \frac{2.052 \pm 8.259}{2}$ $x = 5.16$ *or* -3.10 Only the solution of 5.16 is possible for a side length.

- When working with the cosine law, students sometimes incorrectly apply the order of operations. When asked to simplify $a^2 = 365 - 360 \cos 70^\circ$, for example, students often write $a^2 = 5 \cos 70^\circ$. To avoid this error, teachers should emphasize that multiplication is to be completed before subtraction.

- Use observation to assess student understanding by providing students with several practice problems that use the sine law and/or the law of cosines. As teachers observe students working through the problems, ask them the following questions:
  - What is the unknown? Is it an angle or a side?
  - How can you isolate the unknown?
  - How can you complete the calculations?
  - Does your conclusion answer the question asked?

- Students may need to use a strategy or a combination of strategies to solve problems represented by one or more than one triangle. Provide students with an example in which a playground, in the shape of a quadrilateral, is to be fenced. Ask them to determine the total length of fencing required.

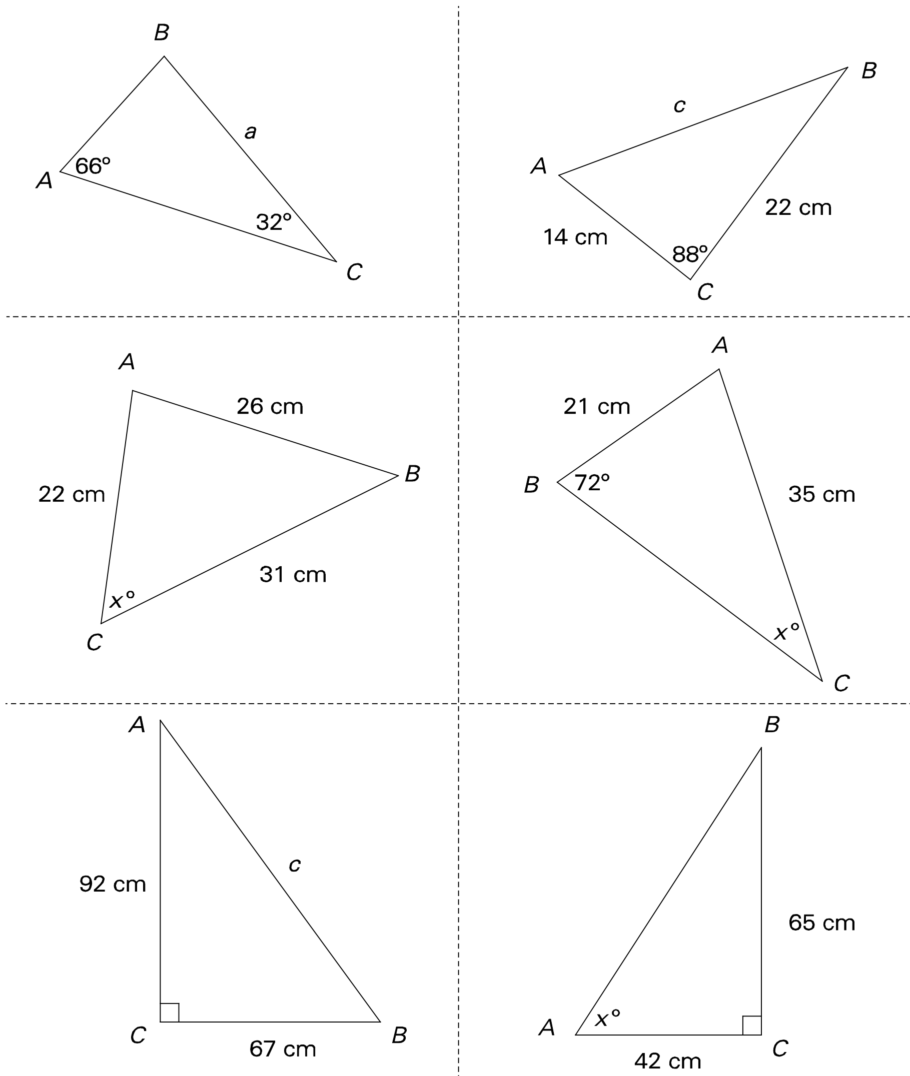


- Discussion around various strategies one might use, as well as which method is the most efficient, is encouraged. Ask students how changing the right angle in the figure to an acute angle, such as 50°, would affect their strategy.

- Ask students to create a graphic organizer. When solving triangles, the organizer can guide students as they decide which is the most efficient method to use when solving for an unknown angle and/or side.

SSS	**ASA**	**SAS**	**SSA**	**Right Triangle**
Law of cosines	Law of sines	Law of cosines	Law of sines (ambiguous case) and/or law of cosines	Pythagorean theorem and/or trigonometric ratios

- In the task Four Corners, students have to think about which method they would use to solve a triangle. Post four signs, one in each corner of the room labelled sine law, cosine law, Pythagorean theorem, and trigonometric ratios. Provide each student with one triangle. Instruct the students to make a decision as to which method they would use to find the missing angle or side and to stand in the corner where it is labelled. Once students are all placed, ask them to discuss why their triangle(s) would be best solved using that particular method. Sample triangles are given below:



Suggested Models and Manipulatives

- protractors
- rulers

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- ambiguous case
- law of cosines
- law of sines

Resources/Notes

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 3.1–3.4, pp. 116–148
  - Sections 4.1–4.4, pp. 162–196

Notes

Logical Reasoning
10 hours

GCO: Students will be expected to develop logical reasoning.

SCO LR01 Students will be expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

LR01.01 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
LR01.02 Explain why inductive reasoning may lead to a false conjecture.
LR01.03 Compare, using examples, inductive and deductive reasoning.
LR01.04 Provide and explain a counterexample to disprove a given conjecture.
LR01.05 Prove algebraic and number relationships, such as divisibility rules, number properties, mental mathematics strategies, or algebraic number puzzles.
LR01.06 Prove a conjecture, using deductive reasoning (not limited to two column proofs).
LR01.07 Determine if an argument is valid and justify the reasoning.
LR01.08 Identify errors in a given proof.
LR01.09 Solve a contextual problem involving inductive or deductive reasoning.

Scope and Sequence

Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
–	**LR01** Students will be expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems.	**LR01** Students will be expected to analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies. (M12)* **T06** Students will be expected to prove trigonometric identities using ▪ reciprocal identities ▪ quotient identities ▪ Pythagorean identities ▪ sum or difference identifies (restricted to sine, cosine, and tangent) ▪ double-angle identities (restricted to sine, cosine, and tangent) (PC12)**

*	M12—Mathematics 12
**	PC12—Pre-calculus Mathematics 12

Background

Students have had no formal instruction with this topic in previous mathematics courses. However, as part of their life experiences, students are expected to have drawn conclusions (conjectures) based on observations.

To develop logical reasoning skills, students will examine situations, information, problems, puzzles, and games. They will form **conjectures** through the use of **inductive reasoning** and prove their conjectures through the use of **deductive reasoning**.

A **conjecture** is a testable expression that is based on available evidence but is not yet proven.

Inductive reasoning is a form of reasoning in which a conclusion is reached based on a pattern present in numerous observations. The premise makes the conclusion likely, but does not guarantee it to be true.

Deductive reasoning is the process of coming up with a conclusion based on information that is already known to be true. The facts that can be used to prove a conclusion deductively may come from accepted definitions, properties, laws, or rules. The truth of the premises guarantees the truth of the conclusion.

Students are introduced to inductive reasoning, through the investigation of geometric situations and are introduced to conjectures through the observation of patterns. They will be required to justify the reasoning used for all conjectures.

Students will be presented with situations to explore and will be encouraged to develop conjectures based on these situations. They will also explore the role that *counterexamples* play in disproving conjectures.

Inductive Reasoning	**Deductive Reasoning**
Begins with experiences or a number of observations.	Begins with statements, laws, or rules that are considered true.
An assumption is made that the pattern or trend will continue. The result is a conjecture.	The result is a conclusion reached from previously known facts.
Conjectures may or may not be true. One counterexample proves the conjecture false.	Conclusion must be true if all previous statements are true.
Used to make educated guesses based on observations and patterns.	Used to draw conclusions that logically flow from the hypothesis.

In this Logical Reasoning unit students are introduced to the notion of formal *proof* including, but not restricted to, the *two-column proof* format.

When constructing proofs, students must use proper mathematical terminology. Students have been exposed to the Pythagorean theorem, the number system, and divisibility rules from previous grades. There are some concepts, however, that will need to be introduced, such as **supplementary angles**, **complementary angles**, and the **transitive property**.

Angles are *supplementary* when their measures add up to 180°. **Note:** Students have been exposed to the term **straight angle** and the concept that if angles *A*, *B*, and *C* make a straight angle then $\angle A + \angle B + \angle C = 180°$. However, they are not familiar with the term **supplementary**. Angles are **complementary** when their measures add to 90°.

The **transitive property** in mathematics and logic, is a statement that if *A* bears some relation to *B* and *B* bears the same relation to *C*, then *A* bears that relation to *C*.

Students will examine arguments and proofs and judge whether or not the reasoning presented is valid. They will determine if there is an error in the reasoning used and, if so, will identify the error.

Contextual problems will be solved using inductive or deductive reasoning.

Deductive reasoning involves drawing a specific conclusion through logical reasoning by starting with general statements that are known to be valid. As a part of the logical reasoning outcome, students will be introduced to geometric proofs using deductive reasoning. However, there will be a much greater emphasis placed on proofs as a part of the geometry outcomes.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Given the visual below, develop an explanation for the possible events that have occurred.

- Determine the next number in the pattern 2, 6, 12, 20, 30, 42, __.
- What is the maximum number of squares that can be created using 20 segments of equal length?
- Using a copy of Pascal's triangle, identify a pattern. (See appendix.)
- How does the way in which optical illusions "trick" your eyes relate to the ideas of valid versus invalid conjectures?

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

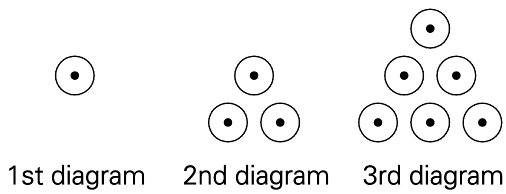
- Draw any quadrilateral. Find the midpoint of each side and connect them. What type of quadrilateral results? Compare your findings with other students and make a conjecture based on your observations.
- Jamal says that if a quadrilateral has four sides of equal length then it must be a square. Is Joe correct? Explain.

- Make a conjecture based on the following statement:
  - Shannon took her umbrella with her when she left for work each of the last four times that rain was in the forecast. Today rain is in the forecast and Shannon is getting ready to leave for work. Are you certain that your conjecture is correct? Explain.

- Find a pattern in a magazine or newspaper and describe it in words. Write a conjecture about the pattern and present your example to the class.

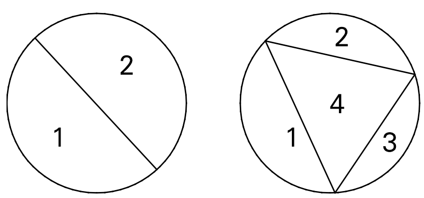
- Describe several situations in your life in which you make conjectures.

- What would the sixth diagram look like in the following sequence?



- Make a conjecture about
  - the sum of the angles in a quadrilateral
  - the product of two consecutive integers

- Points are placed on the circumference of a circle and adjacent points are joined with a straight line. Make a prediction about the number of regions formed when six points are used.



- Complete the following conjecture that holds for all the equations: The sum of two odd numbers is always ...
  3 + 7 = 10
  11 + 5 = 16
  9 + 13 = 22
  7 + 11 = 18

- Write a conjecture about combining two numbers using a mathematical operation. Show a number of examples supporting your conjecture.

- Provide an example in which a limited number of observations might lead to a false conjecture.

- Mike had breakfast at Tarek’s Cafe, a local restaurant, for three Saturdays in a row. He saw Saura there each time. Mike told his friends that Saura always eats breakfast at Tarek’s on Saturdays.
  (a) What type of reasoning did Mike use?
  (b) Is his conclusion valid? Explain.

- Provide a counterexample, if possible, for each of the following conjectures:
  (a) If a number is divisible by 2, then it is divisible by 4.
  (b) If $x + 4 > 0$, then x is positive.
  (c) For all positive numbers n, $\frac{1}{n} \le n$

- Give a counterexample, if possible, to the following conjecture: The difference between two positive numbers is always a positive number.

- For all real numbers x, the expression x^2 is greater than or equal to x. Do you agree? Justify your answer.

- Construct different sized polygons (triangle, quadrilateral, pentagon, hexagon, and octagon). Measure the interior angles. Next, find the sum of the interior angles for each polygon. Make a conjecture to determine the sum of the interior angles for a polygon with n sides.

- If $x < 0$, then $x < -x$. Do you agree? Justify your answer.

- Prove that the sum of a two-digit number and the number formed by reversing its digits will always be divisible by 11.

- Prove that the difference between an odd integer and an even integer is an odd integer.

- Do the following number puzzles and prove them deductively:
  (a) Fabulous Five:
    - Choose a number from 1 to 10.
    - Double the number.
    - Add 10 to your new number.
    - Divide the total by 2.
    - Subtract the original number that you started with.
    - Your answer will always be 5.

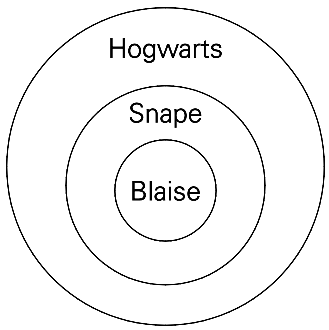
  (b) Is This Your Number?
    - Think of any number and write it on a small sheet of paper. Fold it up and place it in your pocket. Remember this number.
    - Find a partner and ask them to think of a number. Make sure they do not tell you.
    - Tell them to double the number they chose.
    - This is where the number on your paper comes in. In your head, double the number you wrote.
    - Tell your partner to add the number you doubled in your head.
    - Then tell them to divide this new number by 2.
    - Next, tell them to subtract the number they currently have from the first one they started with, or vice versa.
    - Pull out the piece of paper in your pocket and give it to your partner. They will be amazed because the number on the paper will be their number!

- Create a number puzzle that always results in a final answer of 4. Prove the conjecture deductively.

- Find an example of a number puzzle and prove it deductively.

- Identify situations in your everyday life in which you use inductive reasoning and in which you use deductive reasoning. Provide examples of each.

- Dean was given the following situation:
  - Snape is a teacher at Hogwarts. Blaise is a student at Hogwarts. Dean concluded that Blaise is a student of Snape's and represented his conclusion in the Venn Diagram shown below. Identify his error and construct a new diagram to represent this situation.

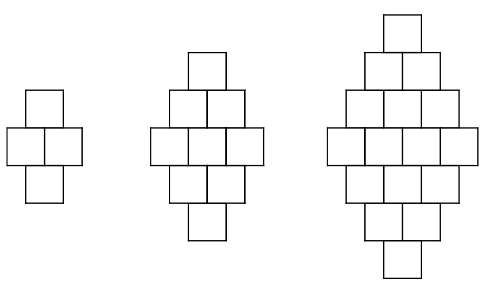


- Ten women meet for a bowling tournament, and each shakes the hand of every other woman. Determine the number of handshakes that occurred. Explain the strategy used to arrive at the answer.

- Look at a monthly calendar and pick any three numbers in a row, column, or diagonal. Using inductive reasoning, make a conjecture about the middle number. Use deductive reasoning to prove your conjecture.

- You have a 5-L bottle and a 3-L bottle, neither with any markings on the sides. How can you measure 4 L of water?

- Predict the number of squares that would be in the next (4th) diagram. Explain how you could predict the number of squares that would be found in the 10th diagram and generalize your explanation to predict the number of squares that could be found in the *n*th diagram.



- Explain how a specific game uses inductive and/or deductive reasoning.

- Margaux creates a series of rectangles, each having different dimensions, and calculated the area and perimeter. She recorded her results as shown.

Length (cm)	**Width (cm)**	**Area (cm^2)**	**Perimeter (cm)**
5	4	20	18
10	3	30	26
6	6	36	24
8	3	24	22

Margaux makes the conjecture that the area of a rectangle is always greater than its perimeter. Do you agree with this conjecture? Justify your reasoning.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Students could be introduced to this unit through the analysis of real-life situations.

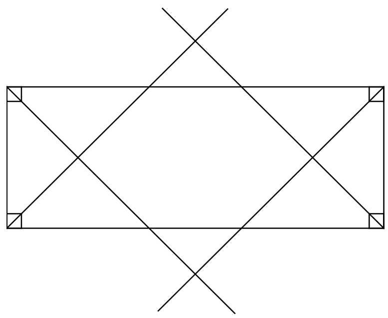
- Show students a video clip of Young Sherlock Holmes to illustrate inductive reasoning. Ask students to note as many examples of inductive reasoning that they hear and identify which ones may lead to a false conjecture.

- Give students an opportunity to make conjectures using a variety of strategies. Encourage them to draw pictures, construct tables, use measuring instruments such as rulers and protractors and/or use geometry software such as FX Draw (Efofex Software 2013) or Geometer's Sketchpad (Key

Curriculum 2013). Dynamic geometry programs, can be used to construct accurate diagrams of geometrical objects. Once students have been exposed to these programs, more complicated conjectures can be investigated which could, in turn, lead to valuable classroom discussions.

- Provide students with specific criteria and encourage them to generate their own conjectures. For example, give students an opportunity to choose a class of quadrilaterals and ask them to identify any properties of the angle bisectors or perpendicular bisectors of the sides of those figures. A sample of answers may include the following:

  – The angle bisectors of a rectangle make a square.

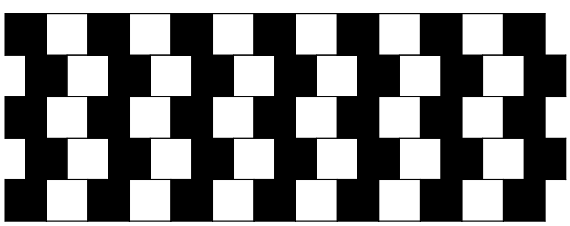


  – In a kite, the point where all the angle bisectors meet is the centre.
  – In an isosceles trapezoid, the two base angle bisectors form an isosceles triangle.

  Students should be encouraged to share their answers with other classmates, to discuss what they observe, and to explain how they came up with their conjectures.

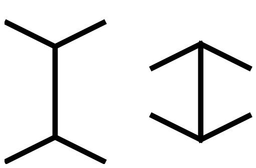
- Students should be presented with situations where they are expected to make a conjecture and then verify or discredit it. They will be expected to gather additional evidence, such as taking measurements, performing calculations, or extending patterns, to determine if a given argument is valid. Consider the following examples:

  – Make a conjecture about the horizontal lines in this diagram.



  Students may initially believe that the lines are slanted. Using a ruler to investigate their conjecture, they will discover that the lines are, in fact, straight.

  – Make a conjecture about the length of the vertical line segments.



In this example, students may make a conjecture that the vertical line segment on the left is longer than the vertical line on the right. If they use a ruler to test their conjecture, they will discover that both lines are actually the same length. Students should then revise their conjecture.

- Students should develop an understanding that inductive reasoning sometimes leads to false conjectures. They can explore situations in which the collection of additional data, for example, refutes an initial conjecture. Consider an example such as the following:
  - A small bakery is building its business. The owner notices that sales are increasing rapidly in the first three weeks of operation. The following table gives a record of the owner's observations:

Week	1	2	3
Number of Cakes Sold	10	15	20

Jaden conjectures that this bakery is selling 5 additional cakes each week. He further conjectures that by the end of the year (52 weeks) this business will be selling more than 250 cakes per week. Are these conjectures valid? Explain.

Students may initially believe that these conjectures are valid, but upon further exploration, they need to recognize that, even if this trend could continue through increased consumer demand, at some point the bakery would reach its maximum capacity. Furthermore, it is unlikely that consumer demand would continue to increase in such a tidy and constant way. While a conjecture might be reasonable for a small domain, and perhaps Jaden's conjectures may be correct for a limited number of weeks, using his conjecture to make projections over a larger domain is likely flawed reasoning.

- When students make a conjecture, they will attempt to either prove their conjecture by constructing a valid argument or they will attempt to disprove their conjecture by finding a counterexample. One counterexample is sufficient to disprove a conjecture. To introduce this topic, students could be asked to come up with counterexamples for statements such as the following:
  - Any piece of furniture having four legs is a table.
  - If a tree has no leaves then it is winter.
  - If the grass is wet, it is raining.

- Students will continue to use strategies, such as drawing pictures or using measuring instruments, when looking for a counterexample. After a counterexample is found, students should revise their conjecture to include the new evidence.

Example
A student may conjecture that all Pythagorean triples can be created by procedure described below.

- Start with an odd number (> 1). This is the first of the three numbers in the triple.
- Square that number, then divide by 2.
- Round the result down and up, giving the remaining two numbers in the triple.

Example
- Start with 5.
- Square to get 25, and divide by 2 to get 12.5.
- Round down to get 12, and up to get 13.
- Triple is 5-12-13.

This pattern works for the Pythagorean triples 3-4-5; 5-12-13; 7-24-25; 9-40-41, and many others. But there are many Pythagorean triples that begin with an even number (such as 8-15-17) and some Pythagorean triples that begin with an odd number and cannot be created this way (such as 33-56-65). These counterexamples might have the students revise their conjecture.

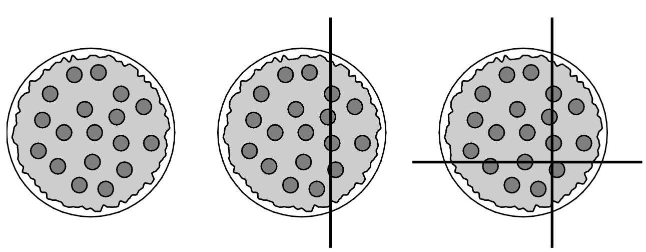
- To reinforce the idea of counterexamples:
  - Provide students with cards containing a conjecture that may be disproved using counterexamples, and have them complete a Quiz-Quiz-Trade task. Student 1 must read the conjecture on the card, and student 2 must provide a counterexample to disprove the statement. When both students have disproved the conjectures, they exchange cards and find a new partner. Examples of conjectures include the following:
    - If a number is divisible by 2, then it is divisible by 4.
    - No triangles have two sides of the same length.
    - All basketball players are more than 6 feet tall.
    - If it is a cell phone, then it has a touch screen.
  - Ask students to participate in the task, Find Your Partner. Half of the students should be given a card with a conjecture on it and the other half should be given a card with a counterexample on it. Students need to move around the classroom to match the conjecture with the correct counterexample. They should then present their findings to the class.

- Have students research either a scientific or mathematical theory that was disproven using counterexamples. Ask them to explain the theory and how it was disproven.

- When testing conjectures involving numbers, students may be unable to find a counterexample because they often use the same types of number. For example, students may only consider natural numbers. They should be encouraged to try various types of numbers, such as whole numbers, fractions, positive numbers, negative numbers, and zero.

- It is important for students to realize that being unable to find a counterexample does not prove a conjecture. The mathematician Christian Goldbach is famous for discovering a conjecture that has not yet been proved to be true or false. Goldbach conjectured that every even number greater than 2 can be written as the sum of two prime numbers. For example, 4 = 2 + 2, 6 = 3 + 3, and 8 = 3 + 5. No one has ever proved this conjecture to be true or found a counterexample to show that it is false, so the conjecture remains just that. As an extension, students can make a list of prime numbers at least up to 100, and use them to test the truth of the conjecture for even numbers up to 100.

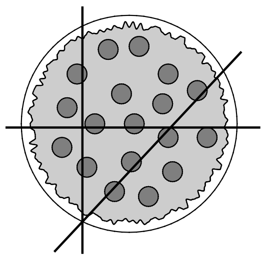
- As teachers observe students work, they should question students about the considerations that have to be taken into account. For example, for the following question:
  - Melissa made a conjecture about slicing pizza. She noticed a pattern between the number of slices of pizza and the number of cuts made in the pizza.



Number of cuts	0	1	2
Number of slices of pizza	1	2	4

Her conjecture was that the number of pizza slices doubled with each cut. Do you agree or disagree? Justify your decision.

- Prompt discussion using the following questions for the conjecture:
  - Is it important to take into account how the pizza is cut?
  - Do all the slices of pizza have to be the same area?
- A possible counterexample for this situation is shown below:



- Ask students to work in groups for various activities. For example:
  - Each group will be given a deck of cards with a variety of conjectures on them. Students will work together to determine which cards use inductive reasoning and which cards use deductive reasoning.

- Create centres in the classroom containing proofs that contain errors. Ask students to participate in a carousel activity in which they move through the centres while identifying and correcting the errors.

- Video clips can be used effectively to discuss logical reasoning. For example, a video clip of a court scene, such as one from *A Few Good Men*, *CSI*, *Lincoln Lawyer*, or another similar show/movie, could be shown in order to prompt a discussion on how inductive and/or deductive reasoning is used in the clip.

- Students could begin their exploration of proof by looking at examples involving the transitive property. Consider the following example: All natural numbers are whole numbers. All whole numbers are integers. Three is a natural number. What can be deduced about the number 3?

- Students should be exposed to the different strategies they can use to prove conjectures.
  - Students can use a visual representation, such as a Venn diagram, to help them understand the information in an example.
  - They can express conjectures as general statements. This involves choosing a variable to algebraically represent the situation. For example, two consecutive integers can be represented as x and $x + 1$.
  - Students can use a two-column proof that contains both statements and reasons. As an alternative, the steps used to prove the conjecture can be communicated in paragraph form.

- It is important for students to recognize that depending on the problem, certain strategies are more efficient than others. This is an opportunity for students to work in groups to investigate a number of proofs given a specific method. After students have completed the task, pose the following questions to promote class discussion:
  - What strategy was used to prove your conjecture?

- Could you use another strategy to prove the conjecture?
- Is one strategy more efficient than another? Why?

- Students should recognize that proving conjectures for varying situations may require different formats. When proving a conjecture involving a geometric figure, for example, a two-column proof or paragraph form is most appropriate. To prove a conjecture involving number properties, divisibility rules, or algebraic number tricks, expressing conjectures as general statements is often the most appropriate method.

- When constructing proofs, including algebraic and number relationships, a common error occurs when students use a numerical value as their example. Students need to understand that a conjecture is only proved when all cases are shown to be true. Consider the following example: Choose a number. Add 5. Double the result. Subtract 4. Divide the result by 2. Subtract the number you started with. The result is 3.

- Students can show inductively, using three or four numbers, that the result is always 3. To prove deductively, students must show this is the case for all numbers by letting the first number be x.

- It is important for students to compare inductive and deductive reasoning through the use of examples. The example below could be used to highlight their differences.
  - The sum of four consecutive integers is equal to the sum of the first and last integers multiplied by two.

Inductive Reasoning	**Deductive Reasoning**
1, 2, 3, 4 1 + 2 + 3 + 4 =10 2(1 + 4) = 10	Let the numbers be x, $x + 1$, $x + 2$, $x + 3$. $x + (x + 1) + (x + 2) + (x + 3) = 4x + 6$ $2[x + (x + 3)] = 2(2x + 3) = 4x + 6$

- Exploring the differences between inductive and deductive reasoning through examples will strengthen student understanding of these concepts. The table below could be used to summarize inductive reasoning versus deductive reasoning.

Inductive Reasoning	**Deductive Reasoning**
Begins with experiences or a number of observations.	Begins with statements, laws, or rules that are considered true.
An assumption is made that the pattern or trend will continue. The result is a conjecture.	The result is a conclusion reached from previously known facts.
Conjectures may or may not be true. One counterexample proves the conjecture false.	Conclusion must be true if all previous statements are true.
Used to make educated guesses based on observations and patterns.	Used to draw conclusions that logically flow from the hypothesis.

- It is beneficial to have students analyze proofs that contain errors. To reinforce their understanding of inductive and deductive reasoning, students should identify errors in a given proof and explain why those errors might have occurred and how they can be corrected. Some typical errors include the following:
  - Proofs that begin with a false statement:
    - All high school students like Facebook. Shoshannah is a high school student; therefore, Shoshannah likes Facebook.
    - Gathering a large quantity of data will strengthen the validity of the conjecture, but to prove a conjecture all cases must be considered. Students should consider if it is possible to ask all high school students if they like Facebook.

  - Algebraic errors

Shelby was trying to prove this number trick:	Shelby wrote the following:
▪ Pick a number.	Let n be your number.
▪ Double your number.	$2n$
▪ Add 20.	$2n + 20$
▪ Divide by 2.	$n + 20$
▪ Subtract the original number.	$n + 20 - n$
▪ The result is 10.	20

Students should identify and correct the error in Shelby's work.

  - Division by zero
    - Pedro claims he can prove that 2 = 5. His work is shown below.

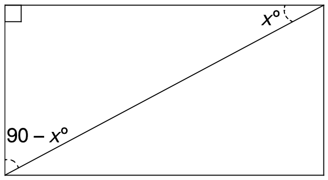
Suppose $a = b$.

$-3a = -3b$	Step 1: Multiplying by –3.
$-3a + 5a = -3b + 5a$	Step 2: Add $5a$ to both sides.
$2a = -3b + 5a$	Step 3: Simplify
$2a - 2b = -3b + 5a - 2b$	Step 4: Subtract $2b$ from each side.
$2a - 2b = 5a - 5b$	Step 5: Simplify
$2(a - b) = 5(a - b)$	Step 6: Factor
$\frac{2(a-b)}{(a-b)} = \frac{5(a-b)}{(a-b)}$	Step 7: Divide by $(a - b)$
$2 = 5$	

This type of question may be more challenging for students because, algebraically, there does not appear to be any mistakes in Pedro's work. Students should analyze Step 6 and Step 7 carefully.

In Step 6, students may use substitution instead of division. Since $a - b = 0$, students may write $2(a - b) = 5(a - b)$ as $2(0) = 5(0)$ resulting in $0 = 0$. In Step 7, students may write $2(a - b) = 5(a - b)$, as $2 = 5$ without realizing they have even divided. Ask students to substitute $a = b$ back into the equation. When working with proofs that involve division, students should check to see if the divisor is zero.

- Circular reasoning
  An argument is circular if its conclusion is among its premises. Darren claims he can prove that the sum of the interior angles in a triangle is 180°. He draws the following rectangle.

  Here is his proof:

  I constructed a rectangle. Next, I drew a diagonal. I knew that all of the angles in a rectangle are 90°. I labelled one of the other angles in the triangle x. Therefore, the other angle must be $180° – 90° – x = 90° – x$. Then, $90° + x + (90° – x) = 180°$.

  Students must realize that they cannot assume a result that follows from what they are trying to prove.

- Students should be exposed to problem-solving situations that require the use of inductive and/or deductive reasoning. They will explore some situations in which they are asked to first show inductively that a pattern exists and then prove it deductively. It is important for students to recognize that inductive and deductive reasoning are not separate entities—they work together. Consider the following example:
  - Hani was investigating patterns on the hundreds chart. He was asked to choose any four numbers that form a 2 × 2 square on the chart. He chose the following:

4	5
14	15

  Hani should be able to use inductive reasoning to make a conjecture about the sum of each diagonal and then use deductive reasoning to prove his conjecture is always true.

- When observing groups of students as they are working on a task you may wish to ask questions such as the following:
  - What strategy did your group use? Why did you choose this strategy?
  - Could you use another strategy? If so, which one? Why?
  - Is one strategy more efficient than another?

Suggested Models and Manipulatives

- protractors
- rulers

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- complimentary angles
- conjectures
- counterexample
- deductive reasoning
- inductive reasoning
- proof
- supplementary angles
- transitive property
- two-column proof

Resources/Notes

Internet

- *Basic-mathematics.com*, "Examples of Inductive Reasoning" (Basic-mathematics.com 2008)
  www.basic-mathematics.com/examples-of-inductive-reasoning.html
  Basic examples of inductive reasoning.

- Movieclips Beta, "Young Sherlock Holmes: Watson Meets Holmes" (Movieclips, Inc. 2014)
  http://movieclips.com/WijX-young-sherlock-holmes-movie-watson-meets-holmes
  The following site provides a clip from *Young Sherlock Holmes*.

- The iPhone/iPod/iPad application called *Crack the Code* is an application involving deductive and/or inductive reasoning.

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  – Sections 1.1–1.7, pp. 6–57

Notes

SCO LR02 Students will be expected to analyze puzzles and games that involve spatial reasoning, using problem-solving strategies. [CN, PS, R, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(It is intended that this outcome be integrated throughout the course by using translations, rotations, construction, deconstruction and similar puzzles and games.)

LR02.01 Determine, explain and verify a strategy to solve a puzzle or to win a game; for example,
- guess and check
- look for a pattern
- make a systematic list
- draw or model
- eliminate possibilities
- simplify the original problem
- work backward
- develop alternative approaches

LR02.02 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.

LR02.03 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

Scope and Sequence

Mathematics 10	Mathematics 11	Mathematics 12
–	**LR02** Students will be expected to analyze puzzles and games that involve spatial reasoning, using problem-solving strategies.	**LR01** Students will be expected to analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.

Background

This outcome is intended to be integrated throughout the course by using puzzles and games focused on translations, rotations, construction, and deconstruction. These puzzles are intended to help enhance spatial reasoning and deepen problem-solving strategies. Numerical reasoning and logical reasoning using puzzles and games will be addressed in Mathematics 12.

Students need time to play and enjoy each game before analysis begins. They can then discuss the game, determine the winning strategies, and explain these strategies by demonstration, orally, or in writing.

A variety of puzzles and games, such as board games, online puzzles and games, and pencil-and-paper games should be used.

Problem-solving strategies will vary depending on the puzzle or game. Some students will explain their strategy by working backwards, looking for a pattern, using guess and check, or by eliminating possibilities, while others will plot their moves by trying to anticipate their opponents' moves. As students play games and analyze strategies, they explore mathematical ideas and compare different strategies for efficiency.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

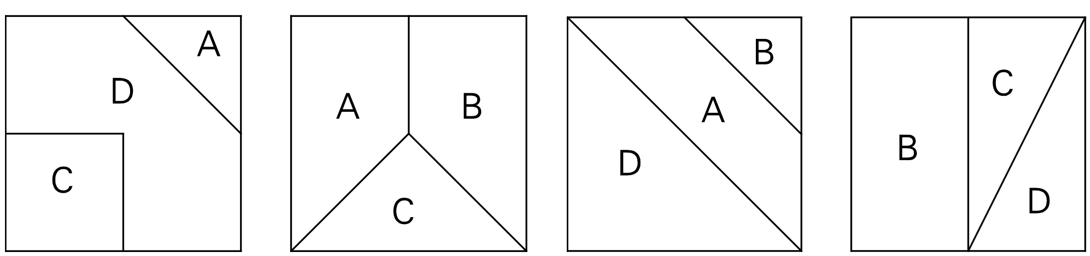
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Name a game or puzzle that you have played.
- Play a game of tic-tac-toe. What strategy do you use?

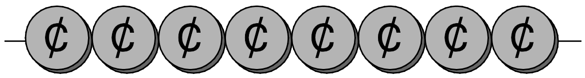
Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

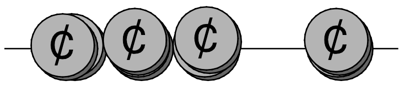
- The squares, shown below, are cut up, and each member of a group of four is given three pieces marked with the same letter. The task for the group is to make four complete squares.



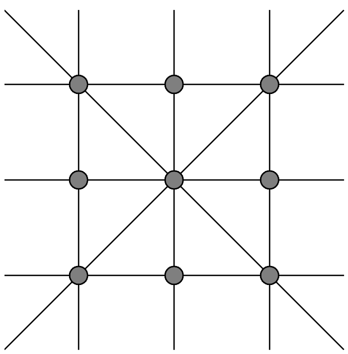
- Take eight coins and arrange them in a row. Your goal is to end up with four piles of two coins each using four moves. Each move consists of picking up a coin, jumping in either direction over two “piles” and landing on the third. Each pile may consist of a single coin or of two coins.
  (a) What strategies did you use?
  (b) What challenges occurred while trying to solve the puzzle?



Goal:

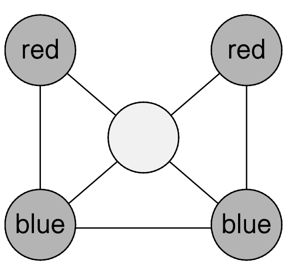


- Take nine coins and arrange them on a 3 × 3 grid as shown to the right. In this configuration, there are eight straight lines that each pass through three coins: three horizontal, three vertical and two diagonal. Now move exactly two coins so that there are ten straight lines, each with three coins. **Note:** In the solution, there are no straight lines with more than three coins, and no coin is placed on top of another.



  (a) What strategies did you use?
  (b) What challenges occurred while trying to solve the puzzle?

- An example of a board game that resembles tic-tac-toe is Pong Hau K’i, which is also played by two players. The game board is made up of five circles joined by seven line segments. Draw a game board.



Rules

- Each player will have two stones (counters or buttons) of different colours. Place two red stones at the top, for example, and two blue at the bottom as shown.
- Players take turns sliding one stone along a line to an adjacent empty circle.
- To win, you have to block the other player so that he or she cannot move.

Play the game several times, taking turns making the first move. Ask the following questions as you play the game:

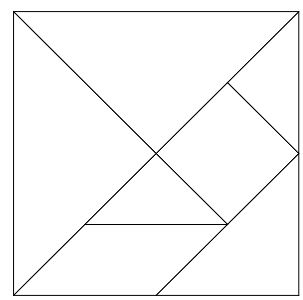
- Where will the first move always be?
- What does the board look like when one player is blocked? Why will this help the player?
- Is it better to have the red stones or blue stones?
- Is it better to make the first move or second move?
- Is anticipating the possible moves ahead an effective game plan strategy? Explain why or why not.

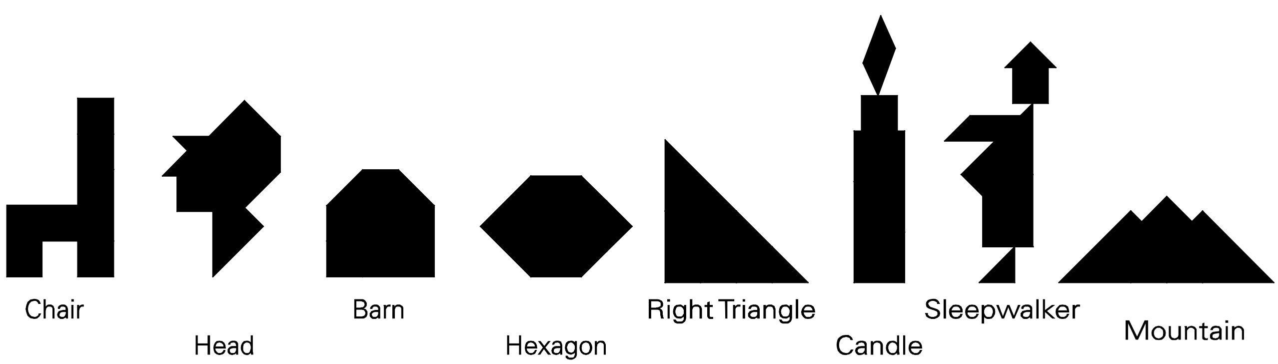
A variation of this game is to start from different positions.

- Decide which colour each player will use and who will place the first stone.
- After the first stone is placed on any circle, the other player places a stone on any of the four remaining circles and so on until all four stones are on the board.
- The game is continued by moving the pieces as previously described.

Play this modified game several times. Is it more challenging than the original game?

- The tangram puzzle was invented in China thousands of years ago. The object is to arrange all seven pieces of the tangram (cut from a square as shown to the right), into various shapes just by looking at the outline of the solution. Try making the shapes shown below, using all seven pieces.

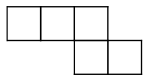
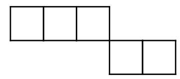




Design other puzzles and challenge your classmates to solve them.

- Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

- Pentominoe Challenges:
  - *Challenge #1*: A pentomino is made from five squares. The squares must be properly joined edge to edge so that they meet at the corner. For example,

     is a pentomino, but  is not a pentomino.

    There are only 12 in the set (because shapes that are identical by rotation or reflection are not counted). Create the set of 12 pentomoinoes.

  - *Challenge #2:* The 12 pentomino pieces fit together to form a rectangle. In fact, there is more than one rectangle that can be created with the 12 pentominoes. Form as many different rectangles as possible with all 12 pieces.

  - *Challenge #3—A Game:* Working with a partner and using pentomino playing grids of 6 × 10, 5 × 12, 4 × 15 and 3 × 20 units, take turns drawing pentominoes in pencil. Co-operative thinking is allowed! When you are satisfied that you have drawn a pentomino that is not already on the grid, colour it (use a different colour for each pentomino). Remember ... no two pentominoes can share a square. Remember ... each of the 12 pentominoes may be used only once.
    Caution: Watch out for translations, reflections, and rotations.
    > The game ends when there is no room available for another pentomino.
    > Record the number of pentominoes created by your team.
    > Play another game using any of the blank pentomino playing grids. The goal is to beat your team score.
    > Continue playing additional games, adjusting the strategy so that the goal is to maximize your score.

  - *Challenge #4:* Take the T-pentomino. Use 9 of the remaining 11 pentominoes to make an enlarged scale model of the T-pentomino.
    > What is the area of the enlarged model?
    > How does this area compare with the area of the T-pentomino?
    > What are the lengths of the each of the sides of the enlarged model?
    > How do these lengths compare with the corresponding sides of the T-pentomino?
    > Repeat this process for the W-pentomino and the X-pentomino.
    > What other pentominoes can you make larger versions of?
    > What strategy can you use to determine which pentominoes can have larger versions created?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

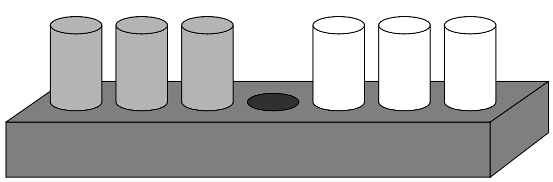
Consider the following sample instructional strategies when planning lessons.

- The following are some tips for using games in the mathematics class:
  - Use games for specific purposes.
  - Keep the number of players in each group to two to four, so that turns come around quickly.
  - Communicate to students the purpose of the game.
  - Engage students in post-game discussions.
  - When students feel that they have obtained a strategy for the solution of a puzzle or game, ask them to share that strategy with the class.

- As students play games or solve puzzles, ask probing questions and listen to their responses. Record the different strategies and use these strategies to begin a class discussion. The following are some possible discussion starters.
  - Thumbs up if you liked the game, thumbs sideways if it was okay, and thumbs down if you did not like it. What did you like about it? Why?
  - What did you notice while playing the game?
  - Did you make any choices while playing?
  - Did anyone figure out a way to quickly find a solution?

  Note: If you are pressed for time, you could use exit cards and then debrief the following day.

- As students play a game, it is important to pose questions and engage students in discussions about the strategies they are using. For example, consider the following peg game.

  The goal of the puzzle is to switch the pegs on the left with the pegs on the right by moving one peg at a time. A peg may only be moved to an open slot directly in front of it or by jumping over a peg to an open slot on the other side of it. You may not move backwards. The game ends when you win or get stuck.

As you observe students playing the game, ask them questions such as the following:

- What did you like about the game? Why?
- What did you notice while playing?
- What was your first move?
- What were your first three moves?
- What problems arose when solving the puzzle?
- What are the minimum number of moves to win this game?
- Did you anticipate the next move?
- Did you notice any patterns?

- Students should be encouraged to explore a simpler version of the game where appropriate. Solving the peg puzzle using two or four pegs, for example, may allow students to complete the game more efficiently.

- Try games in advance as instructions to games are not always clear.

- Have students look for patterns and then develop strategies to fit these patterns. Sometimes these patterns are numerical, such in magic squares and Sudoku.

- Have students develop a game for classmates to play and/or use a known game. Students can then change a rule or parameter and then explain how it affects the outcome of the game.

- Find a game online and critique the quality of the game.

SUGGESTED MODELS AND MANIPULATIVES

- board games such as Blokus and Tetris with an emphasis on spatial reasoning
- coins or counters
- grid paper
- pentominoes (may be made with card stock using the diagrams from the appendix)
- tangrams (may be made with card stock using the diagrams from the appendix)

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- reflection
- rotation
- translation

Resources/Notes

Internet

There are numerous games and puzzles available on the Internet. What follows are just a few suggestions of spatial games and puzzles, available for free online. Many can also be acted out, done on paper, or done with models.

Note: It is expected that the teacher confirm the validity of the site prior to directing students to it.

- Math Is Fun site
  www.mathsisfun.com
  - Towers of Hanoi (2011)
    www.mathsisfun.com/games/towerofhanoi.html
  - Tic-Tac-Toe (2012)
    www.mathsisfun.com/games/tic-tac-toe.html
  - Dots and Boxes Game (2012)
    www.mathsisfun.com/games/dots-and-boxes.html
  - Four in a Line (2012)
    www.mathsisfun.com/games/connect4.html

- Illuminations: Resources for Teaching Math, “Calculation Nation” (National Council of Teachers of Mathematics 2014)
  http://calculationnation.nctm.org/Games
  - Nextu—In this game, players alternate claiming shapes on a tessellation with shapes of greater value than the adjoining shape.
  - Flip-n-Slide—This is a game in which triangles are translated and reflected to capture ladybugs. It is spatially challenging to envision the final position of the triangle.

- *Nrich: Enriching Mathematics* (University of Cambridge 2014)
  http://nrich.maths.org
  - Sprouts is a game for two players, and can be played with paper and a pencil. The rules are simple, but the strategy can be complex. The site discusses the game, its history, and strategies for solving the puzzle.
  - Frogs and Toads is a well-known puzzle in which frogs and toads must change places in as few turns as possible.

- Jill Britton (personal site)
  http://britton.disted.camosun.bc.ca/nim.htm
  - Nim is an ancient game that can be played online, on paper, or using sticks. The winner is the player to not pick up the last stick (Common Nim) or to pick up the last stick (Straight Nim).

- Cool Math
  www.coolmath-games.com/0-b-cubed/index.html
  - Cubed is a game where students must pass over each block prior to reaching the final red block. Other spatial games on this site include Pig Stacks, Aristetris, and Bloxorz.

- Board Games
  - Blockers
  - Blokus
  - Chess
  - Free Flow
  - Rushhour
  - Towers of Hanoi

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Section 1.7, pp. 52–57

Notes

Statistics
20–25 hours

GCO: Students will be expected to develop statistical reasoning.

SCO S01 Students will be expected to demonstrate an understanding of normal distribution, including standard deviation and *z*-scores. [CN, PS, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

S01.01 Explain, using examples, the meaning of standard deviation.
S01.02 Calculate, using technology, the population standard deviation of a data set.
S01.03 Explain, using examples, the properties of a normal curve, including the mean, median, mode, standard deviation, symmetry, and area under the curve.
S01.04 Determine if a data set approximates a normal distribution and explain the reasoning.
S01.05 Compare the properties of two or more normally distributed data sets.
S01.06 Explain, using examples that represent multiple perspectives, the application of standard deviation for making decisions in situations such as warranties, insurance, or opinion polls.
S01.07 Solve a contextual problem that involves the interpretation of standard deviation.
S01.08 Determine, with or without technology, and explain the *z*-score for a given value in a normally distributed data set.
S01.09 Solve a contextual problem that involves normal distribution.

Scope and Sequence

Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
–	**S01** Students will be expected to demonstrate an understanding of normal distribution, including standard deviation and *z*-scores.	–

Background

In Mathematics 7, students explored the measures of central tendency, range, and the effect of outliers on a given data set (7SP01). As part of this statistics outcome, students should be exposed to situations where the measures of central tendency are not always sufficient to represent or compare sets of data.

Descriptive statistics describe the population we are studying. The data could be collected from either a sample or a population, but the results help us organize and describe data. Descriptive statistics can only be used to describe the group that is being studied. That is, the results cannot be generalized to any larger group.

Descriptive statistics are useful and serviceable if you do not need to extend your results to any larger group. However, many uses of statistics in society tend to draw conclusions about segments of the population, such as all parents, all women, all victims, etc.

Inferential statistics are used to make predictions or inferences about a population from observations and analyses of a sample. That is, we can generalize the statistics from a sample to the larger population that the sample represents. In order to do this, however, it is imperative that the sample is representative of the group to which it is being generalized.

In this Statistics unit, students will also consider that the measure of dispersion they learned in Mathematics 7, range, does not provide any information about the variation within the data values themselves since it is based on the two extreme values of the data set. Students will be introduced to standard deviation, which takes into account every score in a distribution, as a useful way to compare two or more sets of data.

In this course, students will be calculating standard deviation of a population represented by the symbol σ.

$$\sigma=\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

Students should be exposed to calculating standard deviation manually to provide an understanding of how standard deviation is determined. Importantly, this should be done using an example with a small amount of data to keep calculations simple. The benefit of using technology to solve contextual problems is that data sets are often large. Students are expected to calculate standard deviation using technology such as scientific calculators, graphing calculators, or computer software programs (e.g., Microsoft Excel).

Students will be introduced to **histograms** and **frequency polygons**. Both provide a pictorial representation of the data. These will serve as the basis for the development of the normal curve.

In this Statistics unit, students will discover that, in many situations, if a controlled experiment is repeated and the data are collected to create a histogram, the histogram becomes more and more bell-shaped as the number of data in the distribution increases. Teachers may find the Free Statistics and Forecasting Software (Wessa.Net 2014) site (www.wessa.net) to be very helpful in illustrating that, when the population is normal and the data are collected randomly, such repeated measurements will produce bell-shaped distributions that exhibit the characteristics of a normal curve.

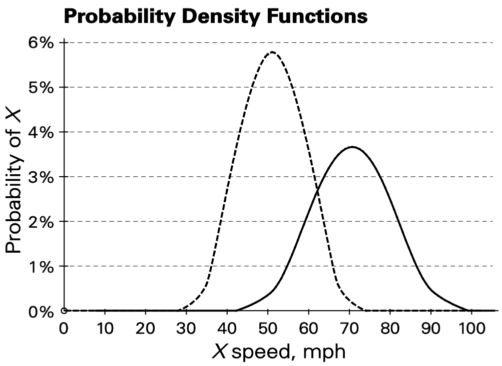
Throughout the unit, it is important to remind students that an attribute may be considered to be normally distributed in a population, but a random sample may not show a normal distribution. Doing a simulation on a calculator will often show students that sample size is relevant. That is, for a small sample size, you may repeat the experiment several times and the histogram will not become bell-shaped but for a very large sample the histogram may appear normal without the need to repeat the experiment multiple times to see the normal curve.

A rule of thumb often suggested is that a sample size of 100, that is obtained using a random sampling method, will be roughly normally distributed if the population it represents is normally distributed.

The heights of adult males are normally distributed. The heights of adult females are also normally distributed. When you consider the heights of adults without the gender differentiation, however, the data tend to be bimodal rather than normal in its distribution. Other examples of variables that are not normally distributed are the length of time that a person stays in a job and a soldier’s age within an army platoon.

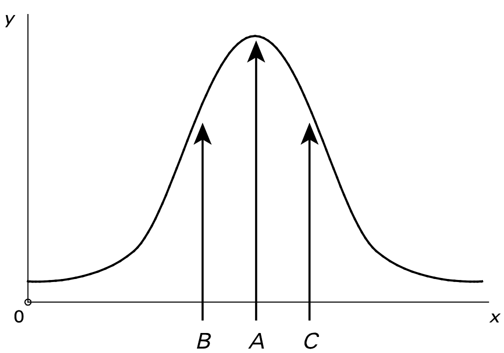
Many variables are normally distributed such as students' marks on examination tests, blood pressure, heights of people, and errors in measurement.

In this unit the focus is on populations that are normally distributed. Students will compare two or more normal distributions by comparing the mean and standard deviation of each normal distribution. The mean will determine the location of the centre of the curve on the horizontal axis. The standard deviation will determine the width and height of the curve.



------ mean = 52. Standard Deviation = 7
_____ mean = 71. Standard Deviation = 11

For populations of data that are normally distributed, when enough data are plotted, the curve will be symmetrical, and it will be concave down for the section that is within one standard deviation of the mean and will be concave up for other areas.



This means that you can very roughly estimate the standard deviation by looking at a carefully constructed normal curve distribution. However, it can be difficult to accurately locate this point visually where the curve changes concavity so caution should be taken when using the graph to determine the standard deviation visually.

Point A: Mean
Point B: 1 standard deviation below mean
Point C: 1 standard deviation to the right of the mean

If two sets of data have the same mean, for example, but different standard deviations, the distribution of data with a lower standard deviation would result in a graph that is taller and narrower.

A ***z*-score** is a standardized value that indicates the number of standard deviations a data value is from the mean. It can only be used when the sample is normally distributed. For example, if $z = 2$, it simply means that the score is 2 standard deviations to the right of the mean. For example, in a normally distributed population with a mean of 25 and a standard deviation of 3, a score of 22 would have a z-score of –1.

A z-score can be calculated using the formula $z = \dfrac{x - \mu}{\sigma}$.

X – data input (value)
μ – population mean
σ – population standard deviation

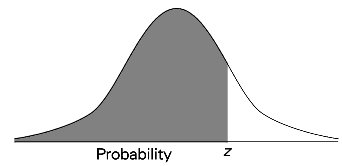
When students convert normally distributed scores into z-scores, they can determine the probabilities of obtaining specific ranges of scores using a z-score table or technology. There are several different z-score tables. Three possible tables are illustrated below. It is helpful for students to have exposure to a variety of these tables. It is also useful for students to use technology such as the graphing calculator to calculate the probabilities associated with *z*-scores.

Standard Normal Cumulative Probability Tables

Cumulative probabilities for *positive* *z*-values are shown in the following table:

Table 1

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.5	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998



Cumulative probabilities for *negative* z-values are shown in the following table:

Table 2

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

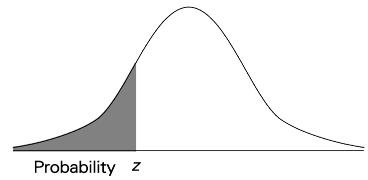
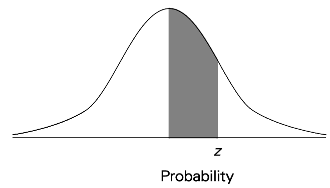


Table 3

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3236	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4367	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4809	0.4911	0.4913	0.4915
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4983	0.4989	0.4989	0.4989	0.4990	0.4990



For this Statistics unit, students will be exposed to situations that require the use of *z*-scores in the discussion of

- particular scores within a set of data
- whether or not a score is above or below average
- how far away a particular score is from the average
- the comparison of scores from different sets of data and figuring out which score is better

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- How can you find the range of a set of data? Create an example to illustrate.
- Could a data set have more than one mean? One mode? One median? Explain, using an example.
- What is an outlier? How does a data set with an outlier differ from one with a gap?
- Give a context in which the mode would be the most suitable measure of central tendency.
- Give a context in which the median would be the most suitable measure of central tendency.
- Give a context in which the mean would be the most suitable measure of central tendency.
- In a science experiment, a group of students tested whether compost helped plants grow faster by counting the number of leaves on each plant. The following results were obtained:

Plant growth without compost (# of leaves per plant)	**Plant growth with compost (# of leaves per plant)**
6	6
4	11
5	1
4	6
8	2
3	4

  (a) Calculate the measures central tendency for each group.
  (b) Calculate the range for each group.
  (c) Which group of plants grew better? Justify your decision.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Calculate the range of each group. Explain why the range, by itself, can be a misleading measure of dispersion.
  Group A: 8, 13, 13, 14, 14, 14, 15, 15, 20
  Group B: 7, 7, 8, 9, 11, 13, 15, 15, 17, 18
- The marks awarded for an assignment for a Grade 11 class of 20 students are given below.

6	7	5	7	7	8	7	6	9	7
4	10	6	8	8	9	5	6	4	8

  Display the data on a histogram and describe how the data are distributed.

- In December, the number of hours of bright sunshine recorded at 36 selected stations was as follows:

16	25	41	20	35	20	16	8	38	23	25	38
41	34	24	39	47	45	17	42	44	47	45	51
35	37	51	39	14	14	40	44	50	40	31	22

  – Create a histogram, ensuring that it includes a title and all labels.

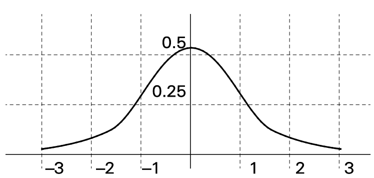
- Two obedience schools for dogs monitor the number of trials required for twenty puppies to learn the *sit and stay* command.

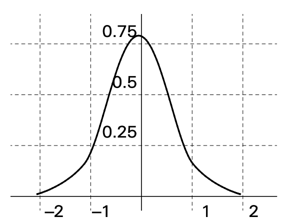
True Companion Dog School	
Number of trials	Number of puppies
7	1
8	2
9	5
10	4
11	4
12	4

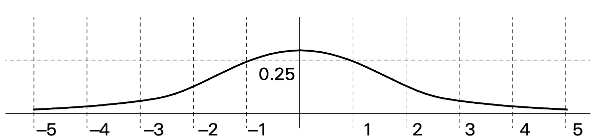
Dog Top School	
Number of trials	Number of puppies
7	4
8	3
9	2
10	3
11	4
12	4

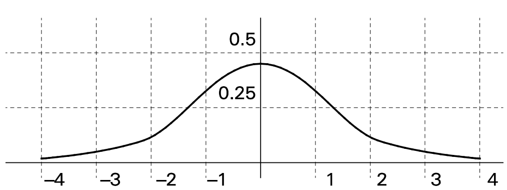
  (a) Determine the mean and standard deviation of the number of trials required to learn the "sit and stay" command.
  (b) Which school is more consistent at teaching puppies to learn the *sit and stay* command? Justify your reasoning.

- Which normal distribution curve in the graphics below has the largest standard deviation? Explain your reasoning.

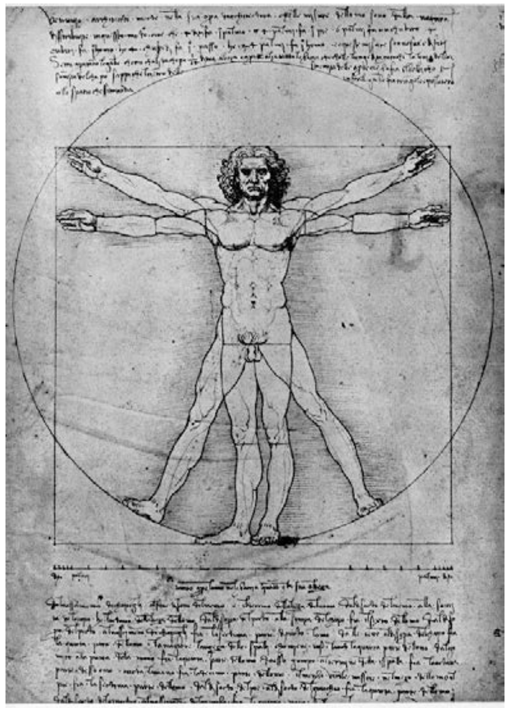








- One of Leonardo da Vinci's well-known illustrations shows the proportions of a human figure by enclosing it in a circle. It appears that the arm span of an adolescent is equal to the height.



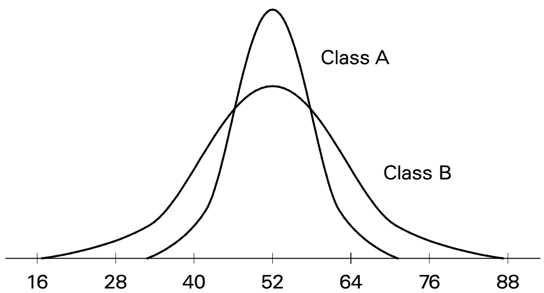
(a) Working with a partner, measure each other's arm span and height. The teacher will collect all the classroom data sheets and create two groups of data sets (one for arm span and one for height).
(b) Use the class data to calculate the mean and standard deviation for each of the two sets of data. Compare the values and explain what you notice.
(c) Calculate the difference between each person's height and arm span. From the data, what would they expect the mean and standard deviation of this data set to be? Check your prediction for ten students in the class.

- A data set of 50 items is randomly selected from a normal population and shown below. The standard deviation of this sample is 1.8.

6	9	6	7	6	7	6	7	6	6
5	8	5	8	6	7	7	4	8	6
5	4	6	8	6	6	7	6	6	8
8	7	6	6	0	4	7	8	5	3
6	7	6	7	1	7	7	4	4	3

(a) What is the value of the mean, median, and mode?
(b) Is the data normally distributed? Explain your reasoning.

- The graph below shows the scores on a standardized test, normally distributed, for two classes.



(a) What do these graphs have in common? How are they different?
(b) Which graph has the greatest standard deviation? Why?
(c) If a student scored 42 on a test, which class are they most likely in? Explain your answer.

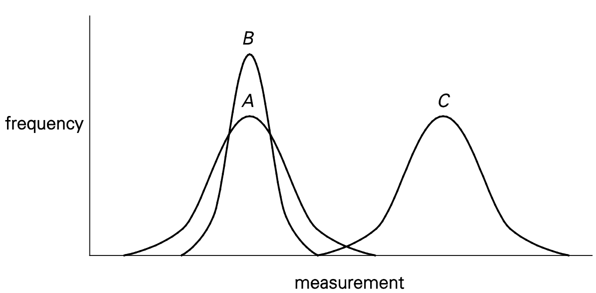
- Sally has a height of 1.75 m and lives in a city where the average height for women is 1.60 m and the standard deviation is 0.20 m. Leah has a height of 1.80 m and lives in a city where the average height for women is 1.70 m and the standard deviation is 0.15 m. If heights are normally distributed, which of the two is considered to be taller compared to their fellow women. Explain their reasoning.

- The average life expectancy of greyhounds was determined to be 12.2 years with a standard deviation of 1.3 years.
  (a) What is the probability that a given greyhound will live less than 14 years?
  (b) In a sample of 200 greyhounds, how many would you expect to live between 10.9 years and 13.5 years?
  (c) What assumption did you make that made your calculation possible?

- On a mathematics placement test at Orchard-view University, the marks were normally distributed with a mean score that was 62 and a standard deviation that was 11. If Mark's *z*-score was 0.8, what was his actual exam mark?

- What does a *z*-score of 1.5 mean? How was it calculated?

- Use the data graphed below to answer the following questions.



(a) What do graphs *A* and *B* have in common? How are they different?
(b) What do graphs *A* and *C* have in common? How are they different?
(c) Is there any relationship between graphs *B* and *C*?
(d) Which graph has the smallest standard deviation? Why?
(e) Why is the graph with the lowest standard deviation also the tallest?
(f) Which graph has the greatest mean? How do you know?

- Battery Inc. claims that its batteries have a mean life of 50 hours, with a standard deviation of six hours. If these data are normally distributed, what is the probability, to two decimal places, that a battery will last 57 hours or longer?

- The speeds of cars on a highway have a mean of 80 kmh with a standard deviation of 8 kmh. If these data are normally distributed, what percent of the cars averaged more than 96 kmh?

- The assessed value of housing in a community in the Annapolis Valley is normally distributed. There are 1200 houses with a mean assessed value of $180,000. The standard deviation is $10,000.
  (a) What percentage of houses have an assessed value between $170,000 and $200,000?
  (b) How many houses are expected to have an assessed value between $170,000 and $200,000?

- Many human attributes are normally distributed. What statistics would a clothing manufacturer of women's sweaters want to know? Would the manufacturer be interested in standard deviation?

- Infant weight gain and growth are normally distributed. Why are these statistics valuable?

- A lineup for the tickets to a concert has an average waiting time of 400 seconds with a standard deviation of 32 seconds. Sketch a normal distribution curve that illustrates the probability that you wait in line for more than six minutes. Do you think that a normal distribution would be reasonable to expect in this situation? Explain.

- What can you say about a data value if you know that its *z*-score is negative?

- What can you say about a data value if you know that its *z*-score is zero?

- The average playing time of a defense player in the NHL is 17.2 minutes per game with a standard deviation of 2.2 minutes.
  (a) What is the probability that a defense player will play between 15.6 and 19.2 minutes?
  (b) What assumption did you make that made your calculation possible?

- Create and solve a question using a real-world example that uses *z*-scores.

- Cars are undercoated as a protection against rust. A car dealer determines the mean life of protection is 65 months and the standard deviation is 4.5 months. Answer the following questions.
  - What guarantee should the dealer give so that fewer than 15% of the customers could potentially return their cars?
  - The dealer creates a fund, based on the guarantee, from which refunds and repairs are made. It is estimated that about 2500 cars will be undercoated annually. The average repair on returned cars is about $165. How much money should be placed in the fund to cover customer returns?
  - What is the probability that an undercoated car, chosen at random, will be returned in five years?

- Consider two different automobiles: the Swiftcar and the Zippycar. For both cars, the mean value of repairs after an accident is $3500. The standard deviation for the Swiftcar is $1200, while the standard deviation for the Zippycar is only $800. If the cost of repairs is normally distributed, determine the probability that the repairs costs will be over $5000 for both cars and then explain how your answer would be expected to impact insurance premiums.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

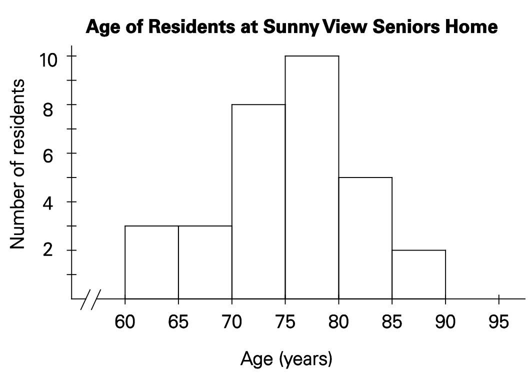
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Graph data that have been organized into **bins** (also called **intervals** or **classes**) resulting in a **histogram**. Students should be made aware that in histograms, the data elements are grouped and form a continuous range from left to right. The data are arranged in a **frequency table**. The **range** of the interval is used to determine the width of the bars that should be neither too narrow nor too broad. Intervals are the same size, and the number of bins is usually kept to between 4 and 10. Often the bins and bin sizes will be determined by the context of the problem and the range of the data.

  Example



The first bar in this graph begins at 60 on the horizontal axis and ends at 65. The second bar starts where the first bar ends, at 65–70. As the intervals are required to be non-overlapping, the convention is that the lower limit of each interval includes the number. For example, in the above graph, the data value 65 would be placed in the 65–70 interval.

To plot each vertex for the frequency polygon, students can determine the midpoint of the interval and then join the vertices with line segments. Ensure students connect the endpoints to the horizontal axis.

- Remind students of range as a measure of dispersion. They should recognize that range only measures how spread out the extreme values are, so it does not provide any information about the variation within the data values themselves. Introduce students to the **standard deviation**. It is useful for comparing two or more sets of data. Consider an example similar to the one shown below:

	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
Tim	60	65	70	75	80
Mary	60	69	70	71	80

The measures of central tendency and range for these two data sets are equal, but there are clearly differences in the two sets of marks.

Once students understand that this difference cannot be described by mean or by range, introduce the concept of standard deviation as a measurement of how far the data values lie from the mean.

Using the test marks as an example, prompt discussion around standard deviation by asking questions such as

- Whose marks are more dispersed? What does this mean in terms of a high or low standard deviation?
- If the data are clustered around the mean, what does this mean about the value of the standard deviation?
- Who was more consistent over the five unit tests?

Students should make the connection that if the data are more clustered around the mean, there is less variation in the data and the standard deviation is lower. If the data are more spread out over a large range of values, there is greater variation and the standard deviation would be higher.

For this specific example:
While both Mary and Tim's marks have a mean of 70 and a range of 20, the data are not the same. This variation can be reflected using the standard deviation. Tim's marks have a standard deviation of 7.1 while Mary's marks have a standard deviation of 6.4.

- Have students explore the meaning of standard deviation through relevant examples. Promote discussion around situations such as the following:
  - *Sports Illustrated* is doing a story on the variation of player heights on NBA basketball teams. The heights, in cm, of the players on the starting lineups for two basketball teams are given in the table below.

Lakers	195	195	210	182	205
Celtics	193	208	195	182	180

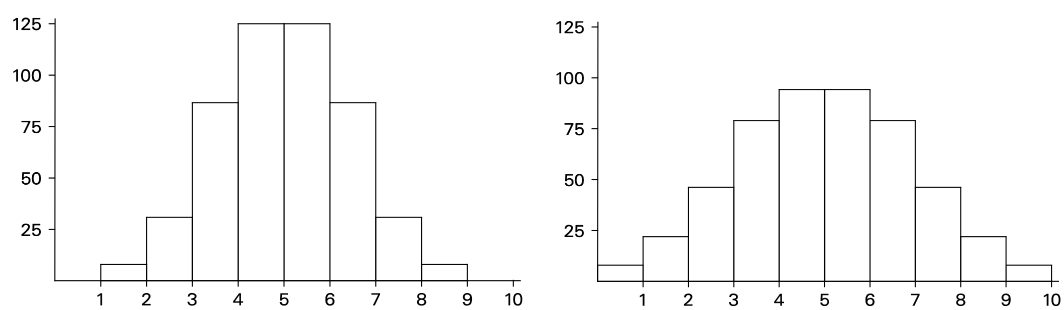
The team with the most variation in height will be selected for the cover of *Sports Illustrated* magazine. Ask students which team will appear on the cover.

- It is also worthwhile to discuss how important it is in manufacturing and packaging to have very small standard deviations. For example, what do you think the standard deviation would be for the mass of a bag of chips or the mass of each cookie in a bag of cookies? It is also valuable to discuss whether the manufacturer would want such data to be normally distributed. If time allows, you could ask students to weigh (using a very accurate scale borrowed from the science department) small bags of chips or granola bars. The data could then be plotted using a histogram. (As you collect more and more data, it is likely that a normal distribution will begin to be revealed.)

- Creating a histogram using Post-it-Note task (3M 2014). The task found at www.post-it.com/wps/portal/3M/en_US/PostItNA/Home/Ideas/Articles/Histogram can be adapted to illustrate the construction of a histogram.

- Choose an attribute common to all students (number of siblings, number of weekly text messages sent or received, numbers of hours watching television, number of minutes spent eating lunch, number of hours spent sleeping, etc.). Ask students to collect the data from the class, select an appropriate bin size, and display the information in a frequency table. They should construct a histogram of the data and then describe the data distribution.

- Ask students which of the distribution of scores in the graphs below has the larger dispersion? They should justify their reasoning.

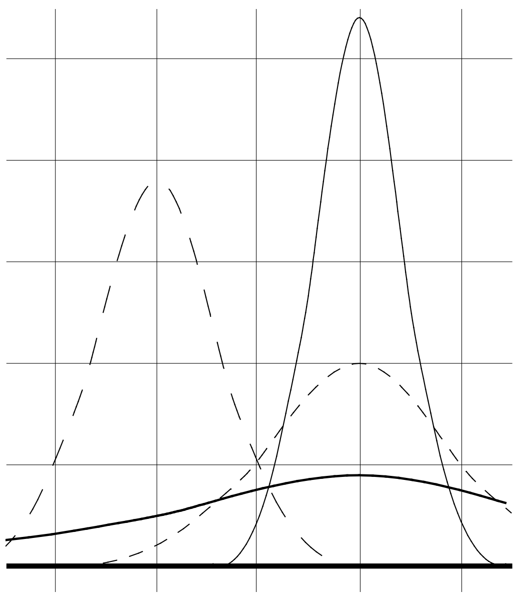


- Introduce students to the shape of the normal curve using an interactive plinko game (Little 2004) (www.math.psu.edu/dlittle/java/probability/plinko/index.html) or Quincunx (MathIsFun.com 2010) (www.mathsisfun.com/data/quincunx.html).

  The general shape of the histogram will begin to approach a normal curve as more trials are added. Students should observe the following characteristics:
  - Normal distributions are symmetrical with a single peak at the mean of the data.
  - The curve is bell-shaped with the graph falling off evenly on either side of the mean.
  - There are a variety of Java applets available online to manipulate the parameters (mean and standard deviation) of data sets to show the characteristics of what and how they affect the shape of the curve. Ask students to investigate these characteristics and answer the following questions:
    (a) How are the standard deviation and the shape of the graph related?
    (b) Does the area under the curve change as the number of standard deviations changes?
    (c) Does the area under the curve change as the value of the standard deviation changes?

- To probe understanding of standard deviation, ask students to answer the following questions:
  (a) Is it possible for a data set to have a standard deviation of 0? Can the standard deviation ever be negative? Explain why or why not.
  (b) If 5 was added to each number in a set of data, what effect would it have on the mean? On the standard deviation? What if each element in the data set was multiplied by –3?
  (c) Identify the characteristics of the graphs that make the standard deviation larger or smaller.
  (d) What features of a histogram seem to have no bearing on the standard deviation? What features do appear to affect the standard deviation?

- Ask students to discuss several groups of graphs similar to that shown to the below. Students should be able to indicate the graph(s) with the largest mean as well as the graph(s) with the larger standard deviation and explain their reasoning.



- Graphing a normal distribution by hand can be tedious and time consuming. Using technology such as graphing calculators and Microsoft Excel, students can much more easily investigate the properties of a normal distribution and work with data that are specifically one, two, and three standard deviations from the mean. Consider the following data sample:

Number of hours playing video games in a week (15-year-old students)									
21.1	21.8	17.7	23.4	14.9	20.8	19.1	18.6	25.6	23.3
18.5	19.1	18.8	18.3	20.2	15.5	21.4	16.7	14.7	20.5
17.9	20.9	24.5	22.2	17.1	24.5	21.1	18.1	21.0	16.2
23.7	18.9	21.3	16.0	20.2	27.3	17.6	20.7	19.7	14.9

Guide students through the problem asking them to do the following:
(a) Determine the mean, standard deviation, median, and mode. Explain what you notice about the values.
(b) Construct a frequency table to generate a histogram. Use an interval width equal to the standard deviation since the spread of the normal distribution is controlled by the standard deviation.
(c) Discuss the symmetry of the histogram.
(d) Draw a frequency polygon and explain its shape.

Students should understand that in a normal distribution the data are distributed symmetrically around the mean. Since the mean lies in the middle of the data, it is also the median. As the mean is located at the point at which the curve is highest, it is also the mode. Therefore the mean, median, and mode are equal for a normal distribution.

- Using data such as those in the previous chart, continue to investigate the area under the curve and the percentage of data that are one, two, and three standard deviations from the mean. Coach students along the investigation using the following directions, and discuss their observations.
  - Approximate the percentage of data that are to the left of the mean and then to the right of the mean.
  - Approximate the percentage of data within three standard deviations. Ask them what they notice.
  - Approximate the percentage of data within one standard deviation.
  - Approximate the percentage of data within two standard deviations.

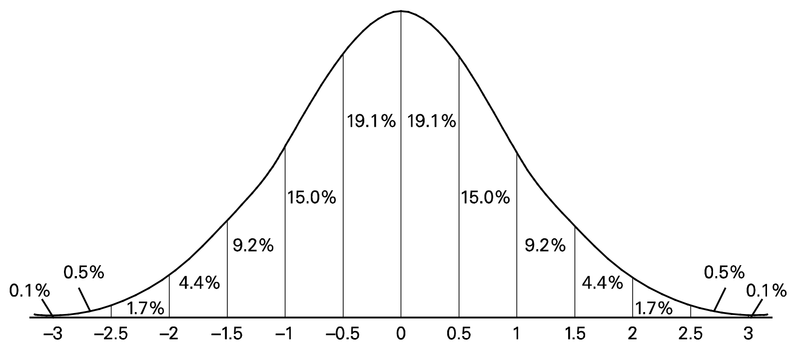
  As students analyze the data, they should recognize that fifty percent of the distribution lies to the left of the mean and fifty percent lies to the right of the mean. Furthermore, approximately 68% of the data lie within one standard deviation of the mean, 95% is within two standard deviations of the mean, and almost all the data are within three standard deviations of the mean.

- Students will have to confirm whether a data set approximates a normal distribution. Observe students as they are working through examples to make sure they understand the concept of normal distribution. Using technology, students should consider the measures of central tendency, the appearance of the histogram, and the percent of data within one, two, and three standard deviations of the mean (i.e., are they close to 68%, 95%, and 99.7%).

- If the data approximate a normal distribution, the properties of the normal distribution can be used to solve problems. Students should recognize that for normally distributed data, the area under the curve between any two values is equal to the probability that a given data value will fall between those two values. Consider a data set of 100 items normally distributed having a mean of 3.4 and a standard deviation of 0.2. If 68% of the area under the curve is within one standard deviation, then students should realize approximately 68 items should be between 3.2 and 3.6. Students should consider how many items would be between 3.2 and 3.6 if the data set has 200 items? 500 items? 750 items? 1000 items? The www.wessa.net website (Wessa 2014) can be used to demonstrate that for a randomly selected sample from a population that is normally distributed as the size of the sample increases, the distribution will approach the shape of a normal curve and the percentages indicated in the diagram of the following task will be approached as well.

- Once students are comfortable working with applications where the scores of interest are one, two, or three standard deviations from the mean, you can introduce a problem where the score is 1.5 standard deviations from the mean. At this point you could use the diagram shown below.

Normal Curve
Standard Deviation

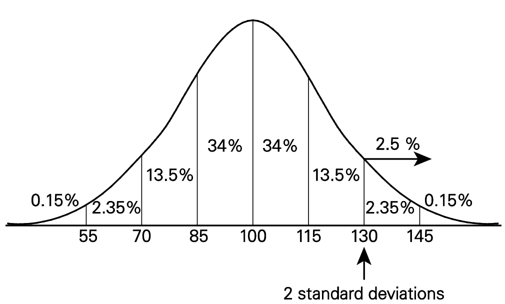


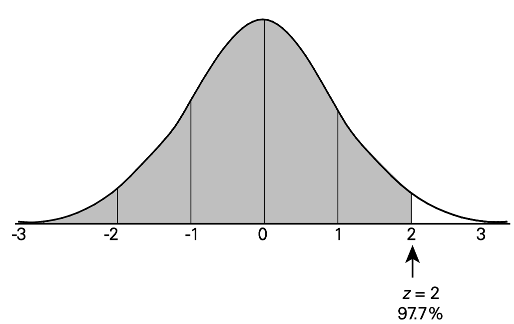
When students encounter a question where the numbers work so neatly, they quickly see the need for a table that is more flexible. This need creates the environment where they are ready to use a *z*-score table.

It is now time to introduce students to the *z*-score formula: $z = \dfrac{x-\mu}{\sigma}$.

- Students will use the properties of the normal curve to solve problems. Encourage students to sketch the normal distribution curve to help them visualize the information. Begin by having students solve problems that fall exactly one, two, or three standard deviations from the mean. Work with these questions and develop a comfort level with these questions prior to introducing *z*-scores and the *z*-score chart.

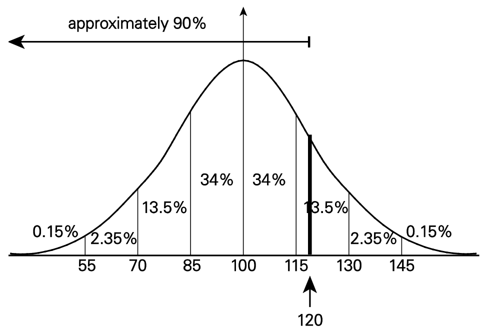
- To emphasize what a *z*-score is, present the following problem:
  The age of a sea turtle is normally distributed with a mean age of 100 years and a standard deviation of 15 years. What percentage of turtles live less than 130 years?
  – Ask students to first draw the normal distribution curve, labelling the mean and standard deviation, then verify their answer using *z*-scores.

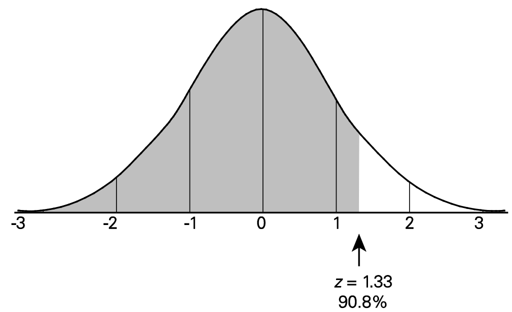




Using a visual representation, students should see that 97.5% of sea turtles would live less than 130 years. Since the score of 130 is two standard deviations away from the mean, students can use the *z*-score table to find the area under the curve to the left of the standard normal curve. This value is 0.9772, which is 97.7%. Discuss with students why there is a small error in the calculation when compared to standard deviation.

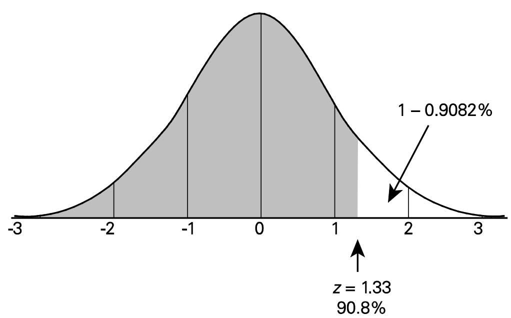
– Using the example, ask students to consider a data value that is between one and two standard deviations from the mean such as an age of 120 years. Ask them to determine the percentage of sea turtles who live less than 120 years? Students should first estimate their answer (using standard deviation and the normal distribution curve) and then verify it using the *z*-score.





– Engage students in a discussion by asking the following questions:
  > Was your estimate reasonable when you compared it to the *z*-score?
  > Why is the *z*-score more reliable than estimating using standard deviation?
  > What percentage of sea turtles lived less than 120 years?

– Students should understand that the *z*-score table can be used to determine the area under the curve. If students were asked to determine the percentage of sea turtles that lived longer than 120 years, they would write 1 – 0.9082 , or 100% – 90.82%.



- The *z*-score can also provide a standard measure for comparing two different normal curves by transferring each to the standard normal distribution curve, having a mean of zero and a standard deviation of one. Students should consider examples such as the following:
  – On her first math test, Tam scored 70%. The mean class score was 65% with a standard deviation of 4%. On her second test, she received 76%. The mean class score was 73% with a standard deviation of 10%. On which test did Tam perform better compared to the rest of her class?

  If students examine Tam's test scores alone, they would probably conclude she performed better on the second unit test. However, this is not necessarily the case. We must take into account the overall performance of the class to compensate for other factors such as the difficulty of the material tested. By calculating Tam's *z*-score values for each test, we can compare her performance on both tests to the same standard.

On her Unit 1 test, Tam's *z*-score was $z = \frac{70-65}{4} = 1.25$, which implies Tam scored 1.25 standard deviations above the mean.

On her Unit 2 test, Tam's *z*-score was $z = \frac{76-73}{10} = 0.3$. On Unit 2, Tam scored 0.3 standard deviations above the mean.

Since both *z*-scores relate to a standard normal distribution, students can compare the values and should conclude that Tam performed better compared to her classmates on her Unit 1 exam.

Suggested Models and Manipulatives

- *z*-score tables (several different types)

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- σ and μ
- bimodal
- bin width
- dispersion
- frequency distribution
- frequency polygon
- histogram
- line plot
- mean
- median
- mode
- normal curve or bell curve
- normal distribution
- outlier
- range
- standard deviation
- *z*-score or standard score

Resources/Notes

Internet

- *Free Statistics Software* (Wessa.Net 2014)
  www.wessa.net
  Data generation site that can be used to produce histograms, count data values and develop normal curve. (Found only on US site.)

- *How to Create a Histogram in Excel* (Kent State University 2013)
  www.math.kent.edu/~honli/teaching/statistics/Chapter2/Excell_Histogram.html
  Creating histogram using Microsoft Excel.

- *Kids'Zone: Learning with NCES*, "Chances" (US Department of Education 2014)
  http://nces.ed.gov/nceskids/chances/index.asp
  An interactive dice game that demonstrates the histogram as you input the number of rolls.

- *Math Is Fun*, "Standard Deviation Calculator" (MathIsFun.com 2012)
  www.mathsisfun.com/data/standard-deviation-calculator.html
  A standard deviation calculator showing all the steps in calculating the population standard deviation.

- *Measuring Usability*, "Interactive Graph of the Normal Curve" (Measuring Usability 2014)
  www.measuringusability.com/normal_curve.php
  An interactive graph of the normal curve.

- *Post-it Products*, "Activity Center: Histograms" (3M 2014)
  www.post-it.com/wps/portal/3M/en_US/PostItNA/Home/Ideas/Articles/Histogram

- *Statistics Calculator: Standard Deviation* (Arcidiacono 2011)
  www.alcula.com/calculators/statistics/standard-deviation
  This website calculates the mean and population standard deviation without showing any calculations.

- *Statistics How To*, "Microsoft Excel" (Statistics How To 2014)
  www.statisticshowto.com/microsoft-excel-for-statistics
  Step-by-step instructions on how to draw a normal distribution curve and create a histogram in Microsoft Excel.

- *Untitled* [Plinko game] (University of Colorado 2013)
  http://phet.colorado.edu/sims/plinko-probability/plinko-probability_en.html
  An interactive plinko game demonstrating the histogram.

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 5.1–5.6, pp. 210–275

Notes

SCO S02 Students will be expected to interpret statistical data, using confidence intervals, confidence levels, and margin of error. [C, CN, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(It is intended that the focus of this outcome be on interpretation of data rather than on statistical calculations.)

S02.01 Explain, using examples, how confidence levels, margin of error, and confidence intervals may vary depending on the size of the random sample.

S02.02 Explain, using examples, the significance of a confidence interval, margin of error, or confidence level.

S02.03 Make inferences about a population from sample data, using given confidence intervals, and explain the reasoning.

S02.04 Provide examples from print or electronic media in which confidence intervals and confidence levels are used to support a particular position.

S02.05 Interpret and explain confidence intervals and margin of error, using examples found in print or electronic media.

S02.06 Support a position by analyzing statistical data presented in the media.

Scope and Sequence

Mathematics 10	Mathematics 11	Grade 12 Mathematics Courses
–	**S02** Students will be expected to interpret statistical data, using confidence intervals, confidence levels, and margin of error.	–

Background

It is intended that the focus of this outcome be on the interpretation of data rather than on mathematical calculations.

In Mathematics 9 (SP02), students have had experience identifying whether a given situation represents the use of a **sample** or a **population**. In this Statistics unit, students will recognize that since an entire population is often difficult to study, we often use a representative population called a sample. They will recognize that, if the sample is truly representative, the statistics generated from the sample will be the same as the information gathered from the population as a whole. It is unlikely, however, that a truly representative sample will be selected. We need to make predictions of how confident we can be that the statistics from our sample are representative of the entire population.

It is always important to keep straight whether a number describes a population or a sample.

A **parameter** is a number that describes a population. A parameter is a fixed number, but in practise we do not know its value.

A **statistic** is a number that describes a sample. The exact value of a statistic can be determined from a sample; but, is variable as it can change from sample to sample. This fact is called sampling variability. We use a statistic to estimate a parameter.

For example, a Gallup Poll asked a Simple Random Sample (SRS) or 515 adults if they believed in ghosts. 160 said “yes.” So the proportion of the sample who said that they believed in ghosts is $\frac{160}{515}=0.31$. This number, 0.31, is a statistic. We can use it to estimate the proportion of all adults who believe in ghosts. This is our parameter of interest.

Surveys are used to draw conclusions from a sample. How well the sample represents the larger population depends on two important values, the **margin of error** and **confidence level**.

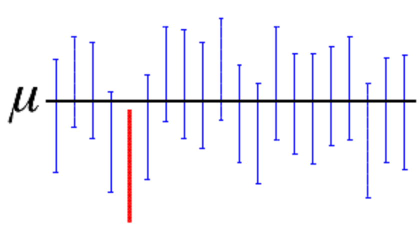
“What the Confidence Interval (CI) Is Not

There are a lot of things that the confidence interval is not. Unfortunately many of these are often used to define confidence interval.

- It is not the probability that the true value is in the confidence interval.
- It is not that we will obtain the true value 95% of the time.
- We are not 95% sure that the true value lies within the interval of the one sample.
- It is not the probability that we correct.
- It does not say anything about how accurate the current estimate is.
- It does not mean that if we calculate a 95% confidence interval then the true value is, with certainty, contained within that one interval.”

(Source: *Statistical Research*, “ When Discussing Confidence Level with Others ...” Blog by Wesley, August 13, 2013. http://statistical-research.com/some-issues-relating-to-margin-of-error/?utm_source=rss&utm_medium=rss&utm_campaign=some-issues-relating-to-margin-of-error)

“In statistics, creating confidence intervals is comparable to throwing nets over a target with an unknown, yet fixed, location. Check out the graphic below, which depicts CI’s generated from 20 samples of the same population. The black line represents the fixed value of the unknown population parameter, the population mean; the 19 blue CI’s contain the true value of the population parameter; the 1 red CI does not.



(For full-colour version, please see original source from which this quote was taken.)

A 95% CI indicates that 19 out of 20 samples (95%) taken from the same population will produce CI's that contain the true population parameter. A 90% CI indicates that 18 out of 20 samples from the same population will produce CI's that contain the population parameter, and so on."

(Source: *The Mimtab Blog*, "What Do Confidence Intervals Have to Do with Rabies?" Carly Barry in Health Care Quality Improvement, September 27, 2012. http://blog.minitab.com/blog/real-world-quality-improvement/what-do-confidence-intervals-have-to-do-with-rabies)

Therefore, the Confidence Level of 95% does mean, in the long run, if we keep on computing these confidence intervals, then 95% of those intervals will contain the true value.

THE MARGIN OF ERROR

A direct component of the confidence interval is the margin of error. This is the number that is most widely seen in the news, whether it be print, TV, or otherwise. Often, however, the confidence level is excluded and not mentioned in these articles. One can normally assume a 95% confidence level, most of the time. What makes the whole thing difficult is that the margin of error could be based on a 90% confidence level, making the margin of error smaller, thus giving the artificial impression of the survey's accuracy.

Consider the following example.

- A1 Car Rentals surveys its customers and finds that 50 percent of the respondents say its customer service is "very good." This information is considered to be accurate to within 3%.

If the confidence level is 95%

This information indicates that if the survey were conducted 100 times, then 95 of those confidence intervals would contain the actual percent of customers who felt that the service is "very good." Furthermore, the confidence interval, based on this sample, would indicate that between 47% and 53% of customers feel that the service is "very good."

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Explain the difference between a population and a sample.
- When would the individuals in your math class be considered a sample? When would they be considered a population?
- Are the Canadian citizens who respond to the federal census a sample or a population? Explain your reasoning.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- A botanist collects a sample of 50 iris petals and measures the length of each. He finds that the mean is 5.55 cm and the standard deviation is 0.57 cm. The botanist then reports that he is 95% confident that the average petal length is between 5.39 cm and 5.71 cm, 19 times out of 20.
  (a) Identify the margin of error, the confidence interval, and the confidence level.
  (b) Explain what information the confidence interval gives about the population of iris petal length.
  (c) How would the length of a 99% confidence interval be different from that of a 95% confidence interval?
  (d) If you do not know the margin of error but you do know that the confidence interval is between 5.39 cm and 5.71 cm, how could you determine the margin of error?
  (e) Was his statement misleading? Explain.

- A report claims that the average family income in a large city is $32,000. It states the results are accurate 19 times out of 20 and have a margin of error of ±2500.
  (a) What is the confidence level in this situation? Explain what it means.
  (b) Explain the meaning of a margin of error of ±2500.

- What is meant by a 90%, a 95%, and a 99% confidence interval. How are these intervals similar? How are they different?

- In a national survey of 400 Canadians aged 20 to 35, 37.5% of those interviewed claimed they exercise for at least four hours a week. The results were considered accurate within 4%, 9 times out of 10.
  (a) Are you dealing with a 90%, 95%, or 99% confidence interval? How do you know?
  (b) How many people in the survey claimed to exercise at least four hours a week?
  (c) What is the margin of error?
  (d) What is the confidence interval? Explain its meaning.
  (e) What are some limitations of this survey?
  (f) If the writers of the article created a 99% confidence interval based on these data, how would it be different? How would it be the same?
  (g) How would the confidence interval change if the sample size were increased to 1000 but the sample proportion remained the same?

- In a recent survey, 355 respondents from a random sample of 500 first-year university students claim to be attending their first choice of university. If the sample size were decreased but the sample proportion remained the same, how would the confidence intervals change?

- The results of a survey show that 71% of residents in Yarmouth own cell phones. The margin of error for the survey was ±2.3%. If there are 7300 people in Yarmouth, determine the range of the number of people that own cell phones.

- When a sample of batteries was tested, it was determined that this brand of battery had a mean life expectancy of 12.6 hours with a margin of error of ±0.7 hours.
  (a) State the confidence interval for this brand of battery.
  (b) If a larger sample of batteries were tested and it was determined that this brand of battery had a mean left expectancy of 12.6 hours, how do you think that the margin of error would compare to 0.7 hours?

- Create a poster that includes an example from print or electronic media that uses confidence intervals or confidence levels to support a position. Your poster should respond to each of the following:
  - Interpret the confidence interval and confidence levels for someone who has no knowledge of statistics.
  - How would the confidence interval change if the sample size used for the study doubled?
  - Do you agree or disagree with any concluding statements that were made about the data from the study? Explain.

- Find an example in the media that uses confidence intervals or confidence levels to support a position relevant to teenagers today. Write two short news articles on the topic. In one article, make the situation sound as disastrous as you can. In the other article, try to minimize the problem.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Begin this discussion by asking a student to choose a number between 0 and 100. Then ask that student if their number is 46? No? Then ask if the number is between 25 and 75? Students will understand that by using a wide interval, the chance of their guess including the actual number that the student selected increases. Discuss how a wider confidence interval is more likely to actually contain the number that the student chose. Students also need to understand that as you increase the confidence level, the interval becomes wider and may lose its usefulness. Basically, students need to understand that a balance is necessary; the interval selected needs to be narrow enough to be useful and wide enough to be accurate.

- The size of a random sample will affect the confidence interval and margin of error. Suppose that the town of Shelburne is trying to determine the location of a new recreation centre. Ask students to make a prediction about the confidence interval and margin of error if 100 people were surveyed, if 1000 people were surveyed, or if all the people in the town were surveyed.
  - If 100 people were surveyed, students should realize the results could be skewed one way or the other. In other words, there is no guarantee that the results will be accurate. This would result in a significant margin of error (due to a small sample size) and would produce a confidence interval with a large range.
  - If 1000 people were surveyed, there would be a greater chance for more accurate results. The increase in sample size would decrease the margin of error and, in turn, decrease the range on the confidence interval, thus, getting us closer to the actual result.
  - If all the people in the town were surveyed, students should recognize they would know the actual result and have no margin of error.

- Provide students with a mean value and a margin of error or the confidence interval and ask them to analyze the data to make inferences. Consider the following example:
  - A recent study reports that 71% of gamers own at least fifteen games. According to the study, if there are 1000 gamers at a conference, what is the range of gamers who own at least fifteen games?
  - Ask students the following questions:
    (a) What is the confidence level?
    (b) What is the confidence interval?
    (c) According to the study, if there are 1000 students, what is the range of students who could own at least 15 games?

- Provide students with an inference based on given confidence intervals and ask them to comment on the validity of the inference. Consider the following example:
  - The city of St. John's is trying to determine whether or not to continue the curbside recycling program. A survey indicated that 50% of residents wanted the program to continue. The survey was reported to be accurate 9 times out of 10, with a margin of error of 16.7%. Based on these results, what course of action should the city take with respect to the curbside recycling program? Explain your answer.

Students need to learn that the confidence level is only 90%, and due to the large margin of error, the range of the confidence interval is also large (33.3% to 66.7%). The city is only 90% confident that between one-third and two-thirds of the population are in favour of the recycling program. Ask students if this is useful information. It should be suggested that the city increase its sample size to reduce the margin of error and use at least a 95% confidence interval to get a better picture of how the people actually feel about the program.

- Students could find a variety of examples, such as quality control or public opinion polls as reported in the newspaper or other news sources, in which confidence levels, confidence intervals, and margins of error are used to report results. They should interpret the meaning of the confidence intervals and levels and its implications in society. These examples could then be used to prompt discussions around interpreting and explaining statistical data leading to forming an opinion on the topic.

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- confidence interval
- confidence level
- margin of error
- parameter
- population
- sample
- sampling variability

Resources/Notes

Internet

- *e! Science News*, "Science news articles about 'confidence interval'" (e! Science News 2014)
  http://esciencenews.com/dictionary/confidence.interval
  Confidence intervals in scientific research.

- *Statistical Research*, "When Discussing Confidence Level with Others ..." (Blog by Wesley, August 13, 2013)
  http://statistical-research.com/some-issues-relating-to-margin-of-error/?utm_source=rss&utm_medium=rss&utm_campaign=some-issues-relating-to-margin-of-error

- *The Mimtab Blog*, "What Do Confidence Intervals Have to Do with Rabies?" (Carly Barry in Health Care Quality Improvement, September 27, 2012)
  http://blog.minitab.com/blog/real-world-quality-improvement/what-do-confidence-intervals-have-to-do-with-rabies

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Section 5.6, pp. 267–275

Notes

SCO S03 Students will be expected to critically analyze society's use of inferential statistics. [C, CN, PS, R, T]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

S03.01 Investigate examples of the use of inferential statistics in society.

S03.02 Assess the accuracy, reliability, and relevance of statistical claims by

- identifying examples of bias and points of view
- identifying and describing the data collection methods
- determining if the data is relevant

S03.03 Identify, discuss, and present multiple sides of the issues with supporting data.

Scope and Sequence

Mathematics 9	Mathematics 11	Grade 12 Mathematics Courses
SP01 Students will be expected to describe the effect on the collection of data of bias, use of language, ethics, cost, time and timing, privacy, and cultural sensitivity. **SP02** Students will be expected to select and defend the choice of using either a population or a sample of a population to answer a question. **SP03** Students will be expected to develop and implement a project plan for the collection, display, and analysis of data by ▪ formulating a question for investigation ▪ choosing a data collection method that includes social considerations ▪ selecting a population or a sample ▪ collecting the data ▪ displaying the collected data in an appropriate manner ▪ drawing conclusions to answer the question	**S03** Students will be expected to critically analyze society's use of inferential statistics.	–

Background

Statistics is the art and science of gathering, analyzing, and making conclusions from data.

In Mathematics 8, the focus of instruction was to critique ways in which data were presented (8SP1). The emphasis in Mathematics 9 was to analyze and critique the data collection process.

There are many factors within the data collection process that have the potential to influence the results. In Mathematics 9, students considered factors such as the method used, the reliability and usefulness of data, and the ability to make generalizations about the population from a sample.

Descriptive statistics describe the population we are studying. The data could be collected from either a sample or a population, but the results help us organize and describe data. Descriptive statistics can only be used to describe the group that is being studying. That is, the results cannot be generalized to any larger group.

Descriptive statistics are useful and serviceable if you do not need to extend your results to any larger group. However, many uses of statistics in society tend to want to draw conclusions about segments of the population, such as all parents, all women, all victims, etc.

Inferential statistics are used to make predictions or inferences about a population from observations and analyses of a sample. That is, we can generalize the statistics from a sample to the larger population that the sample represents. In order to do this, however, it is imperative that the sample is representative of the group to which it is being generalized.

For this unit, students will collect examples of inferential statistics, working in groups to consider both articles that make a statistical claim that is valid as well as those that make claims that may be invalid or questionable.

Students will extend their understanding of bias, points of view, and objectivity for various common data collection methods. Sample types that will be considered are as follows:

- **Simple random sampling (SRS**)
  A simple random sample requires that
  - all selections must be *equally likely*
  - all combinations of selections must be *equally likely*

  A random sample may not end up being representative of the population, but any deviations are due only to chance.

- **Systematic random sampling**
  A systematic random sample is used when sampling a fixed percent of the population. A random starting point (e.g., individual, household, or object) is chosen and then every *n*th individual selected for the study, where *n* is the sampling interval.

- **Stratified random sampling**
  When using a stratified random sample, the population is divided into groups called *strata* (e.g., geographic areas, age groups, places of work). A simple random sample of the members of each stratum is then taken. The size of the sample for each stratum is proportionate to the stratum's size.

- **Cluster random sampling**
  Cluster samples require that the population be organized into groups (e.g., schools, communities, companies). A random sample of groups would be chosen. All the members of the chosen groups would then be surveyed.

- **Multi-stage random sampling**
  Multi-stage samples require that the population be organized into groups. A random sample of groups is chosen and a random sample of members of the chosen groups is taken.

- **Destructive sampling**
  Samples from which the selected elements cannot be reintroduced into the population is called **destructive sampling** (e.g., light bulbs tested for quality control).

Types of sampling bias such as non-response bias, household bias, and response bias are considered.

- **Sampling bias**—The chosen sample does not accurately represent the population.

- **Non-response bias**—The results are influenced because surveys are not returned.

- **Household bias**—One type of respondent is overrepresented because groupings of different sizes are polled equally.

- **Response bias**—Factors in the sampling method influence the result (e.g. leading questions, social desirability, volunteers used in sample).

Types of media bias related to points of view are considered.

- **Bias by omission**—Leaving one side out of an article or a series of articles over a period of time to strengthen a point of view.

- **Bias by selection of sources**—Including more sources that support one view over another.

- **Bias by spin**—Bias by spin occurs when the story has only one interpretation of an event or policy, to the exclusion of the other; spin involves tone—it's a reporter's subjective comments about objective facts; makes one side's ideological perspective look better than another.

- **Bias by story selection**—A pattern of highlighting news stories that coincide with the agenda of either the left or the right, while ignoring stories that coincide with the opposing view.

- **Bias by placement**—Story placement is a measure of how important the editor considers the story. Studies have shown that, in the case of the average newspaper reader and the average news story, most people read only the headline. Bias by placement is where in the paper or in an article a story or event is printed; a pattern of placing news stories so as to downplay information supportive of either conservative views or liberal views.

- **Bias by labelling**—Bias by labelling comes in two forms. The first is the tagging of conservative politicians and groups with extreme labels while leaving liberal politicians and groups unlabelled or with more mild labels, or vice versa. The second kind of bias by labeling occurs when a reporter not only fails to identify a liberal as a liberal or a conservative as a conservative, but describes the person or group with positive labels, such as "an expert" or "independent consumer group." In so doing, the reporter imparts an air of authority that the source does not deserve. If the "expert" is properly called a "conservative" or a "liberal," the news consumer can take that ideological slant into account when evaluating the accuracy of an assertion. When looking for bias by labelling, remember that not all labelling is biased or wrong. Bias by labelling is present when the story labels the conservative but not the liberal, or the liberal but not the conservative; when the story uses more extreme sounding labels for the conservative than the liberal ("ultra-conservative," "far right," but just "liberal" instead of "far left" and "ultra-liberal") or for the liberal than the conservative ("ultra-liberal," "far left," but just "conservative" instead of "far right" and "ultra-conservative"; and when the story misleadingly identifies a liberal or conservative official or group as an expert or independent watchdog organization.

In this unit students will check for whether information has been presented in an objective manner to decide whether or not if it is free from bias. Some indications that the information is reasonably objective are as follows:

- All relevant data are presented even when it does not support the preferred point of view.
- All views of an issue are presented and none are preferred.
- All views of an issue are presented even though one is preferred.
- The topic is presented in a clear and logical manner.
- Assertions, statements, opinions, etc., are documented.
- A variety of reliable sources are used to support the point being made.
- The purpose is clearly stated.

Some indications that information may not be objective are as follows:

- Only one view of an issue is presented.
- Other views of an issue are attacked or ignored.
- Not all data are presented; only data supporting the preferred point of view are presented.
- Assertions, statements, and opinions are presented as facts without adequate documentation.
- Emotion-arousing language is used to persuade the audience of a point without any accompanying documentation.
- Derogatory language is used.
- The presentation is illogical or contains logical fallacies.
- The purpose is not clearly stated or is hidden.
- Converting the audience to a particular point of view is the primary purpose.

For this outcome each student creates a written report of their findings for an original article they collected and discussed in their groups. Students will need to be given a period of time over which the report must be written.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Your friend is unclear what the term **bias** means. Develop an example to help explain the term.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Do you think that the conclusions of the Windsor, Ontario, police chief are reasonable based on the survey results described in the link at http://blackburnnews.com/windsor/windsor-news/2013/09/27/survey-shows-confidence-in-police? Explain.

- Make some statements based on the following two survey results, published in 2013, found at the following links:

  - "Nearly half of Workers are in Dream Jobs" (Linda Nguyen, *The Canadian Press*, Wednesday, September 25, 2013, MSN News) http://news.ca.msn.com/money/nearly-half-of-workers-are-in-dream-jobs-1
  - "Survey shows significant increase in Canadian Netflix subscribers" (Canadian Reviewer, Monday, September 23, 2013; original source, *Maclean's*) www.canadianreviewer.com/cr/2013/9/23/survey-shows-significant-increase-in-canadian-netflix-subscr.html)

- Explain how it is possible that both the following statements about global warming could be correct.
  - "Global mean surface temperature over the past 20 years (1993–2012) rose at a rate of 0.14 ± 0.06°C per decade (95% confidence interval)." (Curry 2013, *Climate Etc.* blog.)
  - "According to the U.S. Global Change Research Program, the temperature in the U.S. has increased by 2 degrees in the last 50 years and precipitation by 5 percent." (DoSomething.org 2014, *11 Facts About Global Warming*)

- You may have heard someone talk about how our ancestors died so much younger than we do today. Sometimes, for example, it's used to help explain why many women and men got married in their teens centuries ago; after all, they had to get started with families early since they would be dead by 40. According to the National Center for Health Statistics, life expectancy for American men in 1907 was only 45 years, though by 1957 it rose to 66. However, this does not mean that our great-grandfathers rarely lived into their fifties. In fact maximum human lifespan—which is not the same as life expectancy—has remained more or less the same for thousands of years. Explain how this is possible.

- Identify the type of sampling that was used in each instance below.
  - Samjai visited each ATV dealership in a city and randomly selected 12 customers from each in order to understand what features were most important when a person decided to purchase an ATV.
  - Nan checked to see that the mushrooms packaged for sale in her store are the appropriate weight. To do this she randomly selected packages from a large shipment and weighed those she had selected.
  - To determine how new members were responding to fitness programs, the director Janine selected a sample of 20 people to survey. When looking at the overall list of new members, she noticed that 60% were female and 40% male. She selected 12 female and 8 males for her survey.
  - Erica randomly selected five counties in Nova Scotia to survey to determine their views on the construction of wind farms as an alternate source of energy in Nova Scotia.

- Explain why it is important for statisticians to use unbiased, rather than biased, data.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Use a jigsaw activity with students to discuss the various types of sampling, sampling bias, and media bias.

- Use a TED Talk or other relevant video to interest students in statistical studies, their impact on our lives, and how they are reported in the media. A variety of these can be found at the TED Talk website at http://new.ted.com.

- Introducing these ideas over a period of weeks may be advisable. This will allow students the time to think critically about articles, discuss them and carefully consider their validity.

- A BBC series of six articles found at the American Statistical Association (2014) website (www.amstat.org/news/blastland_bbcprimer.cfm) could be used as a basis for discussion. A jigsaw would be an effective strategy for considering these articles.

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- bias by labelling
- bias by omission
- bias by selection of sources
- bias by spin
- bias by story selection
- cluster random sampling
- descriptive statistics
- destructive sampling
- household bias
- inferential statistics
- multi-stage random sampling
- non-response bias
- sampling bias
- simple random sample (SRS)
- stratified random sample
- surveys and polls
- systematic random sample

Resources/Notes

Internet

Use some of the free online resources available on financial literacy such as

- *11 Facts About Global Warming* (DoSomething.org 2014)

- *BBC Six-Part Primer on Understanding Statistics in the News* (American Statistical Association 2014) www.amstat.org/news/blastland_bbcprimer.cfm

- *Climate Etc.* blog (Curry 2013) http://judithcurry.com/2013/08/28/overestimated-global-warming-over-the-past-20-years.

- "Nearly half of Workers are in Dream Jobs" (Linda Nguyen, *The Canadian Press*, Wednesday, September 25, 2013, MSN News) http://news.ca.msn.com/money/nearly-half-of-workers-are-in-dream-jobs-1

- "Survey shows significant increase in Canadian Netflix subscribers" (*Canadian Reviewer*, Monday, September 23, 2013; original source, *Maclean's*) www.canadianreviewer.com/cr/2013/9/23/survey-shows-significant-increase-in-canadian-netflix-subscr.html)

- *RSS Presidential Address 2013* (Royal Statistical Society 2013)
  www.youtube.com/watch?feature=player_embedded&v=tsYUXBr4CEg#t=4

- *TED Talk* (TED Talk)
  http://new.ted.com

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Supplementary Resource

Notes

Relations and Functions
30–35 hours

GCO: Students will be expected to develop algebraic and graphical reasoning through the study of relations.

SCO RF01 Students will be expected to model and solve problems that involve systems of linear inequalities in two variables. [CN, PS, T, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF01.01 Model a problem, using a system of linear inequalities in two variables.
RF01.02 Graph the boundary line between two half planes for each inequality in a system of linear inequalities, and justify the choice of solid or broken lines.
RF01.03 Determine and explain the solution region that satisfies a linear inequality, using a test point when given a boundary line.
RF01.04 Determine, graphically, the solution region for a system of linear inequalities, and verify the solution.
RF01.05 Explain, using examples, the significance of the shaded region in the graphical solution of a system of linear inequalities.
RF01.06 Solve an optimization problem, using linear programming.

Scope and Sequence

Mathematics 10	Mathematics 11	Pre-calculus 11
RF01 Students will be expected to interpret and explain the relationships among data, graphs, and situations. **RF05** Students will expected to determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. **RF06** Students will be expected to relate linear relations to their graphs, expressed in ▪ slope-intercept form ($y = mx + b$) ▪ general form ($Ax + By + C = 0$) ▪ slope-point form ($y - y_1) - m(x - x_1$) **RF09** Students will be expected to represent a linear function, using function notation.	**RF01** Students will be expected to model and solve problems that involve systems of linear inequalities in two variables.	**RF07** Students will be expected to solve problems that involve linear and quadratic inequalities in two variables. **RF08** Students will be expected to solve problems that involve quadratic inequalities in one variable.

Mathematics 10 (continued)	Mathematics 11 (continued)	Pre-calculus 11 (continued)
RF10 Students will be expected to solve problems that involve systems of linear equations in two variables, graphically and algebraically.		

Background

In Mathematics 9, students solved single variable linear inequalities with rational coefficients (PR04). In Mathematics 10, students solved problems that involved systems of linear equations in two variables, both graphically and algebraically.

In Mathematics 11, students will test points to decide if a given ordered pair satisfies a given inequality. For a given linear inequality, students will graph the appropriate region and verify. Students will be introduced to linear inequalities in two variables, extending the graphical model of a linear inequality from a one-dimensional linear number line to a two-dimensional coordinate plane and the solution set from points along a linear number line to points in a solution region.

Terms such as those in the table below are used with reference to inequalities.

greater than	less than	greater than or equal to	less than or equal to	at most
no more than	no less than	at least	more than	fewer than
maximum	minimum	optimal		

Students will learn how to represent a simple problem situation algebraically as a system of two linear inequalities and then graph the system to represent the solution set. They will graph systems of linear inequalities that include the possibility of equality, focusing on the intersection of the solution regions and how the common region represents the solution to the system.

Students must define any variable that they use so that its meaning is evident. For example, to say “let *d* represent distance” would be insufficient. They would rather need to say “let *d* represent the distance travelled since time, $t = 0$”. Otherwise d could represent the distance that you are from Halifax or home or school. This clear definition of a variable is very important to establish with students.

Students will represent problem situations algebraically as systems of two linear inequalities and then graph the systems to represent the solution sets. Students will verify their solution using test points and/or graphing technology.

New terminology is introduced and should be both developed and encouraged in student discussions. **Constraints** describe limiting conditions and restrictions on the problem situation and are represented by linear inequalities. The **feasible region** is the region for a system of linear inequalities within which any point is a feasible or valid solution. The **objective function** is the optimization equation in the linear programming model.

It is essential that students develop a solid understanding of constraints and their specific meaning to the context of a problem before putting them to use in the solution of linear programming questions.

For example: A potter is making cups and plates. It takes her 6 minutes to make a cup and 3 minutes to make a plate. Each cup uses $\frac{3}{4}$ lb. of clay and each plate uses 1 lb. of clay. The potter has an order to fill for 10 cups. She has 20 hours available for making the cups and plates and has 250 lb. of clay on hand. She makes a profit of $2 on each cup and $1.50 on each plate. How many cups and how many plates should she make in order to maximize her profit?

Students would begin by clearly defining variables. Let *c* represent the number of cups the potter makes from the clay she has. Let *p* represent the number of plates the potter makes from the clay she has.

Once the students have clearly defined variables, give them the constraints $c \geq 10$ and $p \geq 0$ and ask them what these represent in the context of the question.

Give the students one of the additional constraints, such as $\frac{3}{4}c + 1p \leq 250$ and ask them to explain what each of the terms represents. We would expect them to say that

- $\frac{3}{4}c$ represents that amount, in pounds, of clay per cup multiplied by the number of cups and, therefore, is the amount of clay used for the production of the cups
- $1p$ represents that amount, in pounds, of clay per plate multiplied by the number of plates and, therefore, is the amount of clay used for the production of the plates
- 250 represents the total amount of pounds of clay available

Now ask students to explain why either $\frac{1}{10}c + \frac{1}{20}p \leq 20$ or $6c + 3p \leq 1200$ could be used for a constraint in this situation. We would expect them to say that $\frac{1}{10}c + \frac{1}{20}p \leq 20$ is a time constraint where time is in hours and $6c + 3p \leq 1200$ is the same time constraint where time is in minutes. It is important that they are able to clearly explain what each term represents in any constraint inequation.

Once students have a good understanding of the meaning of the variables and the meaning of the constraints, they are ready to begin finding the constraints from the context of the question.

Students will explore the feasible region by examining the coordinates of the points and discover that optimal values of the objective function are found at the vertices of the feasible region, though the vertex of a feasible region is not necessarily an optimal solution. Although students should be aware of these situations, the level of sophistication that is required to locate the optimal point in these situations is beyond this grade level and is not an expectation in the course.

Technology can be used effectively to graph inequalities. While some software programs allow inequations in any form, students may need to rearrange an equation to obtain its slope intercept form in order to use technology such as the inequality application on the TI-84 calculator.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

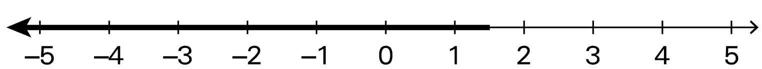
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

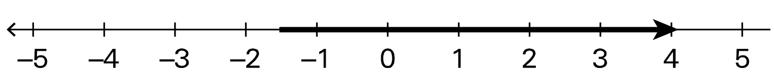
- Explain to another student how you would solve the following system of linear equations. Justify the method you chose.
  (a) $4x - 7y = -39$
  (b) $3x + 5y = -19$

  Create a situation relating to coins that can be modelled by the linear system shown below and explain the meaning of each variable.
  (a) $x + y = 24$
  (b) $0.25x + 0.05y = 4.50$

- Solve each of the following inequalities:
  (a) $2x + 5 \leq 8$
  (b) $5 - 3x \leq 8$

- What inequality is represented by each of the following?



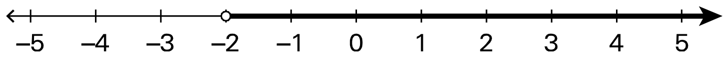


WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

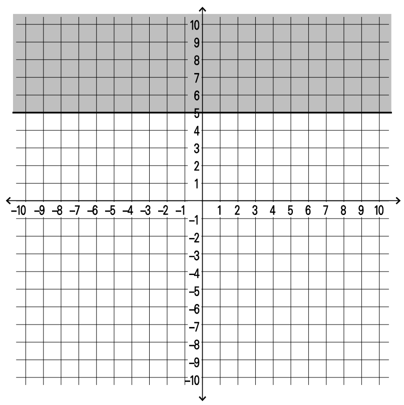
- State the coordinates of a point that satisfies the inequality $3x + 2y \leq 4$?
- Is the point (2, 1) a solution for the inequation $3x - 2y > 8$?
- Explain, using the number line, what is meant by the inequation $x \geq 2$.

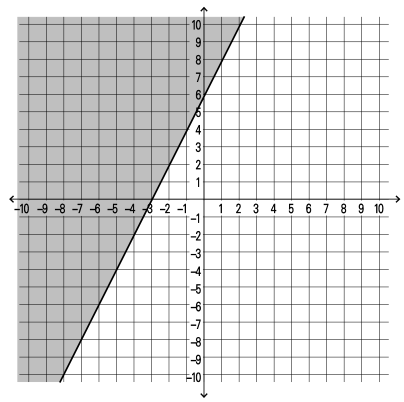
- What inequation is represented by the following diagram?



- Explain, using the Cartesian coordinate system, what is meant by the inequation $x \geq 2$.

- What inequation is represented by each of the following diagrams?



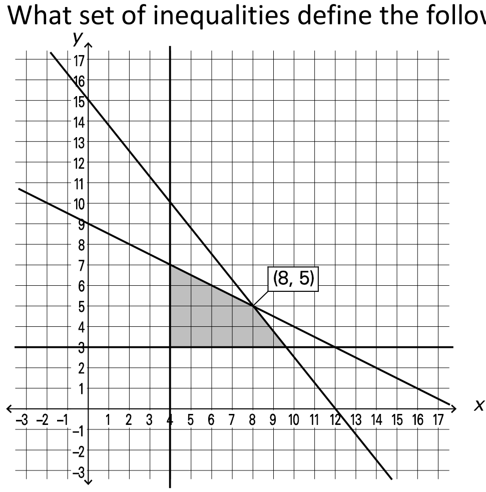


- A farmer has 100 acres on which to plant oats or corn. Each acre of oats requires \$18 of capital. Each acre of corn requires \$36 of capital. The farmer has \$2100 available for capital. If the revenue is \$55 from each acre of oats and \$125 from each acre of corn, what combination of plants will produce the greatest total profit?

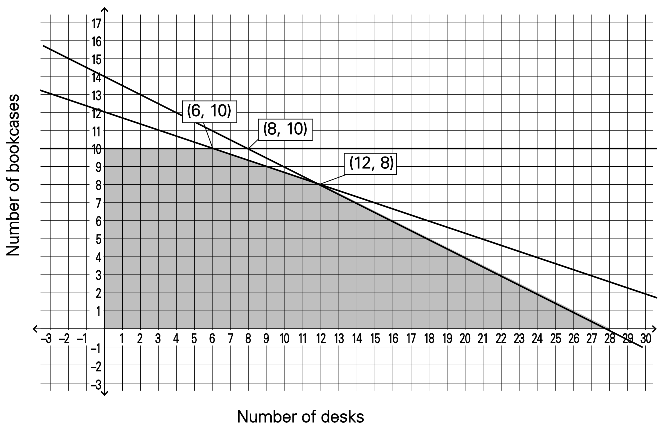
- Midtown Manufacturing Company makes plastic plates and cups, each of which requires time on two different machines. Manufacturing a unit of plates requires one hour on machine A and two hours on machine B. Producing a unit of cups requires three hours on machine A and one hour on machine B. Each machine is operated for at most 15 hours per day. Using what you have learned about systems of inequalities, and the restrictions given above, determine the maximum number of plates and cups the company can produce each day.

- Sketch the solution that corresponds to each set of inequalities.
  (a) $y \leq 6 - x$ and $y \geq x - 2$
  (b) $3x - y < 4$ and $x - 2y > 3$

- What set of inequalities define the following shaded region?



- A local furniture construction company has agreed to certain conditions in order to receive government funding. They also recognize that certain conditions exist due to the size of their workplace and the number of their employees. The shaded region on the graph below indicates all the possible number of desks and bookcases that the company can make each month.



  Each desk can be sold for a price that will yield a profit of \$120 and each bookcase can be sold and yield a profit of \$90. Write the objective function that could be used to determine the number of each type of item that this company should produce each month in order to maximize their profit and determine the number of desks and bookcases that should be produced in order to maximize their weekly profit.

- An artist is producing handmade necklaces and bracelets. She makes more necklaces than she does bracelets. She can make up to a total of 24 bracelets and necklaces per week.
  (a) Define the variables in this situation.
  (b) Write the inequations that describe this situation.
  (c) Graph the inequations and determine the feasible region.
  (d) What additional information is needed if you are asked to determine how many bracelets and necklaces the artist should make in order to maximize her profit?

- Sean has two summer jobs. He works no more than a total of 34 hours per week. Both jobs allow him to have flexible hours. At one job, Sean works at least 12 hours and earns \$11.25 per hour. At the other job, he works no more than 25 hours and earns \$11.50 per hour.
  (a) Sean wrote the inequations shown below. Explain what the variables represent and what each of the inequations represents.
    (i) $A \geq 12$
    (ii) $B \leq 25$
    (iii) $B \geq 0$
    (iv) $A \leq 34$
    (v) $A + B \leq 34$
  (b) Sketch the feasible region.
  (c) What combination of numbers of hours will allow Sean to maximize his earnings? What can he expect to earn?

- A calculator company produces both a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific calculators and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific calculators and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped every day. Each scientific calculator sold results in a $3 profit, but each graphing calculator produces a $5 profit. Let *S* represent the number of scientific calculator produced per day. Let *G* represent the number of graphing calculator produced per day.

(a) Write the inequalities to represent the constraints for this company.
(b) Determine the number of each type of calculator that this company should produce each day in order to maximize their daily profit.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

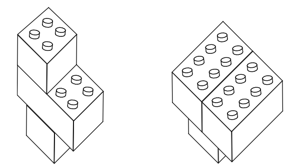
Consider the following sample instructional strategies when planning lessons.

- Begin with a situation that builds understanding of inequalities, such as the following: Two car rental agencies in the UK have the following rate structures for a subcompact car (prices in Canadian Dollars).
  - Ace Cars charges $20 per day plus 15¢ per km.
  - Big Ben Cars charges $18 per day plus 16¢ per km.

  (a) If you rent a car for one day, for what number of kilometres will the two companies have the same total charge?
  (b) Under what conditions, if any, should you rent from Ace Cars? Explain your reasoning.

  Since the answer to this second question is an inequality, this context will build on the students' understanding that while the solution of two linear equations is a point, the solution to a question may be conditional.

- As an introduction to linear programming and constraints use Lego, sticky dots, and chart paper. Have students work in pairs. Each pair gets a Ziploc bag of eight small pieces and six large pieces and is given the following challenge.
  - You are the owner of a furniture shop, which specializes in full-size Lego-inspired furniture. Your supplier can deliver eight small components and six large components each day. A chair requires the use of two small and one large component and has a profit of $10. A table requires the use of two small and two large components and has a profit of $18. How do you make the most profit?



Once students have determined how they would make the most profit for this situation, suggest that the chair be redesigned to have a back that is higher so that the chair now uses three small and one large component, the table remains unchanged. How would this design change impact the number of tables and chairs that should be made to maximize the profit?

After students have worked with the Lego chairs and tables, present the following situation and ask students to agree on the variables to be used and together define these variables.

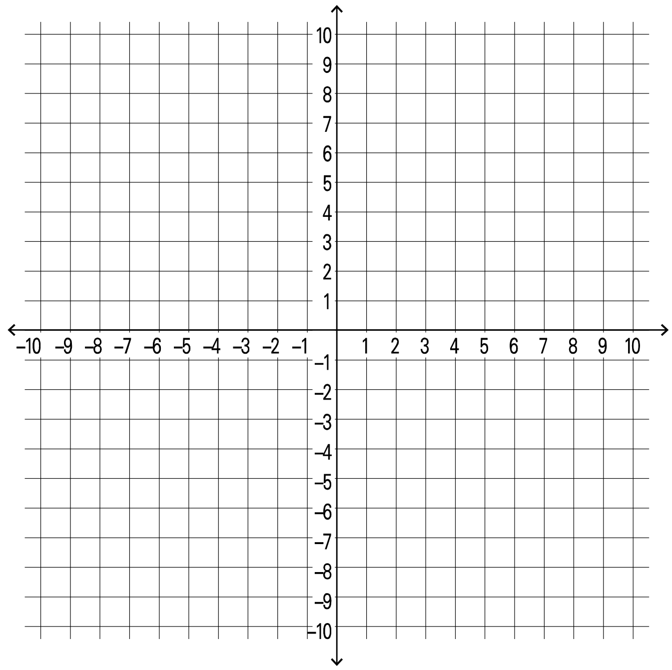
  - Several years later, your shop is now a large chain. You have updated your product line. Your supplier can deliver 1200 small components and 800 large components each day. How do you make the most profit?

Allow groups time to think about this problem and how they would approach it.

Once the class has agreed on the constraints of $2C+2T \leq 1200$ and $1C+2T \leq 800$ or $3C+2T \leq 1200$ and $1C+2T \leq 800$ depending on the furniture design being used, make sure that they are able to clearly explain what each of these equations represents. You might ask them what the constraints $2C+1T \leq 1800$ and $3T \leq 1200$ would represent in terms of a different furniture design.

When you feel confident that the students understand the meaning of the constraints, choose a set of constraints for one particular furniture design. Give each group of students 8–12 points to sort and determine if they satisfy one of the constraints (red dot) or the other constraint (yellow dot). If the point satisfies both constraints use a (blue dot). If the point does not satisfy either equation, do not assign it a dot.

The students will then place the appropriate coloured dots on a grid you have prepared. You can then project or draw the inequations that represent the constraints to illustrate the feasible region.



Insisting that students clearly define the variables that they choose to use can often reduce any confusion that they may experience when obtaining inequations and interpreting their solutions.

- Play a game where students have to match inequalities to their graphs. One such game can be found online at the Quia website at www.quia.com/rr/79715.html?AP_rand=571246363.

- Some students will find the use of a chart, as in the following example, will assist them in organizing their information for a linear programming problem.
  - Carmella spins yarn and then weaves it to produce handmade wall hangings and afghans. A wall hanging requires one hour of spinning and one hour of weaving; an afghan, two hours of spinning and four hours of weaving. Over several days, Carmella spends eight hours spinning and 14 hours weaving. Express this situation as a system of inequalities.

It is important to ensure that students first clearly define the variables that they will be using. Let *w* represent the number of wall hangings that are created. Let *a* represent the number of afghans that are created. Once students have clearly defined the variables, they need to list any implied restrictions, such as the fact that it is impossible to create a negative number of items. We know that $w \geq 0$ and $a \geq 0$.

Creating a chart such as the one below, simplifies obtaining the other in-equations.

	Spinning	**Weaving**
Wall hanging	1 hour	1 hour
Afghan	2 hours	4 hours
Total number of hours	8 hours	14 hours

Constraints

$w \geq 0$ $\quad$ $a \geq 0$ $\quad$ $1w + 2a \leq 8$ $\quad$ $1w + 4a \leq 14$

- Take time to ensure that students
  - clearly define variables
  - explain the meaning of what each term in the various constraints represents (For example, for the inequality $1w + 4a \leq 14$ from the question above, students should be to state that $1w$ represents the fact that a wall hanging takes one hour of weaving; $4a$ represents the fact that one afghan takes four hours of weaving and the 14 represents the total maximum number of hours weaving.)

SUGGESTED MODELS AND MANIPULATIVES

- chart paper
- Lego pieces
- sticky dots

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- constraints
- feasible region
- inequality
- objective function
- profit equation
- verify
- vertex

Resources/Notes

Internet

- For students with netbooks, tablets, or access to the computer lab, use Internet sites such as the following for practice.
  - *Algebra 2 Online!,* "Module-Solving Systems of Linear Equations and Inequalities" (Henrico County Public Schools 2006)
    http://teachers.henrico.k12.va.us/math/hcpsalgebra2/3-4.htm
    A PowerPoint lesson on how to solve systems of inequalities.
  - *Algebra-class.com*, "Systems of Inequalities Word Problems" (Karin Hutchinson 2013)
    www.algebra-class.com/systems-of-inequalities.html
    A good example of an inequality question with the solution worked out. At the bottom of the page there is a link to systems of inequalities practice problems.

- *Linear Inequalities Game* (Quia 2014)
  www.quia.com/rr/79715.html?AP_rand=571246363

Print

- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 6.1–6.6, pp. 294–346

Notes

SCO RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. [CN, PS, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(It is intended that completion of the square not be required.)

RF02.01 Determine, with or without technology, the intercepts of the graph of a quadratic function.
RF02.02 Determine, by factoring, the roots of a quadratic equation, and verify by substitution.
RF02.03 Determine, using the quadratic formula, the roots of a quadratic equation.
RF02.04 Explain the relationships among the roots of an equation, the zeros of the corresponding function, and the x-intercepts of the graph of the function.
RF02.05 Explain, using examples, why the graph of a quadratic function may have zero, one, or two x-intercepts.
RF02.06 Express a quadratic equation in factored form, using the zeros of a corresponding function or the x-intercepts of its graph.
RF02.07 Determine, with or without technology, the coordinates of the vertex of the graph of a quadratic function.
RF02.08 Determine the equation of the axis of symmetry of the graph of a quadratic function, given x-intercepts of the graph.
RF02.09 Determine the coordinates of the vertex of the graph of a quadratic function, given the equation of the function and the axis of symmetry, and determine if the y-coordinate of the vertex is a maximum or a minimum.
RF02.10 Determine the domain and range of a quadratic function.
RF02.11 Sketch the graph of a quadratic function.
RF02.12 Solve a contextual problem that involves the characteristics of a quadratic function.

Scope and Sequence

Mathematics 10	Mathematics 11	Pre-calculus 11
AN05 Students will be expected to demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially, and symbolically. **RF01** Students will be expected to interpret and explain the relationships among data, graphs, and situations. **RF02** Students will be expected to demonstrate an understanding of relations and functions.	**RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry.	**RF03** Students will be expected to analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts.

Mathematics 10 (continued)	Mathematics 11 (continued)	Pre-calculus 11 (continued)
RF04 Students will be expected to describe and represent linear relations, using words, ordered pairs, tables of values, graphs, and equations. **RF05** Students will be expected to determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. **RF06** Students will be expected to relate linear relations to their graphs, expressed in ▪ slope-intercept form ($y = mx + b$) ▪ general form ($Ax + By + C = 0$) ▪ slope-point form $(y - y_1) - m(x - x_1)$ **RF09** Students will be expected to represent a linear function, using function notation.		**RF04** Students will be expected to analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including vertex, domain and range, direction of opening, axis of symmetry, *x*- and *y*-intercepts, and to solve problems. **RF05** Students will be able solve problems that involve quadratic equations.

Background

It is important to note that students in Mathematics 11 will **not** be expected to “complete the square” to change the form of a quadratic function. Completing the square will be covered in a future mathematics course.

In this outcome, students will be introduced to quadratic functions. They will examine quadratic functions expressed in the following three forms.

Standard Form: $f(x) = ax^2 + bx + c$
Factored Form: $f(x) = a(x - r)(x - s)$
Vertex Form: $f(x) = a(x - h)^2 + k$

Students will explore the general impact that changes in **parameters** have on the graphs, and will sketch graphs using characteristics such as *x*- and *y*-intercepts, vertex (as a maximum or minimum point), axis of symmetry, and domain and range. Terminology such as “a vertical stretch of 2” is *not* an expectation. In this course students will talk about the shape of the quadratic in more general terms.

Before students are exposed to the standard form of a quadratic, they need to become familiar with the shape of a quadratic function and how one identifies a quadratic function. The terms **quadratic** and **parabola** are new to students. This will be their first exposure to functions that are non-linear. Students should have an opportunity to investigate what makes a quadratic function.

In Mathematics 9, students used a table of values to graph linear relations. They will now extend this strategy to quadratic equations to determine the vertex and its connection to the axis of symmetry.

When graphing the quadratic function $y = x^2 + 4x + 3$, for example, students can create the following table of values.

x	–4	–3	–2	–1	0	1	2	3
y	3	0	–1	0	3	8	15	24

From the graph that students create, they can identify the characteristics of the quadratic function—the coordinates of the vertex, direction of opening, and the *x*- and *y*-intercepts.

Students will be required to determine the domain and range of the function. The concept of domain and range was introduced in Mathematics 10 using set notation or interval notation (RF05). Students should recognize that all non-contextual quadratic functions have a domain of $\{x \mid x \in R\}$, whereas the range depends on the vertex and the direction of the opening.

Students will first explore quadratic functions in standard form. Students should be introduced to the quadratic term, linear term, and constant of a quadratic equation. They should distinguish between the quadratic term and the coefficient of the quadratic term. For example, students often mistake the quadratic term as the value of a instead of ax^2.

Students will generate graphs of quadratic functions using a table of values; they will then determine the equation of its axis of symmetry. Students will observe that the axis of symmetry passes through the vertex and that this *x*-value is the *x*-coordinate of the vertex. Students will also observe that the *y*-coordinate of the vertex is the function's maximum or minimum value.

Students work with two basic ideas that both involve symmetry in a variety of settings.

- The first idea is that given two points with the same *y*-coordinate on a parabola, the equation of the axis of symmetry (and then the vertex) can be located by averaging the *x*-coordinates of the points.
- The second is that a table of values and/or a graph can directly reveal the parabola's axis of symmetry.

Students will also learn a technique for determining the vertex of a quadratic called **partial factoring**. The technique of removing a partial factor will make sense for students who understand that points that have the same *y*-value must be equidistant from the axis of symmetry. The **zero product property** will have to be introduced, to ensure that understanding is emphasized for this approach.

For example,
$f(x) = 2x^2 - 6x + 7$
$f(x) = 2x(x - 3) + 7$

When $2x(x - 3) = 0$, we know that $f(x) = 7$, and we can find two points on the quadratic that both have the same y value.
$2x = 0$ and $x - 3 = 0$
$x = 0$ and $x = 3$

The points (0, 7) and (3, 7) are on the parabola and the line of symmetry is $x = \frac{0+3}{2} = 1.5$.

Therefore the vertex is found where $x = 1.5$ and the vertex is $(1.5, f(1.5))$ or (1.5, 2.5).

Students should discover, by exploration, that the value of $x = \frac{-b}{2a}$ is the *x*-coordinate of the vertex, and the connection should be made to the equation of the axis of symmetry. The *y*-coordinate of the vertex can be found by substitution of the *x*-coordinate into the quadratic function.

Students should observe the following about a quadratic in standard form, $y = ax^2 + bx + c$.

- The value of the coefficient of the quadratic term will indicate whether the quadratic is opening up ($a > 0$) or opening down ($a < 0$)
- The value of the constant will be the *y*-intercept.
- The *x*-coordinate of the vertex can be determined using partial factoring.
- The values of *a* and *c* may allow the student to determine how many *x*-intercepts the quadratic will have. For example, if $a > 0$ and $c < 0$, then the *y*-intercept is negative and the quadratic opens up and there must be two *x*-intercepts.

After students have worked with the standard form of quadratic functions, they will work with quadratic functions in factored form. They should discover, by exploration, the connection between the factored form of a quadratic function and the x-intercepts of its graph. Thus the function $y = 6x^2 + x - 2$ would be factored as $y = (2x - 1)(3x + 2)$ and would have *x*-intercepts of $(\frac{1}{2}, 0)$ and $(-\frac{2}{3}, 0)$ when graphed. While its factored form of a quadratic is given as $y = a(x - r)(x - s)$, it is **not** expected that students would have to rearrange the equation to express it in this form as follows.

$y = 6x^2 + x - 2$

$y = (2x - 1)(3x + 2)$

$y = (2)(x - \frac{1}{2})(3)(x + \frac{2}{3})$

$y = 6(x - \frac{1}{2})(x + \frac{2}{3})$

Writing the quadratic as the product of two linear factors would be sufficient for students to determine the *x*-intercepts, vertex, and the axis of symmetry.

In Mathematics 10, when students were introduced to common factors and trinomial factoring, they modelled the factoring concretely and pictorially and recorded the process symbolically. Some students will continue to benefit from concrete representations. Manipulatives, such as algebra tiles, should therefore be available for use.

Students will be required to use factoring methods developed in Mathematics 10 to determine the zeros. It is important for students to distinguish between the terms **zeros**, **roots**, and ***x*-intercepts**, and to use the correct term in a given situation. The **zero product property** states that if the product of two real numbers is zero, then one or both of the numbers must be zero. This will be new for students. It is important for them to make the connection that the values of *x* at the zeros of the function are also the *x*-intercepts of the graph.

Students will relate the factored form of a quadratic function to the properties of its corresponding parabola (intercepts, axis of symmetry, vertex, and direction of opening). They will use these features to sketch the corresponding parabola and determine the function’s domain and range.

Students should observe the following about a quadratic in factored form, $y = a(x - r)(x - s)$.

- The value of the coefficient of the quadratic term will indicate whether the quadratic is opening up ($a > 0$) or opening down ($a < 0$)
- The x-intercepts will be $(r, 0)$ and $(s, 0)$.
- The x-coordinate of the vertex and the axis of symmetry will be midway between the values of r and s.
- The y-intercept can be found by letting $x = 0$ and determining $y = a(-r)(-s) = ars$.

Once students are familiar with both the standard and factored forms of quadratics functions they will investigate the vertex form.

Students will graph a quadratic function in vertex form $y = a(x - h)^2 + k$, and will relate the characteristics of the graph to its function. Students were exposed to idea of a parameter earlier in this unit. They will now explore the components h and k and how they relate to the graph of the quadratic. This exploration is most efficiently done using graphing technology.

Students should observe the following about a quadratic in factored form, $y = a(x - h)^2 + k$.

- The value of the coefficient of the quadratic term will indicate whether the quadratic is opening up ($a > 0$) or opening down ($a < 0$)
- The vertex is the point (h, k).
- The maximum or minimum value for the quadratic is $y = k$.
- The axis of symmetry is $x = h$.
- The values of a and the vertex will allow the student to determine how many x-intercepts the quadratic will have. For example, if $a > 0$ and the vertex is $(2, 6)$, then the vertex is above the x-axis and the quadratic opens up and there are no x-intercepts.

Given a quadratic function in standard, vertex, or factored form, students will be able to sketch a graph of the function and then use that graph to solve contextual problems in which the equation of the quadratic is given.

Specifically,

- if the equation is in standard form, students will be able to get a rough sketch of its graph by plotting the y-intercept and using the direction of opening
  - If the context requires the vertex of the parabola, students can use the method of partial factoring or the formula $x = \frac{-b}{2a}$ to find its x-coordinate and then substitute into the equation to determine the y-coordinate of the vertex.
  - If the context requires the x-intercepts, the student will factor the quadratic to obtain its factored form.
- if the equation is in factored form, students will be able to get a rough sketch of its graph by plotting the x-intercepts and the direction of opening
  - If the context requires the y-intercept, students will let $x = 0$ and solve.
  - If the context requires the vertex of the parabola, students will use the symmetry of the quadratic to obtain the x-coordinate and then substitute into the equation to determine the y-coordinate of the vertex.
- if the equation is in vertex form, students will be able to get a rough sketch of its graph by plotting the vertex and the direction of opening
  - If the context requires the y-intercept, students will let $x = 0$ and solve.
  - If the context requires the x-intercepts, students will let $y = 0$ and solve.

Students are also expected to work from a graph or data set with vertex or x-intercepts and one additional point given to obtain the equation of the quadratic function.

This is a process in which students work backwards to obtain the equation of the quadratic function in its factored form or to obtain the equation of a quadratic function in its vertex form.

Examples:

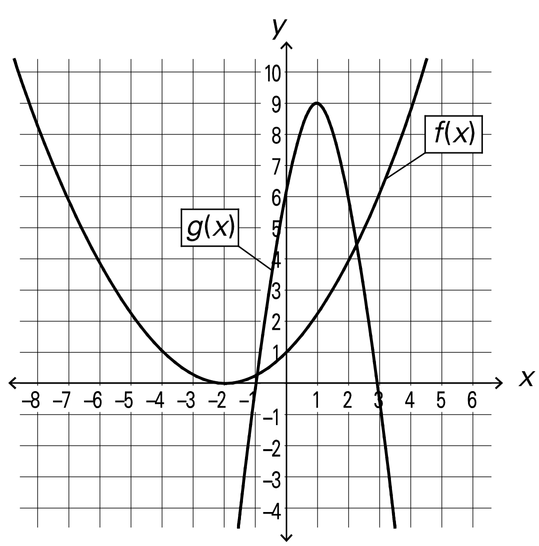
From a data set where you can see the x-intercepts.

x	y	
0	6	Since this has two x-intercepts at (2, 0) and (3, 0), you know that the equation is $y = (a)(x - 2)(x - 3)$; substitution of one of the other points yields the following: (0, 6): $6 = (a)(0 - 2)(0 - 3)$ $6 = a(-2)(-3)$ $6 = 6a$ $1 = a$ $\therefore\ y = 1(x - 2)(x - 3)$
1	2	
2	0	
3	0	
4	2	
5	6	

From a data set where you can see the vertex:

x	y	
0	6	Since the vertex of (1, 4) can be read from this data set, we know that the equation is $y = a(x - 1)^2 + 4$; substitution of one of the other points yields the following: (0, 6): $6 = (a)(0 - 1)^2 + 4$ $6 = a(-1)^2 + 4$ $6 = 1a + 4$ $2 = a$ $\therefore\ y = 2(x - 1)^2 + 4$
1	4	
2	6	
3	12	
4	22	

From a graph where both the x-intercepts and the vertex can be observed:



Factored Form	Vertex Form
$f(x) = k(x + 2)^2$ $f(0) = 1$ $1 = k(0 + 2)^2$ $1 = k(4)$ $\frac{1}{4} = k$ $f(x) = \frac{1}{4}(x + 2)^2$	$f(x) = k(x + 2)^2 + 0$ $f(0) = 1$ $1 = k(0 + 2)^2 + 0$ $1 = k(2)^2 + 0$ $1 = 4k$ $\frac{1}{4} = k$ $f(x) = \frac{1}{4}(x + 2)^2$
$g(x) = k(x + 1)(x - 3)$ $f(2) = 6$ $6 = k(2 + 1)(2 - 3)$ $6 = k(3)(-1)$ $6 = -3k$ $-2 = k$ $g(x) = -2(x + 1)(x - 3)$	$g(x) = k(x - 1)^2 + 8$ $f(2) = 6$ $6 = k(2 - 1)^2 + 8$ $6 = k(1)^2 + 8$ $-2 = 1k$ $-2 = k$ $g(x) = -2(x - 1)^2 + 8$

From a context:

An arrow is shot from a height of 1 m, it follows a parabolic path, and reaches a height of 5 m two seconds after it is shot.	The vertex is (2, 5) and the initial height yields the point (0, 1). Therefore the equation in vertex form is $y = a(x - 2)^2 + 5$; substituting in the point (0, 1) yields the following: (0, 1): $1 = (a)(0 - 2)^2 + 5$ $1 = a(-2)^2 + 5$ $1 = 4a + 5$ $-4 = 4a$ $-1 = a$ $\therefore\ y = -1(x - 2)^2 + 5$

Encourage students to work with and compare the various methods, decide when each of the three methods is most convenient to use, and explain why.

Students need to understand the characteristics of each quadratic form in addition to the limitations of each. Students should be provided with examples in which they have to select which form of the quadratic function best suits the information provided. For example, a key piece of information of the vertex form is knowing the vertex, and the benefit of the factored form is knowing the *x*-intercepts. Even though students initially start with the quadratic in vertex or factored form, they can always use their distributive property to rewrite the equation in standard form.

Using the form of the quadratic function which would most appropriately apply to the situation, students will solve contextual problems where

- the equation is given
- the vertex is evident and one other point is available
- the *x*-intercepts and one other point are given

Students will **not** be expected to generate a quadratic from data or a situation where the vertex or *x*-intercepts are not evident.

For example, although questions such as the following may be found in the selected resource, in this course students will not be expected to solve questions such as the following:

- Ataneq takes tourists on dogsled rides. He needs to build a kennel to separate some of his dogs from the other dogs in his team. He has budgeted for 40 m of fence. He plans to place the kennel against part of this home, to save on materials. What dimensions should Ataneq use to maximize the area of the kennel?

- A career and technology class at a high school operates a small t-shirt business. Over the past few years, the shop has had monthly sales of 300 T-shirts at a price of $15 per T-shirt. The students have learned that for every $2 increase in price, they will sell 20 fewer T-shirts each month. What should they charge for their T-shirts to maximize revenue?

- Once students have become familiar with working with the three forms of the quadratic function, they will be expected to solve a quadratic equation.
  - It is expected that students will solve quadratic equations by factoring and will verify their solutions using substitution. Students will use their knowledge of common factoring, trinomial factoring, and difference-of-squares factoring from Mathematics 10, to solve quadratic equations algebraically.

- Students will also solve quadratic equations by using the quadratic formula. (The quadratic formula is presented to the students at this point without derivation.) They will express their answers as either decimal approximations or exact values as the contexts warrant. In Mathematics 10, students have worked with radicals, and they have simplified entire radicals such as $\sqrt{98}$ to obtain $7\sqrt{2}$. While the term **discriminant** is not introduced in Mathematics 11, a discussion about how the radicand in the quadratic formula can be used to determine if a quadratic is factorable or not is expected. It is also expected that students can use the radicand to determine the number of *x*-intercepts the quadratic will have.
- If the equation is in factored form, students are most likely going to change it to standard form and then use the quadratic formula to solve the equation.
- If the equation is in vertex form and students are able to efficiently solve it without changing forms, they should solve it using that form.
- Students will also examine the graphical methods for solving quadratic equations (with and without technology) and discuss the advantages and disadvantages of a graphical approach. They will make connections between the roots of the equation, the zeros of the corresponding function, and the *x*-intercepts of the corresponding parabola. Students will solve contextual problems that require the use of quadratic equations.

- Students will also be expected to consider if the solution(s) are admissible or not. An **inadmissible** solution is a root of a quadratic equation that does not lead to a solution that satisfies the original problem. These will be tied to the context of the question. For example, if you determine that an arrow hits the ground at $t = -1$ sec. and $t = 3$ sec., where t was defined as the time since the arrow was shot, then the $t = -1$ is not admissible, since it could not hit the ground before the arrow was shot.

  Students will **not** be expected to generate a quadratic from data or a situation where the vertex or *x*-intercepts are not evident. In situations such as the following, students in this course would be provided with the related equation before being expected to solve the quadratic equation that represents the situation.

  For example, although questions such as the following may be found in the selected resource, in this course students will not be expected to solve questions such as the following unless they were given the equation as well as the question.
  - Determine three consecutive odd integers, if the square of the largest integer is 33 less than the sum of the squares of the two smaller integers.
  - At noon, a sailboat leaves a harbour and travels due west at 10 kmh. Three hours later, another sailboat leaves the same harbour and travels due south at 15 kmh. At what time, t, to the nearest minute, will the sailboats be 40 km apart?
  - Francis is an artist. She wants the area of the mat around her new panting to be twice the area of the panting itself. The mat that she wants to use is available in only one width. The outside dimensions of the same mat around another painting are 80 cm × 60 cm. What is the width of the mat?

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

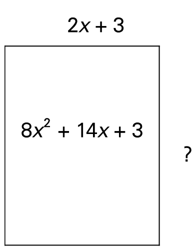
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Given $2x^2 + 6x - 40$, find and correct the mistakes in the factoring below.
  $2x^2 + 6x - 40$
  $= 2(x^2 + 3x - 40)$
  $= 2(x - 8)(x + 5)$

- The area of a rectangle is represented by the product $8x^2 + 14x + 3$ square units. If one dimension of the rectangle is $(2x + 3)$ units, determine the other dimension. How did knowing one of the factors help you determine the other factor?



WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Complete the chart:

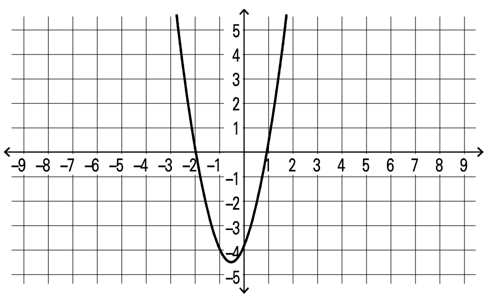
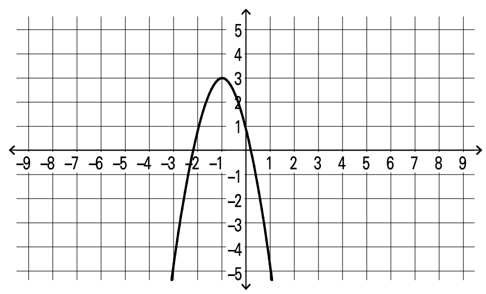
Polynomial Function	Classification	True or False	Explain/Justify
$y = 5(x + 3)$	Linear		
$y = 5(x^2 + 3)$	Quadratic		
$y = 5^2(x + 3)$	Quadratic		
$y = 5x(x + 3)$	Linear		
$y = (5x + 1)(x + 3)$	Quadratic		
$y = 5(x + 3)^2 + 2$	Quadratic		

- The daily profit, P, (dollars) for Swift Shot, a company that makes tennis racquets, is given by $P = -n^2 + 240n - 5400$ where n is the number of racquets produced per day.
  - How many tennis racquets must be produced per day to have a maximum profit?
  - What is the maximum profit?
  - What profit is made when 75 racquets per day are produced?
  - How many tennis racquets must be produced to break even?

- Use algebra to write the quadratic function, $y = 5(x-1)^2 - 8$, in the form $y = ax^2 + bx + c$. Show the work that leads to your solution.

- Sketch each of the following functions, labelling the values you use to draw each of them.
  $f(x) = -3x^2 + 24x - 50$
  $g(x) = 2(x-3)(x+1)$
  $p(x) = 4(x-5)^2 - 2$

- State the equation that would describe each of the graphs shown below.

- Create a quadratic equation that has roots $x = -2$ and $x = 3$. Is it possible to have more than one quadratic that has these roots? Explain.

- What is the minimum value of $y = x^2 - 8x - 9$?

- Create a quadratic equation that has roots of 5 and 7 and a maximum value of 4. Is it possible to have more than one quadratic that satisfies these conditions? Explain.

- For the graph of $y = x^2 - 2x - 35$,
  (a) determine the coordinates of the x-intercepts
  (b) find the vertex
  (c) state the coordinates of the y-intercept
  (d) sketch a graph (Label completely clearly indicating all intercepts, the vertex, axis of symmetry, and scales of the axes.)
  (e) state the domain and the range of this function

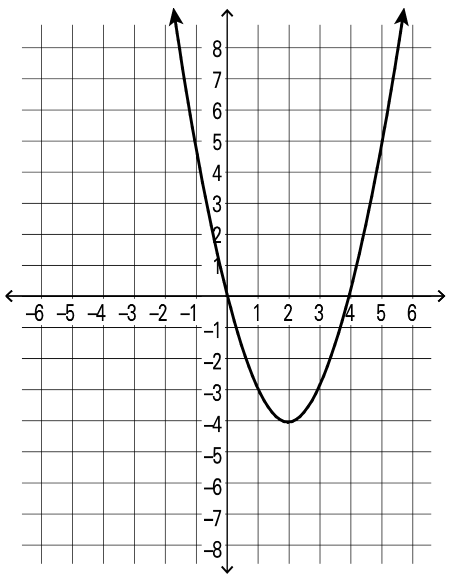
- Explain how you could determine whether the y-coordinate of the vertex of a quadratic function is a maximum or a minimum without graphing.

- Lina was asked to describe how the graph of $y = 2x^2$ compares to the graph of $y = x^2$. She said that the graph of $y = 2x^2$ will be narrower than the graph of $y = x^2$. Is Lina is correct? Explain your reasoning.

- Explain how changing the parameter a affects the graph of the function form $y = ax^2 + bx + c$.

- State the vertex and the equation of the line of symmetry for each of the following:

(a)



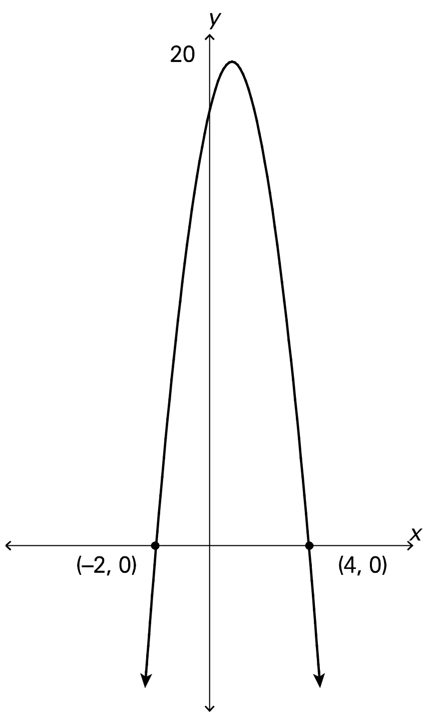
(b)

x	–1	0	1	2	3
y	10	1	–2	1	10

(c) $y = 2x^2 - 8x + 7$

- Given the graph of $y = -2x^2 + 8x - 3$, state
  (a) the coordinates of the y-intercept
  (b) the coordinates of the x-intercept(s)
  (c) the direction of opening
  (d) the vertex
  (e) the equation of the axis of symmetry

- Find the equation of the axis of symmetry for the parabola shown in the graph below.



- How can you determine the coordinates of the vertex given the factored form of a quadratic function?

- How can you obtain the coordinates of the vertex given the standard form of a quadratic function?

- Determine the quadratic function with factors $(x + 3)$ and $(x – 5)$ and a y-intercept of –5.

- What are the minimum requirements to sketch a unique quadratic graph? Explain.

- If the factors of a quadratic function are identical, what does this information tell you about the equation and the graph?

- Based on the following information, which form of the quadratic equation would you prefer to write? Why?
  (a) The vertex of the parabola is (5, –2) and passes through the point (–1, –4).
  (b) The factors are $(x – 1)$ and $(x + 3)$ and passes through the point (–2, 3).
  (c) The quadratic has a y-intercept of 10 and passes through the points (1, 12) and (2, 10).

- Find the equations represented by the following situations:
  (a) The vertex of the parabola is (5, –2) and passes through the point (–1, –4).
  (b) The factors are $(x – 1)$ and $(x + 3)$ and passes through the point (–2, 3).
  (c) The quadratic has a y-intercept of 10 and passes through the points (1, 12) and (2, 10).
  (d) A quarterback throws the ball from an initial height of six feet. It is caught by the receiver 50 feet away and at a height of six feet. The ball reaches a maximum height of 20 feet during its flight. Determine the quadratic function that models this situation, and state the domain and range.

- Is $x = –2$ a root of $x^2 + 5x + 6 = 0$?

- If $x = –3$ is one root of the equation $3x^2 + mx + 3 = 0$, what is the value of m, and what is the other root?

- The path of a model rocket can be described by the quadratic function $y = x^2 – 12x$, where y represents the height of the rocket, in metres, at time x seconds after takeoff. Identify the maximum height reached by the rocket, and determine the time at which the rocket reached its maximum height.

- What do you know about the function $y = x^2 + 6x + c$, if $c = 0$?

- For what values of c will the function $y = x^2 + 6x + c$ have
  (a) two x-intercepts
  (b) one x-intercept
  (c) no x-intercepts

- For what values of c is the function $y = x^2 + 6x + c$ factorable?

- Demonstrate, using alge-tiles or an area model, how you can determine the value(s) of b in the function $y = x^2 + bx + 12$ that create a factorable quadratic.

- Demonstrate, using alge-tiles or an area model, how you can determine the value(s) of b in the function $y = 2x^2 + bx + 6$ that create a factorable quadratic.

- The quadratic formula was used by a student to solve the equation $x^2 + x - 12 = 0$ as shown below. Identify and correct the error in the following:

  Step 1 $\quad x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-12)}}{2(1)}$

  Step 2 $\quad x = \dfrac{-1 \pm \sqrt{49}}{2}$

  Step 3 $\quad x = \dfrac{-1 \pm 7}{2}$

  Step 4 $\quad x = -1 \pm \dfrac{7}{2}$

- Solve the following quadratic equations using a variety methods:
  (a) $5x^2 + 5x + 5 = 35$
  (b) $2x^2 + 7x + 5 = 9$
  (c) $1 = x^2 - 6x$
  (d) $2(x - 3)^2 = 18$
  (e) $4(x - 2)(x + 1) = 0$
  (f) $(x + 2)(x - 1) = 4$

- The daily revenue of y dollars for a ski resort can be modelled by the equation $y = -16x^2 - 480x + 6400$, where x represents the temperature in degrees Celsius.
  (a) At what temperature does the daily revenue reach its maximum value? What is the maximum revenue?
  (b) What is the revenue when the temperature is –10°C?
  (c) What do you think the temperature was when the ski resort made $8704?

- A parabolic archway is 10 m wide. If it is 2 m high 1 m from where it touches the ground, what is the maximum height of the archway?

- A football is thrown so that the height of the ball above the ground, y, can be modelled by the function $y = -4.9x^2 + 24x + 1$, where x is the time in seconds after the ball was thrown.
  (a) State the range of this function to the nearest tenth of a metre, remembering the context of the question.
  (b) Assuming that the football is caught at a height of 1.5 m, how far was the ball thrown?
  (c) Sketch a graph of this situation and use the graph to check your answer to part (b).

- An arrow is shot into the air and its path given by the equation $h = -4.9(t - 1)^2 + 6$ where h is the height of the arrow above the ground, in metres, and t is the time since the arrow was shot, in seconds.
  (a) From what height was the arrow shot?
  (b) When does the arrow hit the ground?

- As part of an air show, an airplane is diving in a parabolic path. Its height above the ground, as recorded over a period of time, is shown below. What was the height of the plane at $t = 0$ seconds.

Time (sec)	1	5	9	13	17
Height above the ground (m)	685	205	45	205	685

  (a) How low to the ground did the plane get in this dive?
  (b) What was the height of the plane at $t = 0$ seconds?
  (c) When does the plain reach a height of 100 m?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Before students are exposed to the standard form of a quadratic, they need to become familiar with the shape of a quadratic function and how to identify a quadratic function. The terms **quadratic** and **parabola** are new to students. As previously discussed, this will be their first exposure to functions that are non-linear.

  Students should have an opportunity to investigate what makes a quadratic function. Have them multiply two linear equations or square a binomial of the form $ax + b$. Consider the examples $y = (x + 1)(x – 4)$ and $y = (3x – 2)^2$. Ask students what they notice in terms of the degree of the polynomial. (**Note:** You may need to define the degree of a polynomial function as the greatest exponent in a polynomial or equation.)

- Projectile motion can be used to explain the path of a baseball or a skier in flight. To help students visualize the motion of a projectile, toss a ball to a student. Ask students to describe the path of the ball to a partner and have them sketch the path of the height of the ball over time (the independent axis represents time and the dependent axis represents the height of the ball). Encourage them to share their graphs with other students. Ask students to think of other examples that might fit the diagrams of parabolas that open upward or downward.

- Characteristics of a parabola, such as vertex and axis of symmetry, should be discussed. It is important for students to recognize that, excepting the vertex, each point on a parabola has a corresponding point on its mirror image, which results in the parabolic shape. Discussions around everyday applications such as projectile motion give students an appreciation of the usefulness of quadratics.

- The characteristics of the quadratic function, $f(x) = ax^2 + bx + c$ should be developed through an investigation in which the parameters a, b, and c are manipulated individually. Technology like graphing calculators, or other suitable graphing software, should be used as students' graphing abilities for quadratics would be limited to the use of a table of values at this point.

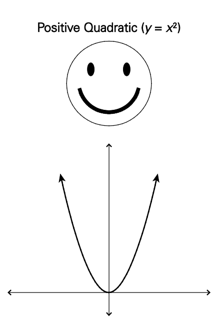
  Note: Students are expected to notice, as they explore the effects of parameter changes, that a impacts direction and shape of the parabola, b changes the location of the line of symmetry and c gives us the y-intercept.

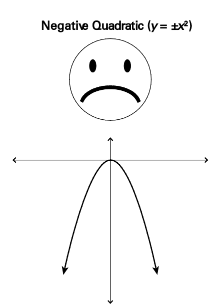
- Consider a task such as the following to examine the effects of manipulating the values of a, b, and c.

  (a) Students will first investigate the effect of changing the value of a by comparing quadratic functions of the form $y = ax^2$. As they compare the graphs of $y = 2x^2$ and $y = -2x^2$, $y = \frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2$, use prompts such as the following to promote student discussion:

  (i) What happens to the direction of the opening of the quadratic if $a < 0$ and $a > 0$?
  (ii) If the quadratic opens upward, is the vertex a maximum or minimum point? Explain your reasoning. What if the quadratic opens downward?
  (iii) Is the shape of the parabola affected by the parameter a? Are some graphs wider or narrower compared to the graph of $y = x^2$?
  (iv) What happens to the x-intercepts as the value of a is manipulated?
  (v) What is the impact on the graph if $a = 0$?

  The idea of the vertex being a maximum or minimum point is determined by the value of a. One way to help students remember how to determine direction of opening is shown here.





(b) Students should then proceed to examine the effects of manipulating the value of *b* and *c* in a similar manner. They should be given time to analyze the graphs of various quadratic functions, such as $y = x^2 + 2x$, $y = x^2 + 5x$, $y = x^2 + 1$, and $y = x^2 - 3$. Encourage them to look for connections between the changes in the graph and the function relative to $y = x^2$. Ask questions such as the following:
   (i) What is the effect of parameter *b* in $y = x^2 + bx$ on the graph of the quadratic? Is the parabola's line of symmetry changing?
   (ii) What is the effect of parameter *c* in $y = x^2 + c$ on the graph of the quadratic? How can you identify the *y*-intercept from the equation in standard form? Is the line of symmetry affected by the parameter *c*?

(c) Once students have explored the characteristics of the quadratic function, $f(x) = ax^2 + bx + c$ as they manipulated the parameters *a*, *b*, and *c*, students can more easily identify the characteristics of the quadratic function—the coordinates of the vertex, direction of opening, and the *x*- and *y*-intercepts, for example—when they have a visual representation. When graphing the quadratic function $y = x^2 + 4x + 3$, for example, students can create the following table of values.

x	−4	−3	−2	−1	0	1	2	3
y	3	0	−1	0	3	8	15	24

Ask students, by looking at this table, if they can recognize the point at which the parabola changes direction. The goal is for students to be able to recognize the symmetry, as the points on the parabola that share the same *y*-coordinate are equidistant from the vertex. Whether students use a graph or a table of values, teachers should promote discussion by asking questions such as the following:
   (i) What connection is there between the axis of symmetry and the coordinates of the vertex?
   (ii) What is the equation of the axis of symmetry?

- Present students with the following questions:
  (a) If the axis of symmetry is provided in addition to the standard form of the quadratic function, how would you determine the *y*-coordinate of the vertex. (Teachers could start the discussion by reminding students of their work with linear relations in Mathematics 10. When given the slope of the line and the coordinates of a point on the line, students determined the y-intercept through the use of substitution. This algebraic method can also work with quadratics by substituting the known *x*-value into the function to find the corresponding *y*-value.)

  (b) If you knew the two *x*-intercepts of the quadratic how could you determine the coordinate of the vertex? (The axis of symmetry can also be linked to the *x*-intercepts of the graph of a quadratic function. Provide students with several graphs of quadratic functions and ask them how the location of the *x*-intercepts and the axis of symmetry are connected. The focus is for students to recognize that, due to symmetry, the axis of symmetry is exactly half-way between the two *x*-intercepts.)

  Be careful as students sometimes incorrectly state the equation of the axis of symmetry as a numerical value rather than an equation. It is important for them to recognize that, when they are graphing quadratic functions, the line of reflection is a vertical line. For the quadratic function $y = x^2 - 6x + 13$, students may mistakenly identify the equation of the axis of symmetry as 3 rather than $x = 3$.

- The characteristics of the quadratic function $f(x) = a(x - r)(x - s)$ should be developed through an investigation in which the parameters a, r, and s are manipulated individually. Technology like graphing calculators, or other suitable graphing software, should be used. Students could work as partners changing the parameters and making conjectures about the impact of those changes.

  Note: Students are expected to notice, as they explore the effects of parameter changes, that a impacts direction and shape of the parabola, r and s will be the values of the x-intercepts of the graph.

  It would be important for you to discuss with students why (r, 0) and (s, 0) are the x-intercepts of the quadratic when it is written in its factored form.

- The method of partial factoring can be used to determine the equation of the line of symmetry for a non-factorable quadratic. The quadratic function $f(x) = 2x^2 - 4x + 7$ is not factorable, but if we consider just factoring the first two terms, $f(x) = 2x^2 - 4x + 7$ and $f(x) = 2x(x - 2) + 7$, we can see that $f(0) = 7$ and $f(2) = 7$, since both these points have the same y-coordinate the line of symmetry must be half way between $x = 0$ and $x = 2$. The equation of the axis of symmetry, therefore, is $x = 1$.

  When using this method, remind students that they are not determining x-intercepts here as they would if they were factoring the quadratic function. A graphical representation may help students visualize the information this method provides.

  It is important for students to recognize that given any two points with the same y-coordinates in a table of values, the equation of the axis of symmetry can be determined by averaging the x-coordinates of the points.

- Students should also be given an opportunity to analyze several tables of values and their corresponding graphs. Use the following questions to help students make a connection between the axis of symmetry and x-intercepts:
  (a) How are the x-intercepts determined using a table of values?
  (b) Can you identify the vertex from the table? Explain.
  (c) How could you determine the x-coordinate of the vertex just from the x-intercepts?
  (d) What happens if the x-intercepts are not evident in a table of values? What strategy can be used to find the axis of symmetry?

- The characteristics of the quadratic function, $f(x) = a(x - h)^2 + k$ should be developed through an investigation in which the parameters a, h, and k are manipulated individually. Technology like graphing calculators, or other suitable graphing software, should be used.

  Note: Students are expected to notice, as they explore the effects of parameter changes, that a impacts direction of opening and the shape of the parabola, h is the x-coordinate of the vertex and k is the y-coordinate of the vertex.

- *Matching graphs task:* Working in groups of two, give one student the graph of a quadratic function. Ask them to turn to their partner and describe, using the characteristics of the quadratic function, the graph they see. The other student will then draw the graph based on the description from the first student. Both students will then check to see if the graphs match.

- To assist the students in making the connection between the factored form of a quadratic function and the x-intercepts of its graph, ask students to graph the following functions using technology and to determine the x-intercepts:
  (a) $y = x^2 - 6x + 9$
  (b) $y = 2x^2 - 4x - 6$
  (c) $y = (x - 2)(x + 3)$
  (d) $y = (x + 1)(x - 4)$

  Lead a discussion focussing on the following questions:
  (a) Can you determine the x-intercepts by looking at a quadratic function? Explain.
  (b) Which form of the quadratic function did you find the easiest to use when determining the x-intercepts?
  (c) What is the connection between the factors and the x-intercepts.
  (d) What is the value of the y-coordinate at the point where the graph crosses the x-axis?
  (e) What would happen if the factors of the quadratic function were identical?
  (f) How can you find the x-intercepts of a quadratic function without the graph or a table of values?

- Encourage students to think about what the graph of the quadratic function should look like before actually sketching the graph. They can discuss features such as the direction of the opening, whether the vertex is a maximum or minimum point, what the value of the y-intercept is, and how many x-intercepts there should be. Consider, for example, the equation $y = -x^2 + 5x + 4$. Since the parabola opens downward (a value is negative), the vertex is a maximum point. The y-intercept (c value) is 4, which is above the x-axis. This results in two x-intercepts.

- Ask students what the important characteristics are to consider when sketching the graph of a quadratic function. Students may initially use a table of values to draw their sketch. Encourage them to use the other methods addressed throughout this outcome. To draw a reasonably accurate sketch, students should plot the vertex and at least two other points on the graph.

- Each form of a quadratic has its own characteristics, and its own benefits.
  - If the quadratic is written is standard form, students can determine the y-intercept and the direction of the opening of the parabola directly from the equation.
  - If the equation is written in factored form, students can determine the x-intercepts of the graph and the direction of the opening of the parabola.
  - If the equation is written in vertex form, students can determine the vertex and the direction of the opening.

- Students should be given a quadratic in standard form and asked to put it in factored form so that they could determine the x-intercepts of its graph without using technology. A common student error occurs when students factor a quadratic and mistakenly ignore the signs when determining the x-intercepts. For example, when students factor $y = x^2 + 2x - 8$ as $y = (x + 4)(x - 2)$, they often state the x-intercepts are 4 and –2. Reinforce to students that they are solving equations $x + 4 = 0$ and $x - 2 = 0$ to obtain the x-intercepts –4 and 2.

- Promote student discussion around the following task:
  - Sketch a parabola that passes through (–4, 0) and (3, 0).
  - Make three more parabolas that are different from the first one, but still have x-intercepts –4 and 3.
  - How many parabolas do you think are possible? Explain.

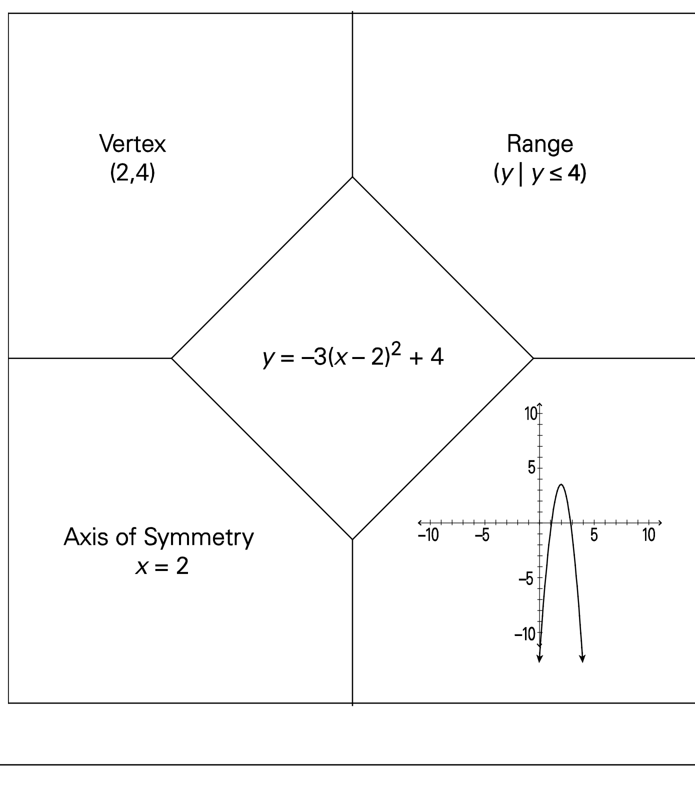
The goal is for students to recognize that a family of parabolas are possible when the *x*-intercepts are given. When provided with an extra point, however, students can narrow down the exact formula for the quadratic equation. In order to determine the multiplier *a* in the factored form $y = a(x - r)(x - s)$, students will choose any point on the parabola and use substitution. They should compare their answers with the class to confirm that the points they chose resulted in the same quadratic function, and then have a discussion about whether some points are easier to work with than others. Using the distributive property, students may write the equation from factored form to standard form.

- *Find your partner:* Provide half the students with cards that show graphs labelled with the vertex or x-intercepts. The other half of the class will be given cards that have the corresponding quadratic functions in a variety of forms. Students move around the classroom trying to locate their matching card.

- *Frayer Model Puzzle:* This task will involve a group of three students. Provide students with a completed Frayer model cut into five puzzle pieces. To make this more challenging, keep some of the puzzle pieces blank. The titles would be Graph, Table of Values, Equation, Characteristics, and Domain and Range. Each student is randomly given one piece of the quadratic function. He or she should move about the classroom locating the corresponding pieces to put the puzzle back together and complete the Frayer model. Once the group is formed, they will have three pieces of the puzzle and as a group will have to complete the remaining two pieces. Teachers can first use this model with standard form and then apply it to the vertex and factored form of a quadratic.

- Students can work in pairs to complete the following quadratic puzzle, investigating the characteristics and graphs of various quadratic functions of the form $y = a(x - h)^2 + k$. They should work with 20 puzzle pieces (four complete puzzles consisting of a function and four related characteristics) to correctly match the characteristics with each function.

  A sample is shown below.



- Students should be provided with examples where characteristics of quadratic functions are applied to conceptual problems. Consider the following example:
  - A ball is thrown from an initial height of 1 m and follows a parabolic path. After 2 seconds, the ball reaches a maximum height of 21 m. Using algebra, determine the quadratic function that models the path followed by the ball, and use it to determine the approximate height of the ball at 3 seconds.

  Encourage students to discuss the following:
  - How is the shape of the graph connected to the situation?
  - What do the coordinates of the vertex represent?
  - What do the x- and y-intercepts represent?
  - Why isn't the domain all real numbers in this situation?

- Students should be provided with a variety of contextual problems that place restrictions on the domain and range. To help them distinguish between situations where restrictions on the domain are necessary, students could be exposed to questions such as the following:
  - State the domain and range for the function $f(x) = -0.15x^2 + 6x$.
  - Now suppose the function $h(t) = -0.15t^2 + 6t$ represents the height of a ball, in metres, above the ground as a function of time, in seconds. What impact does this context have on domain and range?

- *Fishing for Quadratic Functions:* Fill two bags with intercepts (in coordinate form) written on pieces of paper that are shaped like fish.
  - **Version A**
    One bag would contain only x-intercepts, and the other bag would contain only y-intercepts. Ask students to fish out two x-intercepts and one y-intercept, and then determine the quadratic function that passes through the three points. For this version of the game, students would write the quadratic in its factored form. (**Note:** It will be necessary to have twice as many x-intercepts as y-intercepts.)
  - **Version B**
    One bag would contain only vertices and the other bag contain a point. For this version of the game, students would write the quadratic function in its vertex form.

- Ask students to play Parabola Math. Provide students with a deck of cards containing the graph of the parabola, the function of the parabola, the vertex of the parabola, and the equation of the axis of symmetry. Each characteristic could be on different colour paper. Students should work in groups to match the various components of the parabola.

- You can challenge students to use the method of partial factoring to show that $x = -\frac{-b}{2a}$ is the x-coordinate of the vertex for the general quadratic $y = ax^2 + bx + c$.

  After you have done this, students may use the formula $x = -\frac{-b}{2a}$ to find the x-coordinate of the vertex for quadratic functions of the form $y = ax^2 + bx + c$. This value can then be used to determine the equation of the axis of symmetry.

- Teachers should promote discussion when students are graphing quadratic functions by asking the following questions:

- Why is the domain the set of all real numbers when only some points are plotted from the table of values?
- How is the range related to the direction of the opening?

- As a review, students can play the following game. Each group should be given a pair of dice (or they can create their own).
  - For $y = ax^2 + bx + c$: On the first die, two of the sides will be labelled a, two sides will be labelled b, and two labelled c. On the second die, there will be various rational numbers. Students should roll the dice and state the effect on the graph of $y = x^2$ of changing the indicated parameter to the given number.
  - For $y = a(x - h)^2 + k$: On the first die, two of the sides will be labelled a, two sides will be labelled h, and two labelled k. On the second die, there will be various rational numbers. Students should roll the dice and state the effect on the graph of $y = x^2$ of changing the indicated parameter to the given number.
  - For $y = a(x - r)(x - s)$: On the first die, two of the sides will be labelled a, two sides will be labelled r, and two labelled s. On the second die, there will be various rational numbers. Students should roll the dice and state the effect on the graph of $y = x^2$ of changing the indicated parameter to the given number.

- Once the students are familiar with the quadratic formula, then teachers may wish to revisit the ambiguous case and the use of the law of cosines.

  Example

  Two sides of a triangle are 10 inches and 16 inches in length. The angle opposite the 10 inch side is 30°. Determine the length of the other side.

Law of Cosines (using quadratics)	
$10^2 = 16^2 + x^2 - 2(16)(x)\cos 30°$ $100 = 256 + x^2 - 27.71x$ $0 = x^2 - 27.71x + 156$ Using the quadratic formula $x = \dfrac{-(-27.71) \pm \sqrt{(-27.71)^2 - 4(1)(156)}}{2(1)}$	$x = \dfrac{27.71 \pm \sqrt{(144)}}{2}$ $x = \dfrac{27.71 \pm 12}{2}$ $x = 19.855$ $x = 7.855$ Therefore, there are two possible triangles that satisfy the listed conditions.

SUGGESTED MODELS AND MANIPULATIVES

- area model
- alge-tiles

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- axis of symmetry
- break-even point
- degree of polynomial
- factored form
- parabola
- parameter
- partial factoring
- quadratic
- roots
- standard form
- vertex form
- vertex
- *x*-intercepts
- *y*-intercept
- zeros

Resources/Notes

Internet

- National Library of Virtual Manipulatives (Utah State University 2010)
  http://nlvm.usu.edu/en/nav/vlibrary.html
  This website allows students to review factoring using algebra tiles.

- Algebra-Class.com, "Solving a Quadratic Equation" (Karin Hutchinson 2013)
  www.algebra-class.com/quadratic-equation.html
  This website has good examples for solving quadratic equations using different methods and graphing quadratic functions.
- Graphic Calculator (Houghton Mifflin Harcourt 2013)
  http://my.hrw.com/math06_07/nsmedia/tools/Graph_Calculator/graphCalc.html
  This website contains an online graphing calculator.

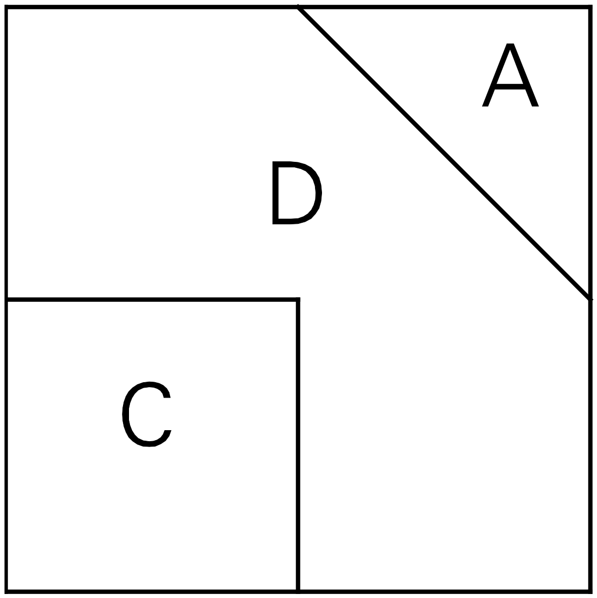
Print

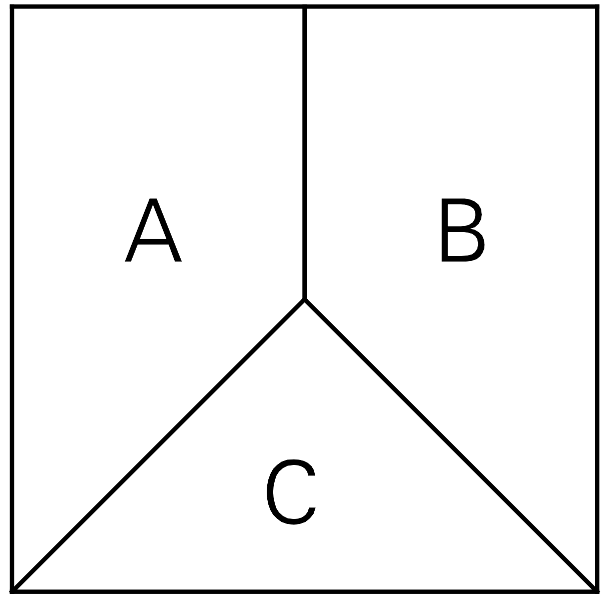
- *Foundations of Mathematics 11* (Canavan-McGrath et al. 2011, Nelson Canada)
  - Sections 7.1–7.8, pp. 358–438

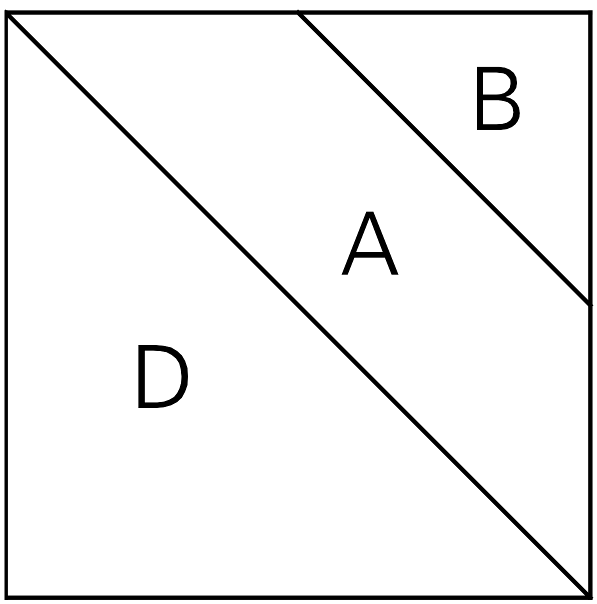
Notes

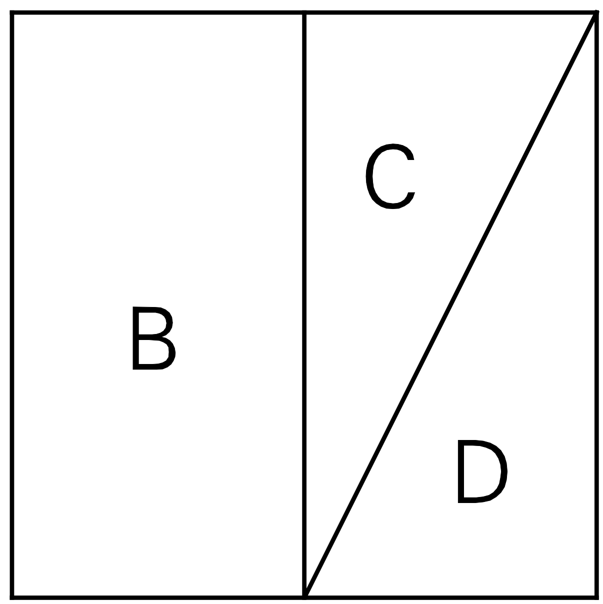
Appendix

Square Puzzles

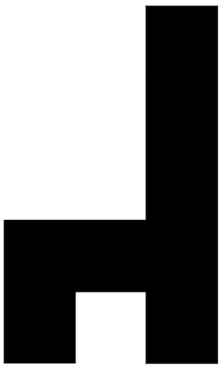




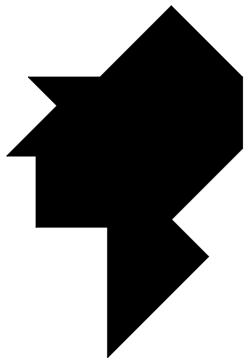




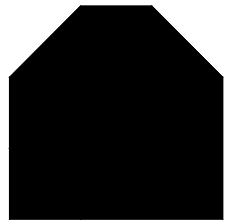
Tangrams



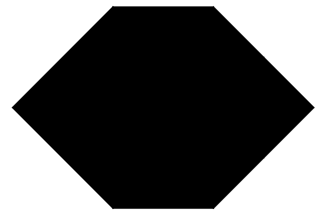
Chair



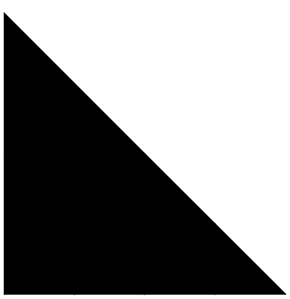
Head



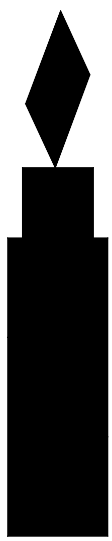
Barn



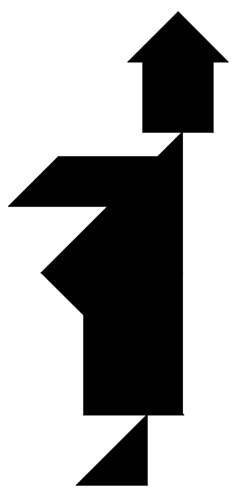
Hexagon



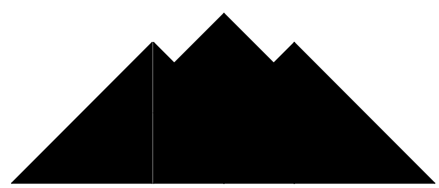
Right Triangle



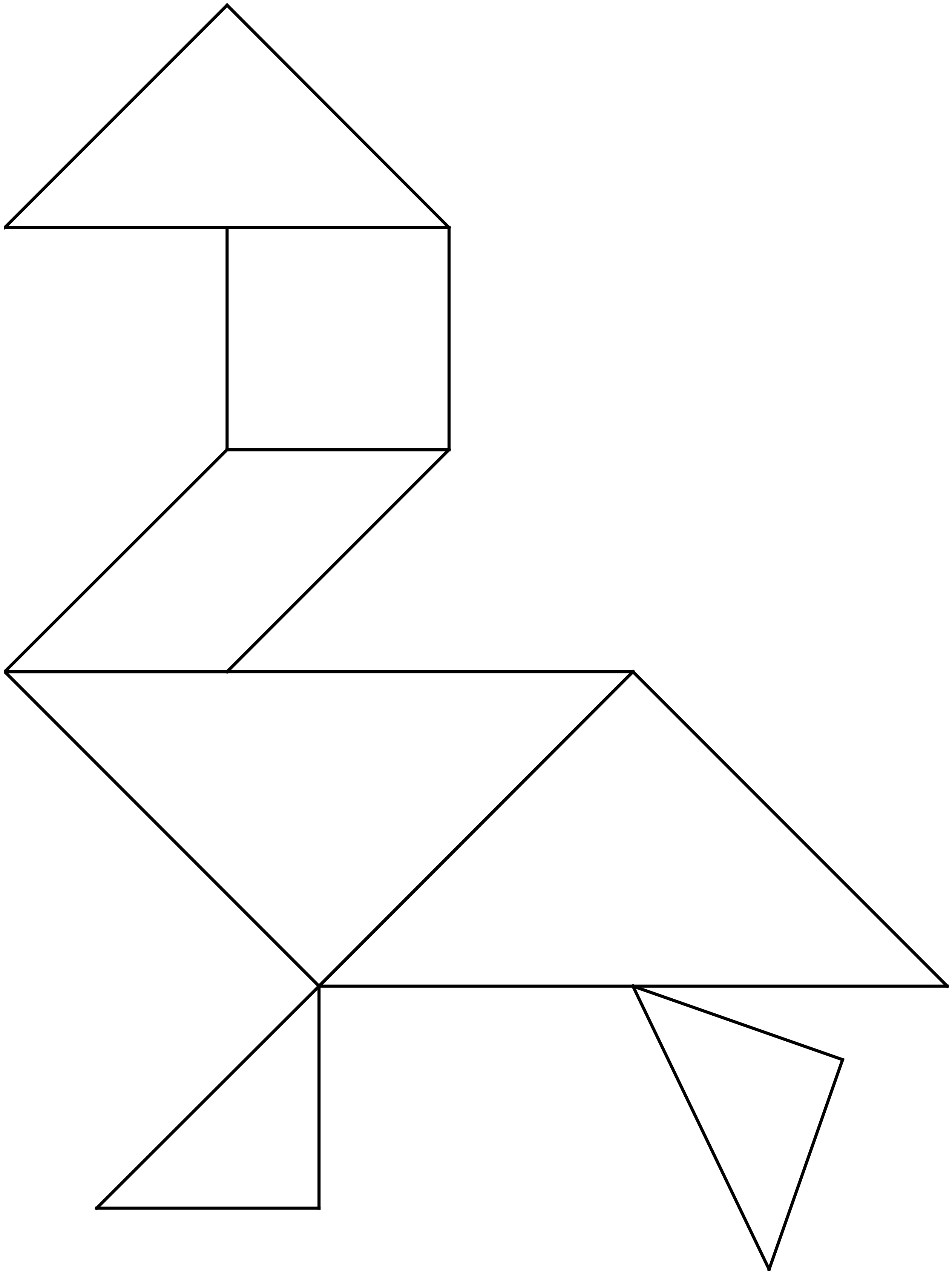
Candle

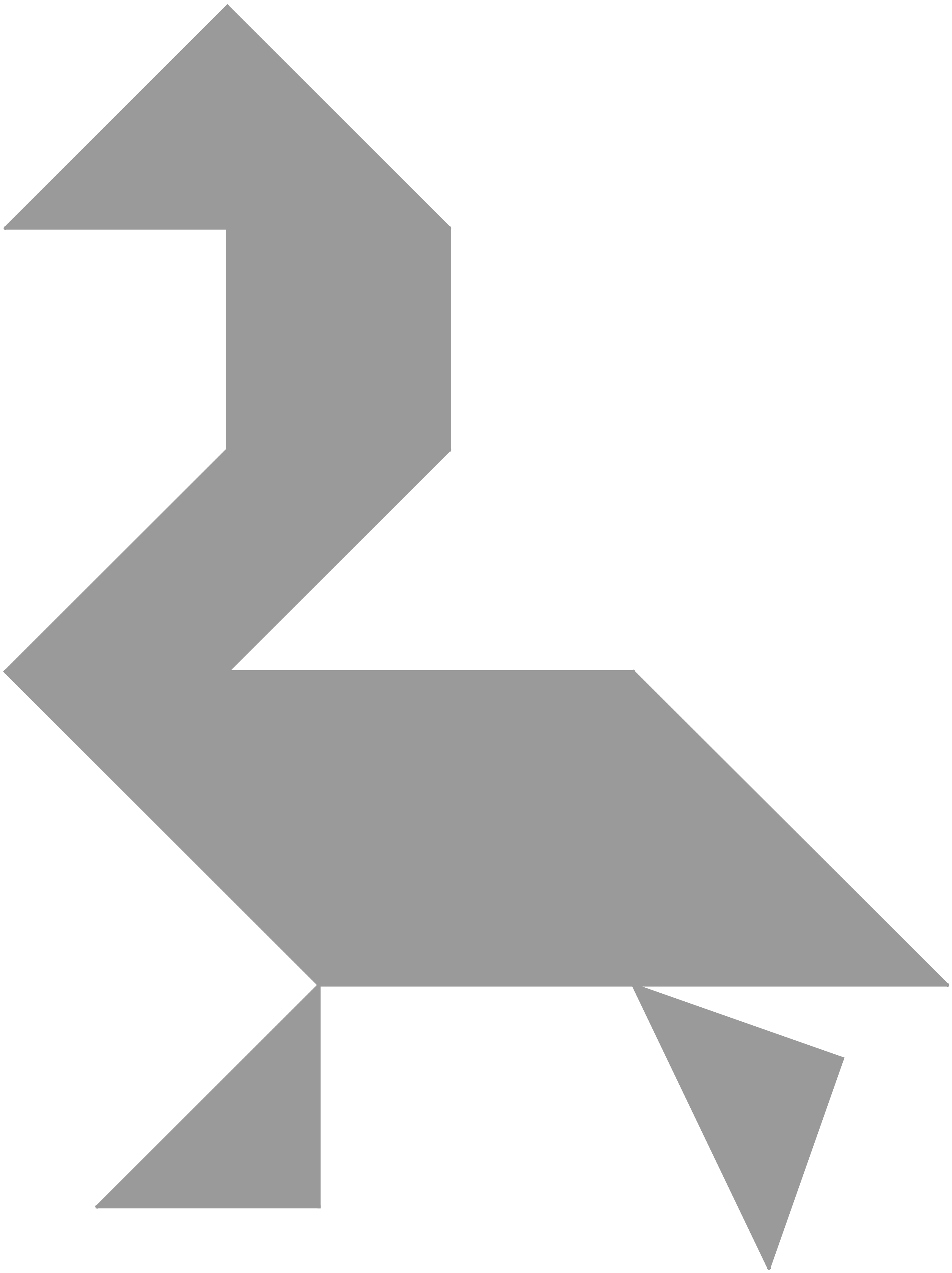


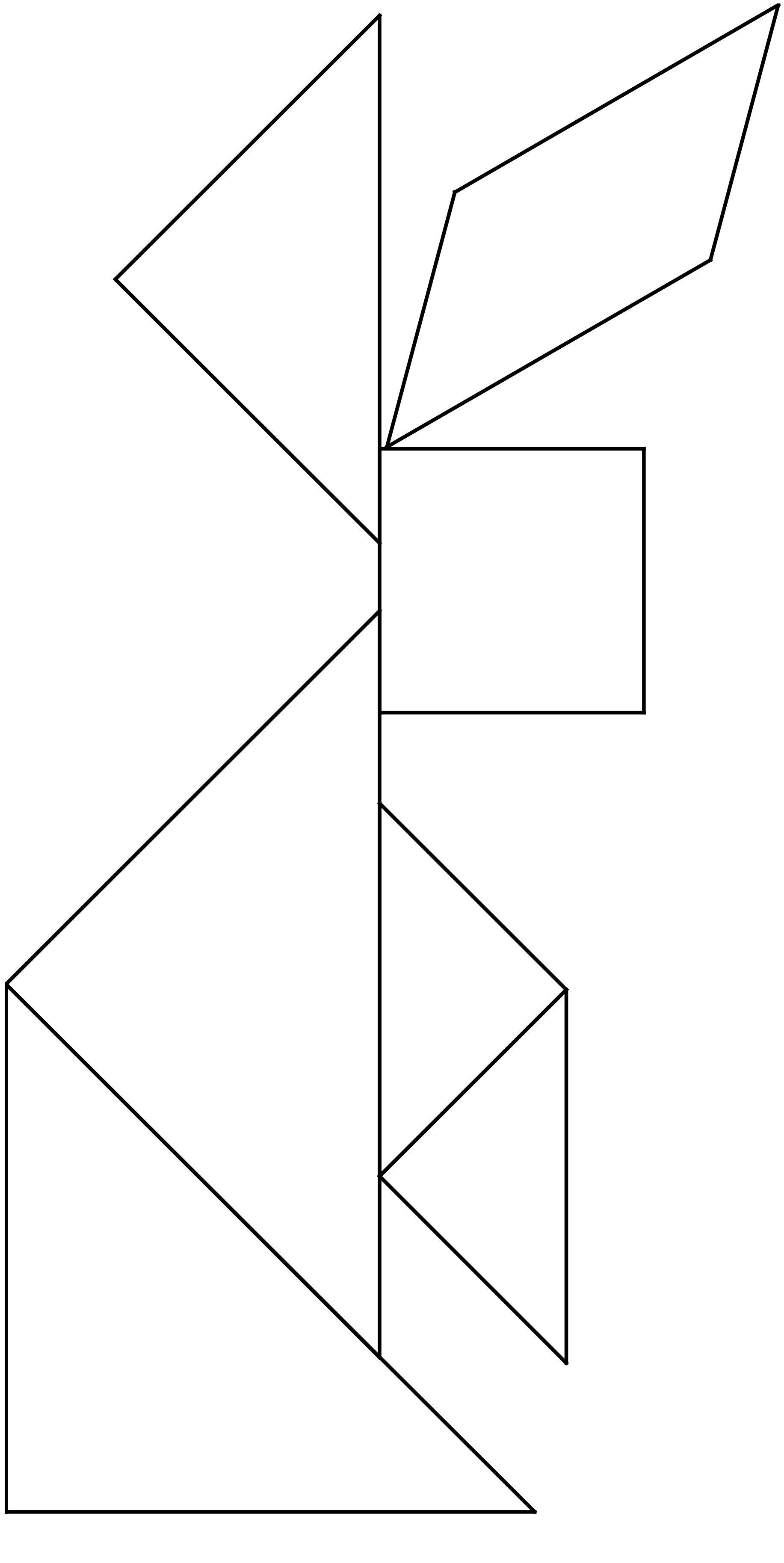
Sleepwalker

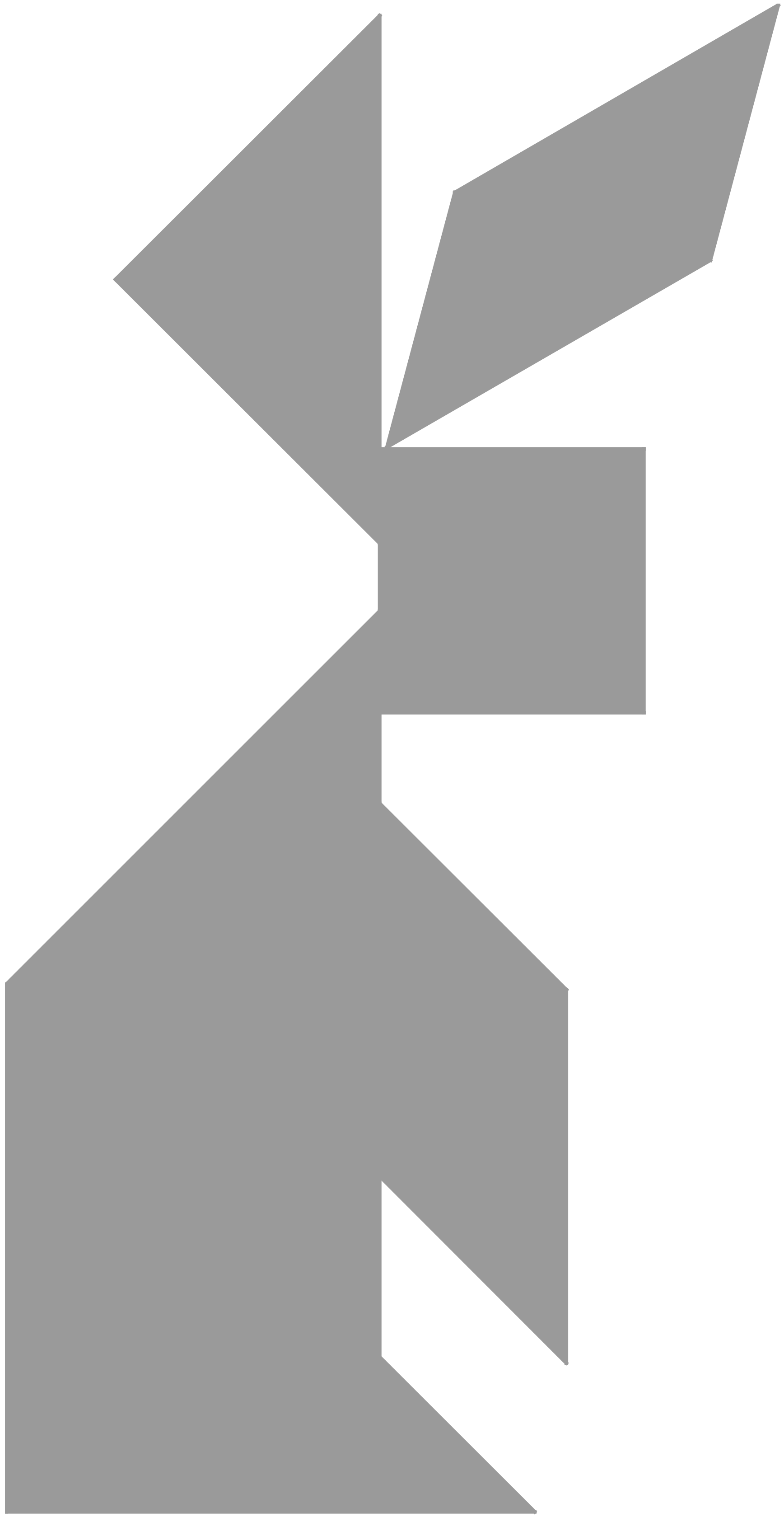


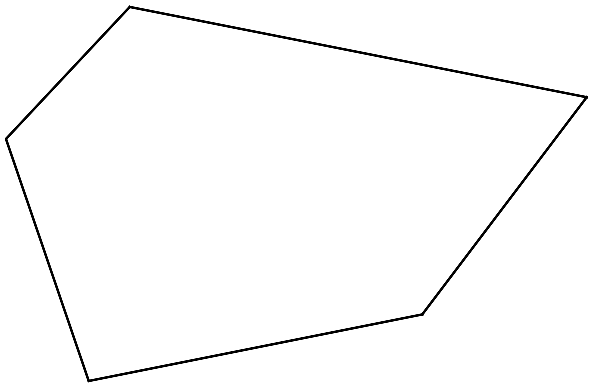
Mountain

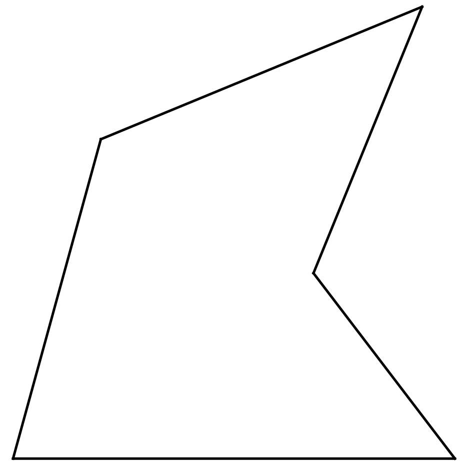


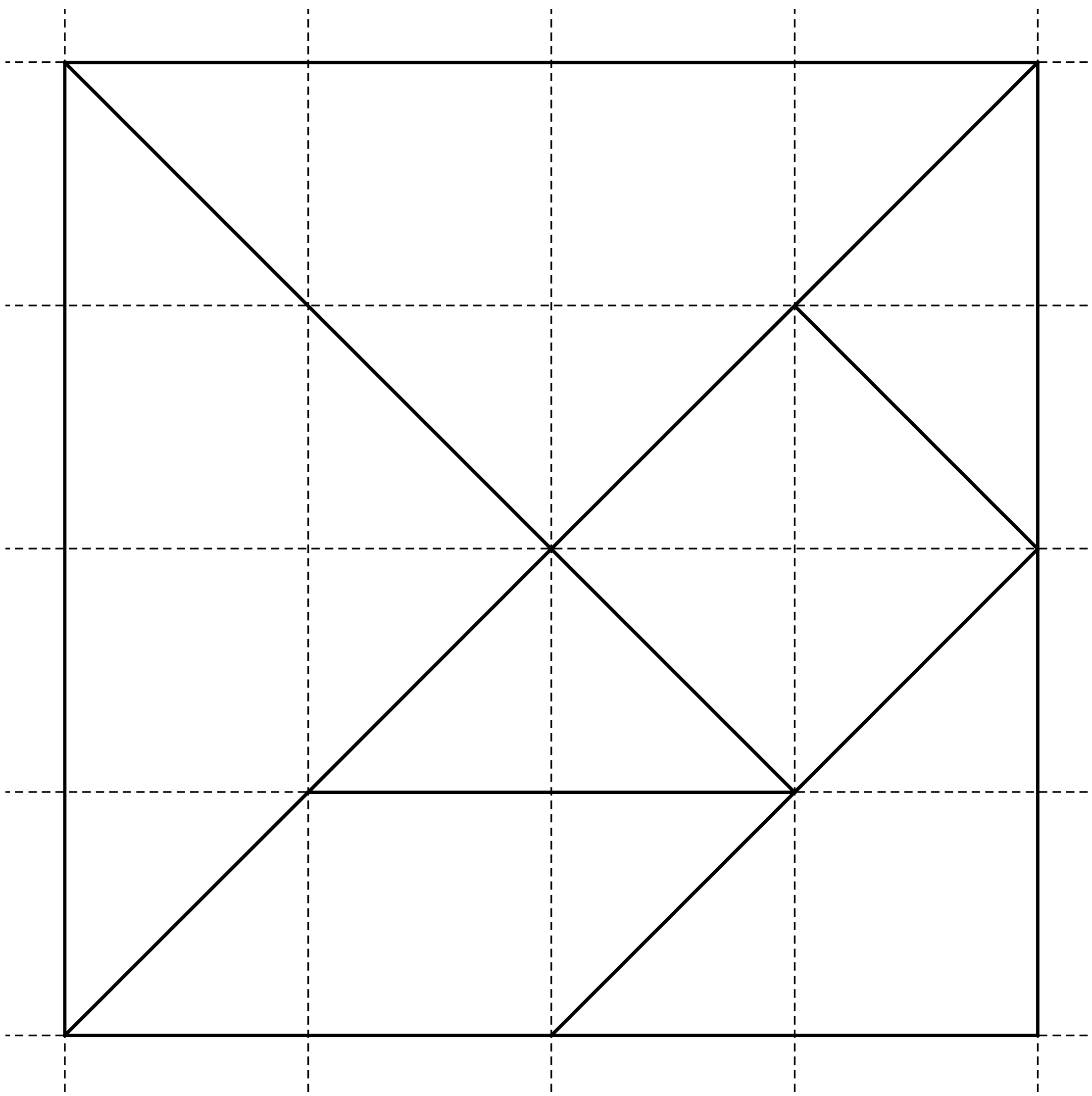


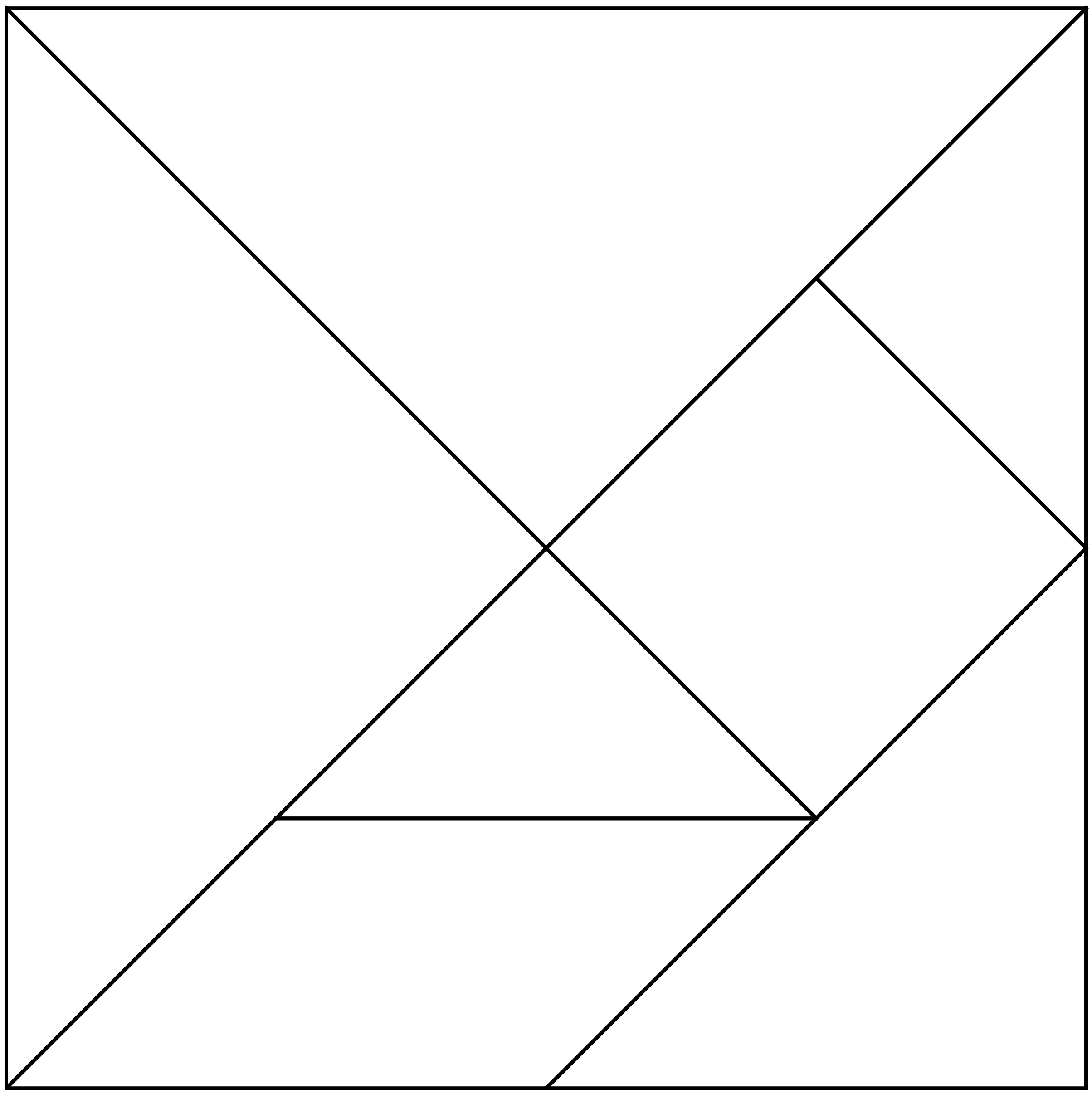












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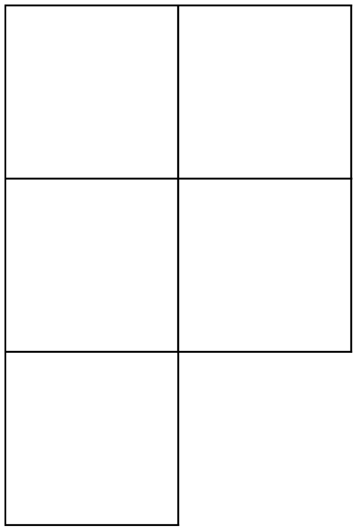

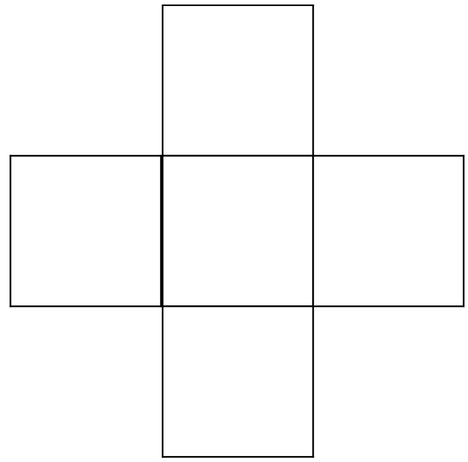

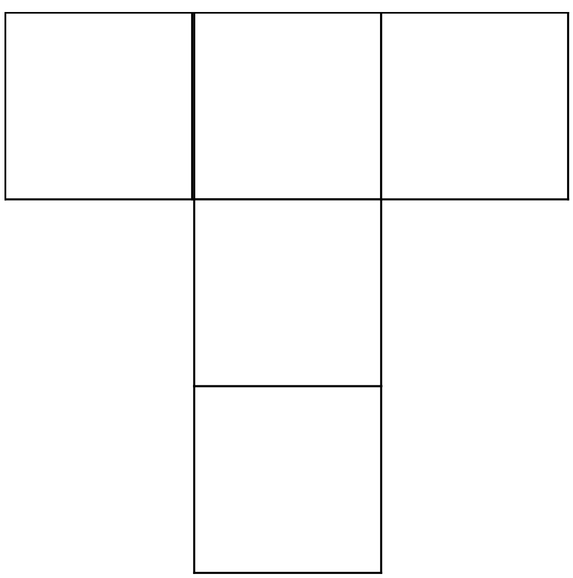

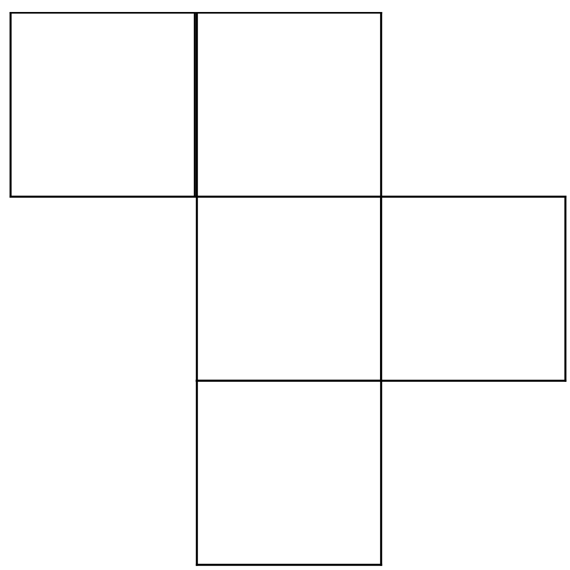

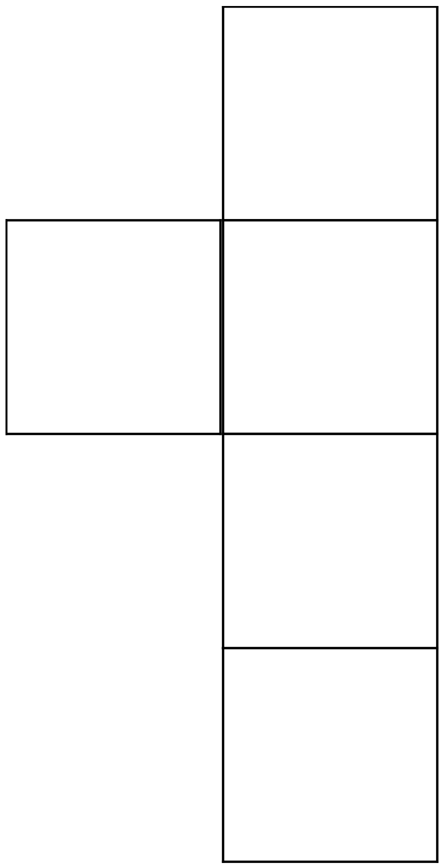

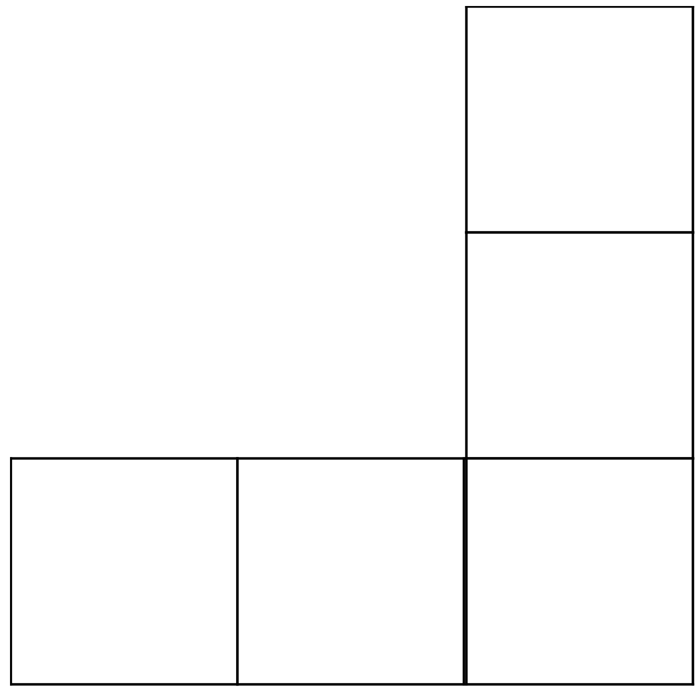

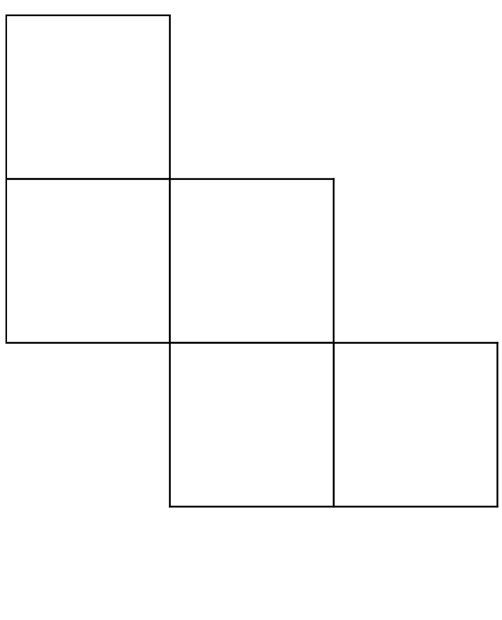

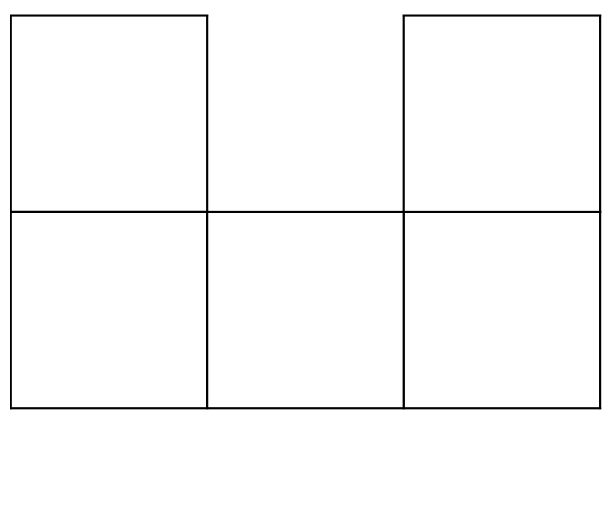

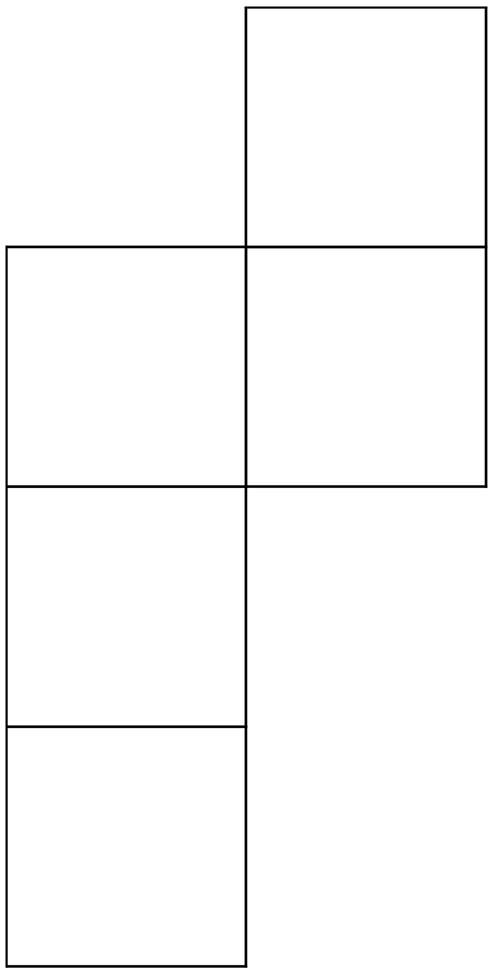

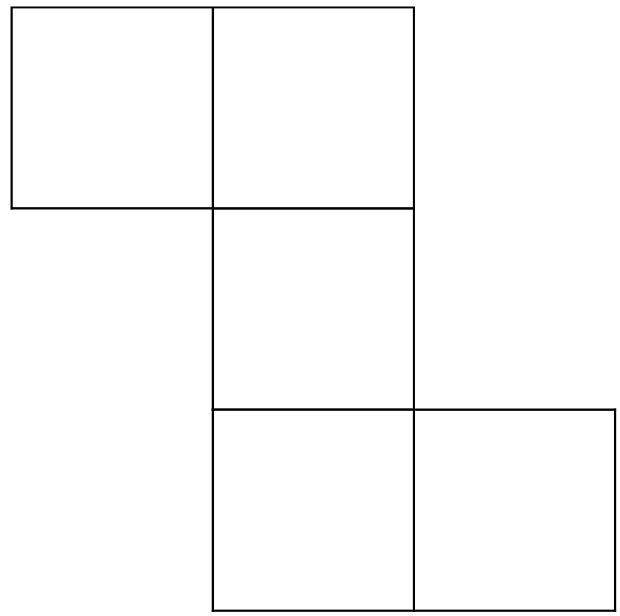

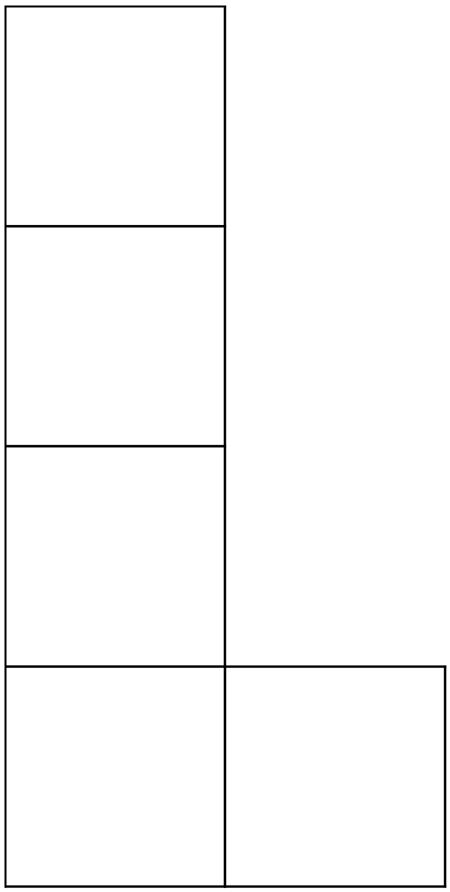

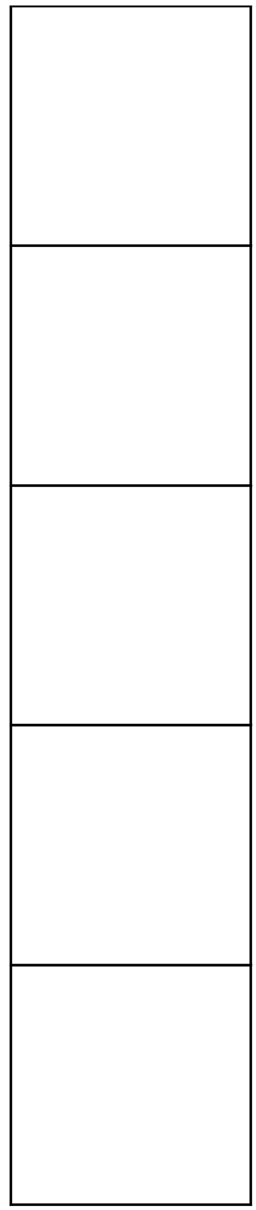

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

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