Mathematics 12

Guide

NOVA SCOTIA
NOUVELLE-ÉCOSSE

2015

Mathematics 12

Prepared by the Department of Education and Early Childhood Development

This is the most recent version of the current curriculum materials as used by teachers in Nova Scotia.

Mathematics 12

Implementation Draft
June 2015

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Contents

Introduction

Background and Rationale

Mathematics curriculum is shaped by a vision that fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society. It is essential that the mathematics curriculum reflect current research in mathematics instruction. To achieve this goal, Western and Northern Canadian Protocol's (WNCP) *The Common Curriculum Framework for Grades 10–12 Mathematics* (2008) has been adopted as the basis for the new mathematics curriculum in Nova Scotia.

The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan, and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and performance indicators agreed upon by the seven jurisdictions. The outcomes and performance indicators have been adapted for Nova Scotia. This document is based on both national and international research by the WNCP and the National Council of Teachers of Mathematics (NCTM).

There is an emphasis in the Nova Scotia curriculum on particular key concepts at each grade that will result in greater depth of understanding and, ultimately, stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

Purpose

This document provides sets of outcomes and performance indicators to be used as a mandated common base for defining mathematics curriculum expectations. This common base should result in consistent student outcomes in mathematics within the province of Nova Scotia. It should also enable easier transfer for students moving within the province or from any jurisdiction that has adopted the WNCP framework. This document is intended to clearly communicate to all education partners across the province the high expectations for students' mathematical learning.

Program Design and Components

Pathways

The *Common Curriculum Framework for Grades 10–12 Mathematics* (WNCP 2008), on which the Nova Scotia Mathematics 10–12 curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics* (WNCP 2006). In Nova Scotia, four pathways are available: Mathematics Essentials, Mathematics at Work, Mathematics, and Pre-calculus.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all four pathways are to provide prerequisite attitudes, knowledge, skills, and understandings for specific post-secondary programs or direct entry into the work force. All four pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents, and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour, and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of the Mathematics Essentials courses was designed in Nova Scotia to fill a specific need for Nova Scotia students. The content of each of the Mathematics at Work, Mathematics, and Pre-calculus pathways has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* (System Improvement Group, Alberta Education 2006) and on consultations with mathematics teachers.

MATHEMATICS ESSENTIALS (GRADUATION)

This pathway is designed to provide students with the development of the skills and understandings required in the workplace, as well as those required for everyday life at home and in the community. Students will become better equipped to deal with mathematics in the real world and will become more confident in their mathematical abilities.

Mathematics at Work (Graduation)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, and statistics and probability.

Mathematics (Academic)

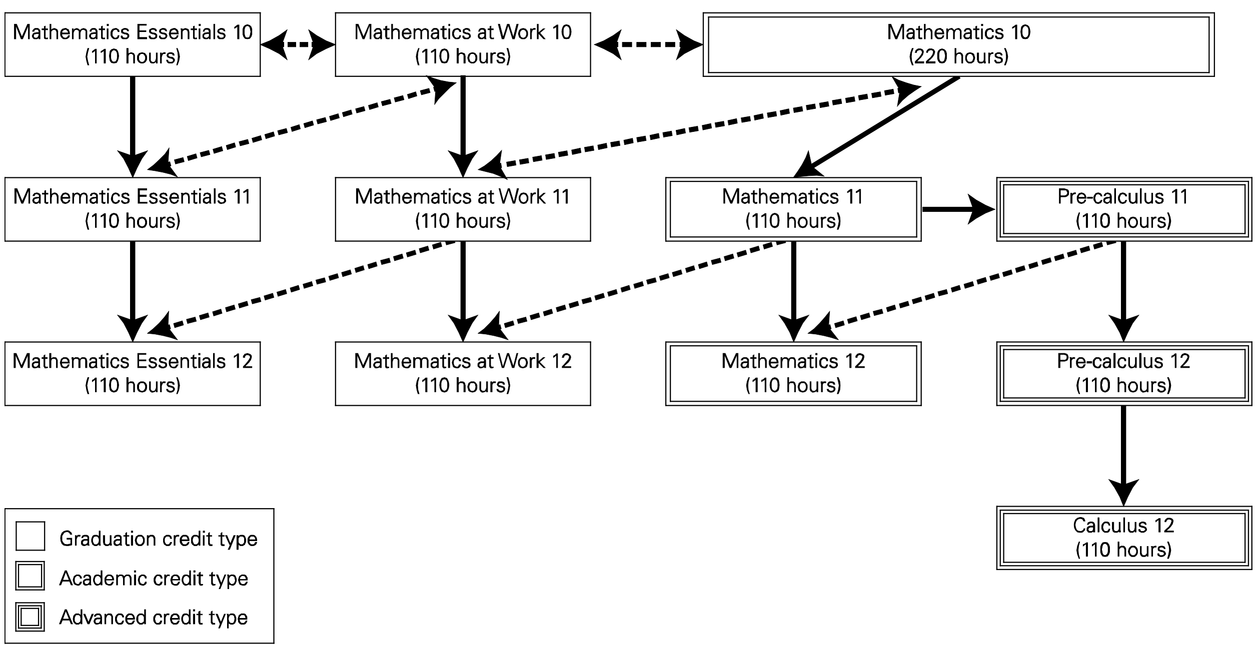
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that require an academic or pre-calculus mathematics credit. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, and statistics and probability. **Note:** After completion of Mathematics 11, students have the choice of an academic or pre-calculus pathway.

Pre-calculus (Advanced)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Topics include algebra and number, measurement, relations and functions, trigonometry, and permutations, combinations, and binomial theorem.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.



Instructional Focus

Each pathway in senior high mathematics pathways is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful.

Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems, and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Assessment

Ongoing assessment for learning is essential to effective teaching and learning. Research has shown that assessment for learning (formative assessment) practices produce significant and often substantial learning gains, close achievement gaps, and build students' ability to learn new skills (Black & Wiliam 1998; OECD 2006). Student involvement in assessment promotes learning. Timely and effective teacher feedback and student self-assessment allow students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes

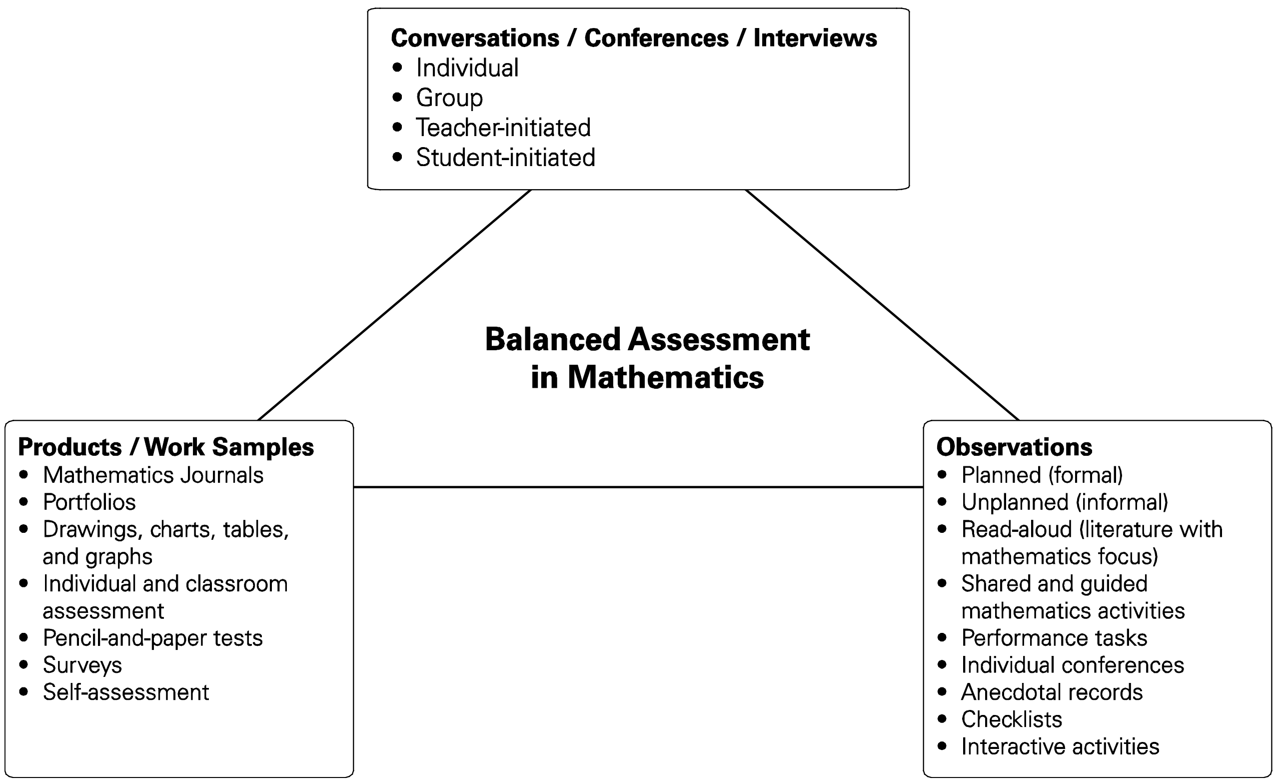
- providing clear goals, targets, and learning outcomes
- using exemplars, rubrics, and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning

(Davies 2000)

Assessment for learning practices act as the scaffolding for learning, which only then can be measured through assessment of learning (summative assessment). Assessment of learning tracks student progress, informs instructional programming, and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning, and produce achievement gains.

Assessment of student learning should

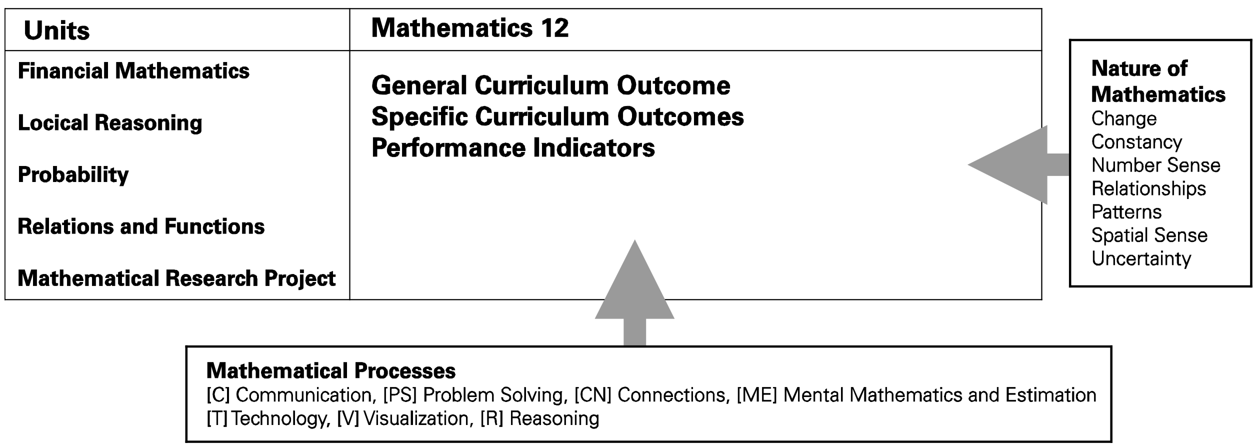
- align with curriculum outcomes
- clearly define criteria for success
- make explicit the expectations for students' performance
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction



Outcomes

Conceptual Framework for Mathematics 10–12

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



(Adapted with permission from Western and Northern Canadian Protocol, *The Common Curriculum Framework for K–9 Mathematics*, p. 5. All rights reserved.)

Structure of the Mathematics 12 Curriculum

Units

Mathematics 12 comprises five units:

- Financial Mathematics (FM) (15–20 hours)
- Logical Reasoning (LR) (20–25 hours)
- Probability (P) (15–20 hours)
- Relations and Functions (RF) (35–40 hours)
- Mathematical Research Project (MRP) (10–15 hours)

Outcomes and Performance Indicators

The Nova Scotia curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes, and performance indicators.

General Curriculum Outcomes (GCOs)

General curriculum outcomes are overarching statements about what students are expected to learn in each strand/sub-strand. The GCO for each strand/sub-strand is the same throughout the pathway.

Financial Mathematics (FM)

Students will be expected to develop number sense in financial applications.

Logical Reasoning (LR)

Students will be expected to develop logical reasoning.

Probability (P)

Students will be expected to develop critical-thinking skills related to uncertainty.

Relations and Functions (RF)

Students will be expected to develop algebraic and graphical reasoning through the study of relations.

Mathematical Research Project (MRP)

Students will be expected to develop an appreciation of the role of mathematics in society.

Specific Curriculum Outcomes (SCOs) and Performance Indicators

Specific curriculum outcomes are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as expected for a given grade.

Performance indicators are samples of how students may demonstrate their performance of the goals of a specific curriculum outcome. The range of samples provided is meant to reflect the scope of the SCO. In the SCOs, the word **including** indicates that any ensuing items *must* be addressed to fully achieve the learning outcome. The phrase **such as** indicates that the ensuing items are provided for clarification only and are **not** requirements that must be addressed to fully achieve the learning outcome. The word **and** used in an outcome indicates that both ideas must be addressed to achieve the learning outcome, although not necessarily at the same time or in the same question.

Financial Mathematics (FM)

FM01 Students will be expected to solve problems that involve compound interest in financial decision making.

Performance Indicators

FM01.01 Explain the advantages and disadvantages of compound interest and simple interest.
FM01.02 Identify situations that involve compound interest.
FM01.03 Graph and compare, in a given situation, the total interest paid or earned for different compounding periods.
FM01.04 Determine, given the principal, interest rate, and number of compounding periods, the total interest of a loan.
FM01.05 Graph and describe the effects of changing the value of one of the variables in a situation that involves compound interest.
FM01.06 Determine, using technology, the total cost of a loan under a variety of conditions (e.g., different amortization periods, interest rates, compounding periods, and terms).
FM01.07 Compare and explain, using technology, different credit options that involve compound interest, including bank and store credit cards and special promotions.
FM01.08 Solve a contextual problem that involves compound interest.

FM02 Students will be expected to analyze costs and benefits of renting, leasing and buying.

Performance Indicators

FM02.01 Identify and describe examples of assets that appreciate or depreciate.
FM02.02 Compare, using examples, renting, leasing and buying.
FM02.03 Justify, for a specific set of circumstances, if renting, buying, or leasing would be advantageous.
FM02.04 Solve a problem involving renting, leasing, or buying that requires the manipulation of a formula.
FM02.05 Solve, using technology, a contextual problem that involves cost-and-benefit analysis.

FM03 Students will be expected to analyze an investment portfolio in terms of interest rate, rate of return, and total return.

Performance Indicators

FM03.01 Determine and compare the strengths and weaknesses of two or more portfolios.
FM03.02 Determine, using technology, the total value of an investment when there are regular contributions to the principal.
FM03.03 Graph and compare the total value of an investment with and without regular contributions.
FM03.04 Apply the Rule of 72 to solve investment problems, and explain the limitations of the rule.
FM03.05 Determine, using technology, possible investment strategies to achieve a financial goal.
FM03.06 Explain the advantages and disadvantages of long-term and short-term investment options.
FM03.07 Explain, using examples, why smaller investments over a longer term may be better than larger investments over a shorter term.
FM03.08 Solve an investment problem.

Logical Reasoning (LR)

LR01 Students will be expected to analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.

Performance Indicators

(It is intended that this outcome be integrated throughout the course by using games and puzzles such as chess, sudoku, Nim, logic puzzles, magic squares, Kakuro, and cribbage.)

LR01.01 Determine, explain, and verify a strategy to solve a puzzle or to win a game; for example,

- guess and check
- look for a pattern
- make a systematic list
- draw or model
- eliminate possibilities
- simplify the original problem
- work backward
- develop alternative approaches

LR01.02 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.

LR01.03 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

LR02 Students will be expected to solve problems that involve the application of set theory.

Performance Indicators

LR02.01 Provide examples of the empty set, disjoint sets, subsets, and universal sets in context, and explain the reasoning.

LR02.02 Organize information such as collected data and number properties using graphic organizers, and explain the reasoning.

LR02.03 Explain what a specified region in a Venn diagram represents, using connecting words (and, or, not) or set notation.

LR02.04 Determine the elements in the complement, the intersection, or the union of two sets.

LR02.05 Explain how set theory is used in applications such as Internet searches, database queries, data analysis, games, and puzzles.

LR02.06 Identify and correct errors in a given solution to a problem that involves sets.

LR02.07 Solve a contextual problem that involves sets, and record the solution, using set notation.

LR03 Students will be expected to solve problems that involve conditional statements.

Performance Indicators

LR03.01 Analyze an “if-then” statement, make a conclusion, and explain the reasoning.

LR03.02 Make and justify a decision, using “what if?” questions, in contexts such as probability, finance, sports, games, or puzzles, with or without technology.

LR03.03 Determine the converse, inverse, and contrapositive of an “if-then” statement; determine its veracity; and, if it is false, provide a counterexample.

LR03.04 Demonstrate, using examples, that the veracity of any statement does not imply the veracity of its converse or inverse.

LR03.05 Demonstrate, using examples, that the veracity of any statement does imply the veracity of its contrapositive.

LR03.06 Identify and describe contexts in which a biconditional statement can be justified.
LR03.07 Analyze and summarize, using a graphic organizer such as a truth table or Venn diagram, the possible results of given logical arguments that involve biconditional, converse, inverse or contrapositive statements.

PROBABILITY (P)

P01 Students will be expected to interpret and assess the validity of odds and probability statements.

Performance Indicators

P01.01 Provide examples of statements of probability and odds found in fields such as media, biology, sports, medicine, sociology, and psychology.
P01.02 Explain, using examples, the relationship between odds (part-part) and probability (part-whole).
P01.03 Express odds as a probability and vice versa.
P01.04 Determine the probability of, or the odds for and against, an outcome in a situation.
P01.05 Explain, using examples, how decisions may be based on probability or odds and on subjective judgments.
P01.06 Solve a contextual problem that involves odds or probability.

P02 Students will be expected to solve problems that involve the probability of mutually exclusive and non-mutually exclusive events.

Performance Indicators

P02.01 Classify events as mutually exclusive or non-mutually exclusive, and explain the reasoning.
P02.02 Determine if two events are complementary, and explain the reasoning.
P02.03 Represent, using set notation or graphic organizers, mutually exclusive (including complementary) and non-mutually exclusive events.
P02.04 Solve a contextual problem that involves the probability of mutually exclusive or non-mutually exclusive events.
P02.05 Solve a contextual problem that involves the probability of complementary events.
P02.06 Create and solve a problem that involves mutually exclusive or non-mutually exclusive events.

P03 Students will be expected to solve problems that involve the probability of two events.

Performance Indicators

P03.01 Compare, using examples, dependent and independent events.
P03.02 Determine the probability of an event, given the occurrence of a previous event.
P03.03 Determine the probability of two dependent or two independent events.
P03.04 Create and solve a contextual problem that involves determining the probability of dependent or independent events.

P04 Students will be expected to solve problems that involve the fundamental counting principle.

Performance Indicators

P04.01 Represent and solve counting problems, using a graphic organizer.
P04.02 Generalize the fundamental counting principle, using inductive reasoning.
P04.03 Identify and explain assumptions made in solving a counting problem.

P04.04 Solve a contextual counting problem, using the fundamental counting principle, and explain the reasoning.

P05 Students will be expected to solve problems that involve permutations.

Performance Indicators

(It is intended that circular permutations not be included.)

P05.01 Represent the number of arrangements of *n* elements taken *n* at a time, using factorial notation.
P05.02 Determine, with or without technology, the value of a factorial.
P05.03 Simplify a numeric or algebraic fraction containing factorials in both the numerator and denominator.
P05.04 Solve an equation that involves factorials.
P05.05 Determine the number of permutations of *n* elements taken *r* at a time.
P05.06 Determine the number of permutations of *n* elements taken *n* at a time where some elements are not distinct.
P05.07 Explain, using examples, the effect on the total number of permutations of *n* elements when two or more elements are identical.
P05.08 Generalize strategies for determining the number of permutations of *n* elements taken *r* at a time.
P05.09 Solve a contextual problem that involves probability and permutations.

P06 Students will be expected to solve problems that involve combinations.

Performance Indicators

P06.01 Explain, using examples, why order is or is not important when solving problems that involve permutations or combinations.
P06.02 Determine the number of combinations of *n* elements taken *r* at a time.
P06.03 Generalize strategies for determining the number of combinations of *n* elements taken *r* at a time.
P06.04 Solve a contextual problem that involves combinations and probability.

Relations and Functions (RF)

RF01 Students will be expected to represent data, using polynomial functions (of degree ≤ 3), to solve problems.

Performance Indicators

RF01.01 Describe, orally and in written form, the characteristics of polynomial functions by analyzing their graphs.
RF01.02 Describe, orally and in written form, the characteristics of polynomial functions by analyzing their equations.
RF01.03 Match equations in a given set to their corresponding graphs.
RF01.04 Graph data and determine the polynomial function that best approximates the data.
RF01.05 Interpret the graph of a polynomial function that models a situation, and explain the reasoning.
RF01.06 Solve, using technology, a contextual problem that involves data that is best represented by graphs of polynomial functions, and explain the reasoning.

RF02 Students will be expected to represent data, using exponential and logarithmic functions, to solve problems.

Performance Indicators

RF02.01 Describe, orally and in written form, the characteristics of exponential or logarithmic functions by analyzing their graphs.

RF02.02 Describe, orally and in written form, the characteristics of exponential or logarithmic functions by analyzing their equations.

RF02.03 Match equations in a given set to their corresponding graphs.

RF02.04 Graph data and determine the exponential or logarithmic function that best approximates the data.

RF02.05 Interpret the graph of an exponential or logarithmic function that models a situation, and explain the reasoning.

RF02.06 Solve, using technology, a contextual problem that involves data that is best represented by graphs of exponential or logarithmic functions, and explain the reasoning.

RF03 Students will be expected to represent data, using sinusoidal functions, to solve problems.

Performance Indicators

RF03.01 Demonstrate an understanding of angles expressed in degrees and radians.

RF03.02 Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their graphs.

RF03.03 Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their equations.

RF03.04 Match equations in a given set to their corresponding graphs.

RF03.05 Graph data and determine the sinusoidal function that best approximates the data.

RF03.06 Interpret the graph of a sinusoidal function that models a situation, and explain the reasoning. Solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning.

Mathematics Research Project (MRP)

MRP01 Students will be expected to research and give a presentation on a topic that involves the application of mathematics.

Performance Indicators

MRP01.01 Collect primary or secondary data (statistical or informational) related to the topic.

MRP01.02 Assess the accuracy, reliability, and relevance of the primary or secondary data.

MRP01.03 Make a statement and justify the statement based on your data.

MRP01.04 Identify controversial issues, if any, and present multiple sides of the issues with supporting data.

MRP01.05 Organize and present the research project, with or without technology.

Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.

Students are expected to

- communicate in order to learn and express their understanding of mathematics (Communication [C])
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines (Connections [CN])
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation [ME])
- develop and apply new mathematical knowledge through problem solving (Problem Solving [PS])
- develop mathematical reasoning (Reasoning [R])
- select and use technologies as tools for learning and solving problems (Technology [T])
- develop visualization skills to assist in processing information, making connections, and solving problems (Visualization [V])

The Nova Scotia curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning. The key to these process standards is presented in a box, as shown on the next page, with each specific outcome within the units.

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics.

Students also need to communicate their learning using mathematical terminology. Communication can help students make connections between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, written and symbolic—of mathematical ideas. Students must communicate *daily* about their mathematics learning. This enables them to reflect, to validate, and to clarify their thinking and provides teachers with insight into students' interpretations of mathematical meanings and ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts.

When students encounter new situations and respond to questions of the type, How would you ... ? or How could you ... ?, the problem-solving approach is being modeled. Students develop their own problem-solving strategies by listening to, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families, or current events.

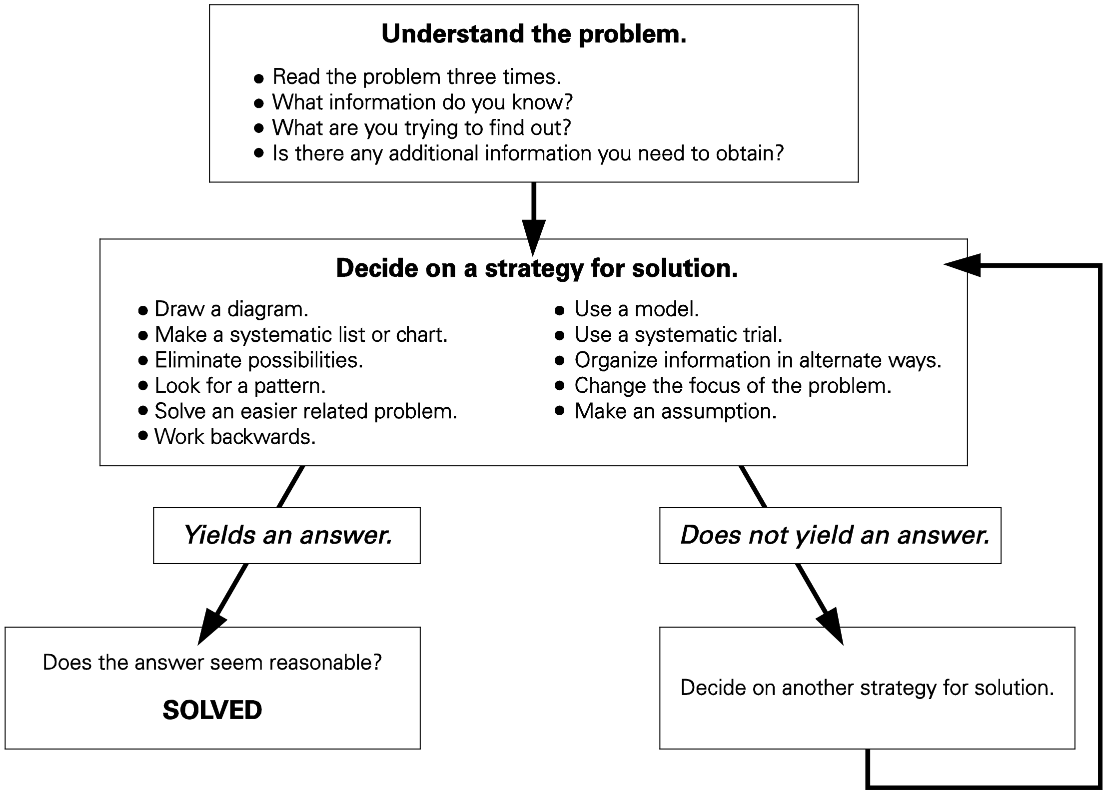
Both conceptual understanding and student engagement are fundamental in molding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill, or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive- and deductive-reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem, they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for, and engage in finding, a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

A possible flow chart to share with students is as follows:



Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to one another or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding. … *Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*." (Caine and Caine 1991, 5).

Mathematics should be viewed as an integrated whole rather than as the study of separate strands or units. Connections must also be made between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, and symbolic. The process of making connections, in turn, facilitates learning. Concepts and skills should also be connected to everyday situations and other curricular areas. For example, when developing literacy skills students learn to make text-to-world, text-to-text, and text-to-self connections. Students can also make connections to make mathematics come alive through math-to-world, math-to-math, and math-to-self connections.

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external aids.

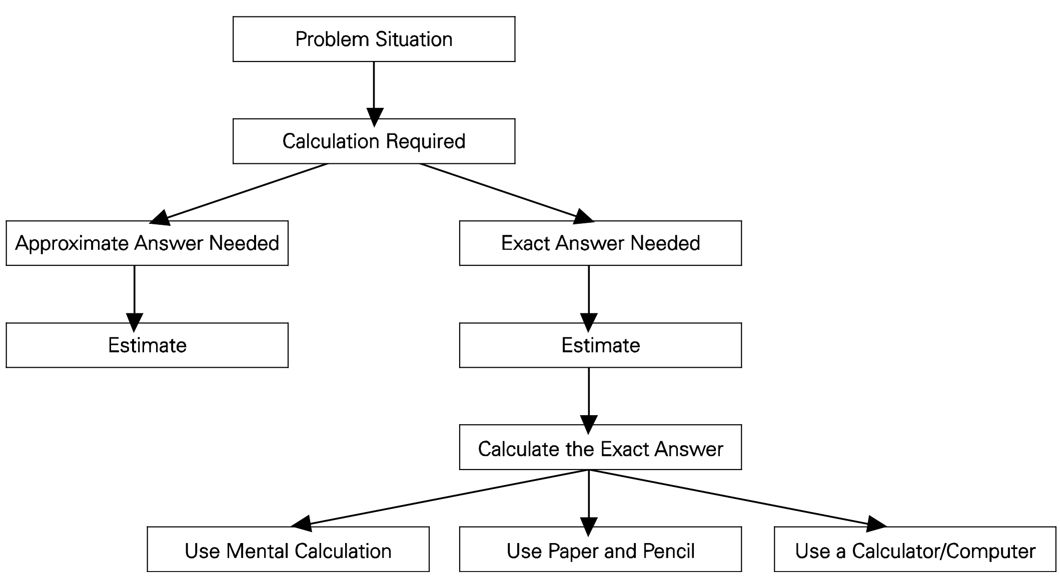
Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

“Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math.” (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers, and are more able to use multiple approaches to problem solving.” (Rubenstein 2001) Mental mathematics “provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers.” (Hope et al. 1988, v)

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when, and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process as illustrated below.



The skill of estimation requires a sound knowledge of mental mathematics. Both are necessary to many everyday experiences, and students should be provided with frequent opportunities to practise these skills.

Technology [T]

Technology can be effectively used to contribute to and support the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Calculators, computers, and other technologies can be used to

- explore and represent mathematical relationships and patterns in a variety of ways
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of foundational concepts
- develop personal procedures for mathematical operations
- simulate situations
- develop number and spatial sense
- generate and test inductive conjectures

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world." (Armstrong 1993, 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and to know several estimation strategies. (Shaw and Cliatt 1989, 150)

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations. It is through visualization that abstract concepts can be understood by the student. Visualization is a foundation to the development of abstract understanding, confidence, and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. Questions that challenge students to think, analyze, and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, Why do you believe that's true/correct? or What would happen if ...

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true.

Mathematics reasoning involves informal thinking, conjecturing, and validating—these help students understand that mathematics makes sense. Students are encouraged to justify, in a variety of ways, their solutions, thinking processes, and hypotheses. In fact, good reasoning is as important as finding correct answers. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics, and these are woven throughout this document. These components include change, constancy, number sense, relationships, patterns, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen 1990, 184)

Students need to learn that new concepts of mathematics as well as changes to previously learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers, and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms **stability**, **conservation**, **equilibrium**, **steady state** and **symmetry** (AAAS–Benchmarks 1993, 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.
- Lines with constant slope.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy. (British Columbia Ministry of Education, 2000, 146) Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities, and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and use benchmarks and referents. This results in students who are computationally fluent, flexible with numbers, and have intuition about numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables, and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory, or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create, and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

Curriculum Document Format

This guide presents the mathematics curriculum so that a teacher may readily view the scope of the outcomes that students are expected to achieve during that year. Teachers are encouraged, however, to examine what comes before and what follows, to better understand how students' learning at a particular grade level is part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes.

When a specific curriculum outcome is introduced, it is followed by the mathematical processes and performance indicators for that outcome. A scope and sequence is then provided, which relates the SCO to previous and next grade SCOs. Also for each SCO, there are background information, assessment strategies, suggested instructional strategies, and suggested models and manipulatives, mathematical vocabulary, and resource notes. For each section, the guiding questions should be used to help with unit and lesson preparation.

Assessment Strategies

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcome and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO

Mathematical Processes		
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections
[ME] Mental Mathematics and Estimation		
[T] Technology	**[V]** Visualization	**[R]** Reasoning

Performance Indicators

Describes observable indicators of whether students have achieved the specific outcome.

Scope and Sequence

Previous grade or course SCOs	Current grade SCO	Following grade or course SCOs

Background

Describes the “big ideas” to be learned and how they relate to work in previous grade and work in subsequent courses.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Sample tasks that can be used to determine students’ prior knowledge.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Some suggestions for specific activities and questions that can be used for both instruction and assessment

FOLLOW-UP ON ASSESSMENT

Planning for Instruction

Choosing Instructional Strategies

Suggested strategies for planning daily lessons.

SUGGESTED LEARNING TASKS

Suggestions for general approaches and strategies suggested for teaching this outcome.

SUGGESTED MODELS AND MANIPULATIVES

MATHEMATICAL LANGUAGE

Teacher and student mathematical language associated with the respective outcome.

Resources/Notes

Contexts for Learning and Teaching

Beliefs about Students and Mathematics Learning

"Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge." (National Council of Teachers of Mathematics 2000, 20).

- The Nova Scotia mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning that have grown out of research and practice. These beliefs include the following:
- Mathematics learning is an active and constructive process.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.
- Learners are individuals who bring a wide range of prior knowledge and experiences and who learn via various styles and at different rates.
- Leaning is most effective when standards of expectation are made clear with ongoing assessment and feedback.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best constructed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals, and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial, and symbolic representations of mathematics. The learning environment should value, respect, and address all students' experiences and ways of thinking so that students are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals of Mathematics Education

The main goals of mathematics education are to prepare students to

- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- become mathematically literate adults, using mathematics to contribute to society
- commit themselves to lifelong learning

Students who have met these goals will

- gain understanding and appreciation of the contributions of mathematics as a science, a philosophy, and an art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding. Students should be encouraged to

- take risks
- think and reflect independently
- share and communicate mathematical understanding
- solve problems in individual and group projects
- pursue greater understanding of mathematics
- appreciate the value of mathematics throughout history

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for student success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations, and engage in reflective practices.

To experience success, students must be taught to set achievable goals and assess their progress as they work toward these goals. Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Engaging All Learners

*"*No matter how engagement is defined or which dimension is considered, research confirms this truism of education: *The more engaged you are, the more you will learn."* (Hume 2011, 6)

Student engagement is at the core of learning. Engagement in learning occurs when students are provided with opportunities to become more invested in their learning. This is critical for teachers to take into account when planning and implementing instruction. Effective instruction engages, embraces, and supports all learners through a range of learning experiences that are both age and developmentally appropriate.

This curriculum is designed to provide learning opportunities that are equitable, accessible, and inclusive of the many facets of diversity represented in today's classrooms. When teachers know their students as individual learners and as individual people, their students are more likely to be motivated to learn, persist in challenging situations, and apply reflective practices.

Supportive Learning Environments

A supportive and positive learning environment has a profound effect on students' learning. Students need to feel physically, socially, emotionally, and culturally safe in order to take risks with their learning. In classrooms where students feel a sense of belonging, see their teachers' passion for learning and teaching, are encouraged to actively participate, and are challenged appropriately, they are more likely to be successful.

Teachers recognize that not all students progress at the same pace nor are they equally positioned in terms of their prior knowledge of particular concepts, skills, and learning outcomes. Teachers are able to create more equitable access to learning when

- instruction and assessment are flexible and offer multiple means of representation
- students have options to engage in learning through multiple ways
- students can express their knowledge, skills, and understanding in multiple ways

(Hall, Meyer, and Rose 2012)

In a supportive learning environment, teachers plan learning experiences that support *each* student's ability to achieve curriculum outcomes. Teachers use a variety of effective instructional approaches that help students to succeed, such as

- providing a range of learning opportunities that build on individual strengths and prior knowledge
- providing all students with equitable access to appropriate learning strategies, resources, and technology
- involving students in the creation of criteria for assessment and evaluation
- engaging and challenging students through inquiry-based practices
- verbalizing their own thinking to model comprehension strategies and new learning
- balancing individual, small-group, and whole-class learning experiences
- scaffolding instruction and assignments as needed and giving frequent and meaningful descriptive feedback throughout the learning process
- integrating "blended learning" opportunities by including an online environment that extends learning beyond the physical classroom
- encouraging students to take time and to persevere, when appropriate, in order to achieve a particular learning outcome

Multiple Ways of Learning

"Advances in neuroscience and education research over the past 40 years have reshaped our understanding of the learning brain. One of the clearest and most important revelations stemming from brain research is that there is no such thing as a 'regular student.'" (Hall, Meyer, and Rose 2012, 2) Teachers who know their students well are aware of students' individual learning differences and use this understanding to inform instruction and assessment decisions.

The ways in which students make sense of and demonstrate learning vary widely. Individual students tend to have a natural inclination toward one or a few learning styles. Teachers are often able to detect learning strengths and styles through observation and through conversation with students. Teachers can also get a sense of learning styles through an awareness of students' personal interests and talents.

Instruction and assessment practices that are designed to account for multiple learning styles create greater opportunities for all students to succeed.

While multiple learning styles are addressed in the classroom, the three most commonly identified are:
- auditory (such as listening to teacher-modelled think-aloud strategies or participating in peer discussion)
- kinesthetic (such as examining artifacts or problem-solving using tools or manipulatives)
- visual (such as reading print and visual texts or viewing video clips)

For additional information, refer to *Frames of Mind: The Theory of Multiple Intelligences* (Gardner 2007) and *How to Differentiate Instruction in Mixed-Ability Classrooms* (Tomlinson 2001).

A Gender-Inclusive Curriculum and Classroom

It is important that the curriculum and classroom climate respect the experiences and values of all students and that learning resources and instructional practices are not gender-biased. Teachers promote gender equity and inclusion in their classrooms when they
- articulate equally high expectations for all students
- provide equal opportunity for input and response from all students
- model gender-fair language, inclusive practices, and respectful listening in their interactions with students
- identify and openly address societal biases with respect to gender and sexual identity

Valuing Diversity: Teaching with Cultural Proficiency

"Instruction that is embedded in socially meaningful contexts, and tasks that are meaningful and relevant to the lives of students, will engage students in high-level problem-solving and reasoning and enhance students' engagement (Frankenstein 1995; Gutstein 2003; Ladson-Billings 1997; Tate 1995)." (Herzig 2005)

Teachers appreciate that students have diverse life and cultural experiences and that individual students bring different prior knowledge to their learning. Teachers can build upon their knowledge of their students as individuals, value their prior experiences, and respond by using a variety of culturally-proficient instruction and assessment practices in order to make learning more engaging, relevant, and accessible for all students. For additional information, refer to *Racial Equity Policy* (Nova Scotia Department of Education 2002) and *Racial Equity / Cultural Proficiency Framework* (Nova Scotia Department of Education 2011).

Students with Language, Communication, and Learning Challenges

Today's classrooms include students who have diverse language backgrounds, abilities, levels of development, and learning challenges. By observing and interacting with students and by conversing with students and/or their families, teachers gain deeper insights into the student as a learner. Teachers can use this awareness to identify and respond to areas where students may need additional support to achieve their learning goals. For students who are experiencing difficulties, it is important that teachers distinguish between those students for whom curriculum content is challenging and those for whom language-based factors are at the root of apparent academic difficulties. Students who are learning English as an additional language may require individual support, particularly in language-based subject areas, while they become more proficient in their English language skills. Teachers understand that many students who appear to be disengaged may be experiencing difficult life or family circumstances, mental health challenges, or low self-esteem, resulting in a loss of confidence that affects their engagement in learning. A caring, supportive teacher demonstrates belief in the students' abilities to learn and uses the students' strengths to create small successes that help nurture engagement in learning and provide a sense of hope.

Students who Demonstrate Exceptional Talents and Giftedness

Modern conceptions of giftedness recognize diversity, multiple forms of giftedness, and inclusivity. Some talents are easily observable in the classroom because they are already well developed and students have opportunities to express them in the curricular and extracurricular activities commonly offered in schools. Other talents only develop if students are exposed to many and various domains and hands-on experiences. Twenty-first century learning supports the thinking that most students are more engaged when learning activities are problem-centred, inquiry-based, and open-ended. Talented and gifted students usually thrive when such learning activities are present. Learning experiences may be enriched by offering a range of activities and resources that require increased cognitive demand and higher-level thinking with different degrees of complexity and abstraction. Teachers can provide further challenges and enhance learning by adjusting the pace of instruction and the breadth and depth of concepts being explored. For additional information, refer to *Gifted Education and Talent Development* (Nova Scotia Department of Education 2010).

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in business education, career education, literacy, music, physical education, science, social studies, technology education, and visual arts.

Financial Mathematics
15–20 hours

GCO: Students will be expected to develop number sense in financial applications.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO FM01 Students will be expected to solve problems that involve compound interest in financial decision making. [C, CN, PS, T, V]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

FM01.01 Explain the advantages and disadvantages of compound interest and simple interest.
FM01.02 Identify situations that involve compound interest.
FM01.03 Graph and compare, in a given situation, the total interest paid or earned for different compounding periods.
FM01.04 Determine, given the principal, interest rate, and number of compounding periods, the total interest of a loan.
FM01.05 Graph and describe the effects of changing the value of one of the variables in a situation that involves compound interest.
FM01.06 Determine, using technology, the total cost of a loan under a variety of conditions (e.g., different amortization periods, interest rates, compounding periods, and terms).
FM01.07 Compare and explain, using technology, different credit options that involve compound interest, including bank and store credit cards and special promotions.
FM01.08 Solve a contextual problem that involves compound interest.

Scope and Sequence

Mathematics 11	**Mathematics 12**
–	**FM01** Students will be expected to solve problems that involve compound interest in financial decision making.

Background

This will be students' first exposure to simple and compound interest. In this unit, the initial focus is on simple interest and its use in problem solving. This is followed by a comparison of simple and compound interest with a focus on the relationship between the two. Students will then work with compound interest, considering the effect of different compounding periods, and use compound interest in problem-solving situations.

In Mathematics 10, students had experience graphing linear functions (RF06). In Mathematics 11, students have graphed quadratic functions (RF02). For this unit students will be expected to graph compound interest situations in order to compare the total interest paid or earned for different compounding periods.

New terminology will include the following:

- **Amortization table —** a table that lists regular payments of a loan and shows how much of each payment goes toward the interest charged and the principal borrowed as the balance of the loan is reduced to zero
- **Collateral —** an asset that is held as security against the repayment of a loan
- **Compound interest —** the interest paid on the principal and its accumulated interest
- **Compounded annually —** when compound interest is determined or paid yearly
- **Compounding period —** the time over which interest is determined; interest can be compounded annually, semi-annually (every 6 months), quarterly (every 3 months), monthly, weekly, or daily
- **Fixed interest rate —** an interest rate that is guaranteed not to change during the term of an investment or loan
- **Future value —** the amount, A, that an investment will be worth after a specified period of time
- **Interest rate —** the percentage charged, usually stated as a per year rate
- **Interest —** money earned on an investment or a fee paid for borrowing money, usually expressed as a percentage
- **Maturity —** the contracted end date of an investment or loan, at the end of the term
- **Mortgage —** a loan, usually for the purchase of real estate
- **Present value —** the amount that must be invested now to result in a specific future value in a certain time at a given interest rate
- **Principal —** the original amount invested or borrowed
- **Rate of return —** the ratio of money earned (or lost) on an investment relative to the amount of money invested
- **Simple interest —** interest calculated as a percentage of the principal
- **Term —** the time in years for an investment or loan

Students are introduced to the concept of simple interest and the use of the simple interest formula, $I = Prt$. P represents the principal, r represents the rate of interest, and t represents the term of the investment. To calculate final or future value, use the formula $FV = P + I$. FV represents the final or future value. (The formula is sometimes written as $A = P + I$ or $A = P + Prt$ or $A = P(1 + rt)$ where A represents the final amount.)

Simpler problems will require that students substitute the values for the principal, rate (as a decimal), and time (in years) into the formula to obtain the interest earned. Other problems may require formula rearrangement, either before or after values are substituted. When given the time in months, students are expected to change it into years before substituting into the interest formula.

Compound interest is much more common than simple interest. To understand compound interest, however, it is important to have an understanding of simple interest first.

The relationship between simple and compound interest should be investigated using investment vehicles such as savings accounts or guaranteed investment certificates (GICs). Initially, students should use an iterative process to compare the effects of simple and compound interest. The tables below illustrate the effects of simple and compound interest on the value of $1000 in a savings account earning annual interest of 5%.

Simple Interest

Year	Principal	Amount of Annual Interest	Total Amount at the End of the Year
1	$1000	1000 × 0.05 = 50	$1050
2	$1000	1000 × 0.05 = 50	$1100
3	$1000	1000 × 0.05 = 50	$1150
4	$1000	1000 × 0.05 = 50	$1200

Compound Interest

Year	Principal plus Interest	Amount of Annual Interest	Total Amount at the End of the Year
1	$1000	1000 × 0.05 = 50	$1050
2	$1050	1050 × 0.05 = 52.50	$1102.50
3	$1102.50	1102.50 × 0.05 = 55.13	$1157.63
4	$1157.63	1157.63 × 0.05 = 57.88	$1215.51

To highlight the effect of compound interest in the short term, larger values for the initial amount of money in the savings account can be used. Alternatively, a longer time period can be used.

Once students have explored the iterative process, they can be introduced to the appropriate compound interest formulas and use them to calculate the value of an investment after a specific amount of time.

Personal loans from family members, for example, may use simple interest. Ask students to consider other possible advantages of borrowing money from family. In some cases family members may offer the money at a very low interest rate or no interest rate at all.

Many products offered by financial institutions, however, typically charge compound interest.
The formula used for compound interest is $A = P(1 + i)^n$ where P is the principle, i is the interest rate per compounding period, n is the number of compounding periods, and A represents the future value of the principle.

For example: If $1000 is invested for 3 years at 6% interest, compounded monthly,

$$P = \$1000, i = \frac{0.06}{12} = 0.005, n = (3)(12) = 36.$$
$$\therefore A = 1000(1 + 0.005)^{36} = \$1196.68$$

To be successful, students should be able to distinguish between biweekly payments, accelerated biweekly payments, and semi-monthly payments. Teachers could ask students to consider a loan payment of $600 per month and work through the payments to help them identify the differences.

- Biweekly → 600 × 12 ÷ 26 = $276.92 paid 26 times per year
- Semi-monthly → 600 ÷ 2 = $300.00, paid 24 times per year (1st and 15th of each month)
- Accelerated Biweekly → 600 ÷ 2 = $300.00 paid 26 times per year

While compound interest is beneficial to the person receiving the interest, it costs the person paying the interest on a loan or credit card a lot of money, as interest is charged on interest.

In this unit, students will solve problems that involve single payment loans and regular payment loans. The use of technology such as spreadsheets or a financial application is important here.

A loan can involve regular loan payments over the term of the loan or a single payment at the end of the term. Students should first work through examples where a loan is paid using a single payment at the end of the term. They should also compare the total interest paid for different compounding periods given the same initial principal, interest rate, and term. When making financial decisions, it is important for students to understand the rate of interest charged, as well as the compounding, as these can create large differences over long periods of time.

Students will be expected to solve a problem similar to the following:

- Mary borrows $1000 at 10% interest, compounded semi-annually.
- Sean borrows $1000 at 10% interest, compounded annually.
- How much interest will each pay at the end of two years?

Students will be expected to notice interest accumulates faster when there is an increase in the frequency of compounding. Encourage students to work through the term of the loan so they have an opportunity to describe and explain how the interest is calculated. They can then progress to using the compound interest formula, $A = P(1 + i)^n$ to determine the value of the loan and use $I = A – P$ to determine the total interest.

Students will be expected to analyze situations where regular loan payments are made over the term of the loan, such as a mortgage or a car loan.

Students will be expected to be able to explain why the frequency of a loan payment is often linked to a payroll schedule. Teachers should provide examples where the payment frequency matches the compounding period. For example, if a loan repayment is occurring monthly, then the interest should be compounded monthly as well.

Students will be expected to investigate the effects that changing a variable or multiple variables has on the total cost of a loan. They should first compare the effects of changing one specific variable. Using a financial application, ask students to complete the following table:

Principal	**Interest Rate** (Annual Compound)	**Loan period** (months)	**Monthly payment**	**Total Cost of Loan**	**Total Interest** (Total Cost-Principal)
$18,000	4%	48			
$18,000	4%	36			
$18,000	3%	48			
$18,000	3%	36			

Students will be expected to notice that by decreasing the amortization period, they are increasing their monthly payment but decreasing their total interest.

Similar conclusions can be made regarding interest rates, compounding periods, and terms.

Students will be expected to investigate the effect that changing multiple conditions has on the total loan amount. For example, students can use a financial application to determine which of the following options for a mortgage results in a lower total loan cost.

- *Option 1:* Cost of mortgage is $300,000, interest rate 2.5%, monthly payments, 25-year amortization.
- *Option 2:* Cost of mortgage is $300,000, interest rate is 3.5%, monthly payments, 20-year amortization.

The variables in compound interest questions include present value, regular payment amount, payment frequency, number of payments, annual interest rate, compounding frequency, total interest, and future value. Tasks should include the following:

- Find the present value, given all other variables.
- Find the number of payments, given all other variables.
- Find the regular payment amount, given all other variables.
- Find the future value, given all other variables.

Students should be able to calculate the total interest at the end of the loan period.

Using technology, students have the power to make decisions about their ability to afford a loan based on their budget.

Example:

- Larry is considering buying a new home that costs $220,000. He needs to know what the monthly payment would be for a 30-year mortgage at 4.75% interest, compounded monthly.

  Using a finance application on a graphing calculator or an online loan calculator, ask students to determine the payment amount (i.e., $1147.62).

  Using the Finance Application on the TI 83+/84:

  Press [APPS], then FINANCE, choose TVM Solver. Press [ENTER].

N = 360 I% = 4.75 PV = 220000 PMT = –1147.62414 FV = 0 P/Y = 12 C/Y = 12 PMT: END BEGIN	N — the number of total periods for compounding I% — the interest rate (usually as annual rate) PV — present value PMT — the periodic payment FV — future value P/Y — payment periods per year C/Y — compounding periods per year
Note: No entry can be left blank. Enter zero for the unknown value (the one you are solving for).	

Students should also be able to determine the maximum interest rate that a purchaser can afford for a car loan.

Example:

- Janet wants to buy a $10,000 car and knows she can only afford a maximum monthly payment of $200. She does not want to have the loan for any longer than five years. What is the maximum interest rate that she could afford to pay?

  Using a finance application on a graphing calculator or an online loan calculator, students conclude the highest rate Janet can afford is 7.4%.

  Using the Finance Application on the TI 83+/84

  Press [APPS], then FINANCE, choose TVM Solver. Press [ENTER].

N = 60 I% = 0 PV = 10000 PMT = –200 FV = 0 P/Y = 12 C/Y = 12 PMT: END BEGIN	With the cursor flashing on the I% = 0 (or any number; it does not matter as long as it is not left blank), press [ALPHA] [SOLVE] SOLVE is over the [ENTER] Key.	N = 60 I% = 7.420095794 PV = 10000 PMT = –200 FV = 0 P/Y = 12 C/Y = 12 PMT: END BEGIN

It is important for students to explore why this information is so important when making financial decisions. Teachers should discuss with students that this information, which is based on the budget of the individual, allows the person to negotiate a rate without exceeding the maximum.

Students should work with forms of credit that include bank loans, credit cards, and special promotions that have various conditions. They should compare credit options with varying interest rates, compounding periods, annual fees, and special limited-time offers such as “no interest” periods. Amortization tables, spreadsheets, and financial applications should be used to determine monthly payments, total cost, total interest, etc., to ultimately determine the most financially sound investment.

Recently a bank offered a 9% interest rate over a three-year period, the fine print revealed that this meant 3% per year for a three-year period. Similarly, a credit card reported its interest rate to be 2% per month rather than 24% per year. It is important to note that interest rates are most often reported as per-year interest rates, but the consumer should always read any document carefully to ensure that they are not misunderstanding interest rates.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students’ prior knowledge.

- As a pre-assessment, ask students to complete a snowball activity.

  (a) Provide all students with a blank sheet of paper.

(b) Ask the following question of the class, What is interest (on a loan, etc.)?

(c) Give students a few minutes to write on their sheet what they think interest is.

(d) Ask students to toss the "snowballs" to the front of the class and then collect one snowball each. They should take turns to read aloud what is written on the snowball they have chosen.

(e) The teacher should then summarize their ideas and generate a discussion on the topic to start the unit.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- How would the graphs illustrating each the following situations be the same and how would they be different?

  Situation A: \$100 invested at 6% compounded semi-annually for 20 years.

  Situation B: \$100 invested at 6% compounded monthly for 20 years.

  Situation C: \$100 invested at 6% compounded weekly for 20 years.

- Suppose Peter borrows \$1000 from his parents.

  (a) How much will he have to pay back in two years if they charge 3% interest per year?

  (b) How much will he have to pay back if they charge 3% interest compounded monthly?

  (c) Explain which of these options is better for Peter.

- Which of the following investments would yield the greater return?

  Option 1: \$1000 at 3.5% annual simple interest

  Option 2: \$1000 at 3% annual compound interest

  Verify your prediction by calculating the value of their investments after 5 years, 10 years, and 15 years.

- Using the formula for compound interest, $A = P(1+r)^n$,

  (a) graph$y = 100(1+0.04)^x$and explain what this graph illustrates in terms of compound interest.

  (b) graph$y = 100(1+x)^2$and explain what this graph illustrates in terms of compound interest.

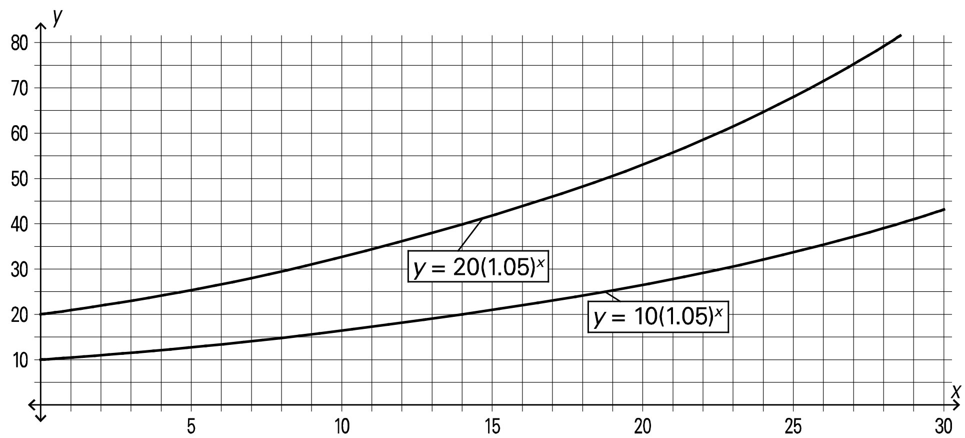
  (c) graph$y = x(1+0.04)^2$and explain what this graph illustrates in terms of compound interest.

- Complete the following spreadsheet.

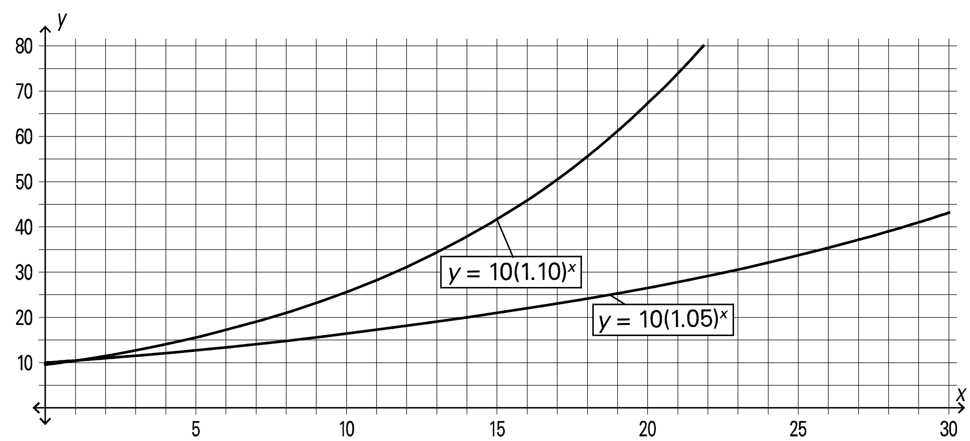
	A	B	C	D	E	F	G	H
1	Principal	Rate per annum %	Compound	Compound periods in one year	Rate per compound period	Term (years)	Number of compound periods	Amount
2	\$1000	12	annually	1	12	2	2	\$1254.40
3			semi-annually					
4			monthly					
5			biweekly					

Which compounding period provides the most return on your investment?

- For each of the graphs shown below, describe the parameter that has changed and what it means in terms of compound interest.



Graph A



Graph B

- Marie takes out a loan for \$5000 at 4% compounded annually.
  (a) How much total interest will she pay if the loan is for six years?
  (b) If the interest is compounded monthly, how much total interest will be paid?
  (c) What if the interest is compounded daily?
  (d) What happens to the amount of interest as the frequency of compounding changes?
- For each of the following, determine the future value of the loan, if simple interest is paid.
  (a) P = \$800, r = 4%, t = 5 years
  (b) P = \$1250, r = 8.5%, t = 6 months
  (c) P = \$3800, r = 9%, t = 60 days
- For each of the following, determine the future value and the total interest earned.
  (a) \$200 invested for 9 years at 9% compounded monthly
  (b) \$750 invested for 12 years at 4% compounded quarterly
- A couple decides to set aside \$5000 in a savings account for a second honeymoon trip. It is compounded quarterly for 10 years at 9%. Find the amount of money they will have in 10 years.

- A 25-year old woman plans to retire at age 50. She decides to invest an inheritance of $60,000 at 7% compounded quarterly. How much will she have at age 50?
- In addition to working and her family's contribution, Jane had to borrow $8000 over the course of six years to complete her education. The simple interest is $4046.40. Find the interest rate.
- To take advantage of a going-out-of-business sale, the College Corner Furniture Store had to borrow some money. It paid back a total amount of $150,000 on a 6-month loan at a simple interest rate of 12%. Find the principal.
- To train employees to use new equipment, Williams Muffler Repair had to borrow $4500 at a simple interest rate of 9.5 %. The company paid $1282.50 in interest. Find the term of the loan.
- Brittany takes out a loan for $100,000 at 6% annual interest. She takes 20 years to repay the loan.

  (a) Use a financial application to determine the amount of each monthly payment.

  (b) How much interest will she have paid at the end of the 20 years?
- Tyler decides to purchase a BMX bike from the Cyclic Shop for $2200, including taxes. He considers the following options:

Option A	**Option B**
▪ pay $2200 in cash	▪ pay an initial administration fee of $30 in cash ▪ no down payment ▪ monthly payment of $191.37 based on 8% annual interest compounded monthly for 1 year

  How much more will Tyler have to pay if he chooses Option B instead of Option A?
- Visit the websites of at least three different financial institutions that offer mortgages (CIBC, Scotiabank, TD, BMO, etc.). Decide which offers the best rate for a five-year closed-term mortgage with a fixed rate. Use the mortgage calculator on the website and determine for a 25-year amortization period

  (a) the monthly payment amount

  (b) total cost of the mortgage

  (d) total amount of interest paid

  Change the payment frequency to accelerated biweekly. How does this affects the three values determined previously? Explain your reasoning.
- Lori is thinking about getting a credit card. She receives a phone call in which a credit card sales representative tells her she has been pre-approved for a credit card. List five questions that Lori should ask about before she accepts the offer.
- Kaelie, who is 25, would like to have $100,000 in an RRSP at age 55 when she retires. If her RRSP pays 5% per year compounded annually, how much money will she have to invest now in order to have her desired amount?
- Karen has invested $20,000 in a Registered Education Savings Plan. She wants her investment to grow to $50,000 by the time her newborn enters university, in 18 years. What interest rate, compounded semi-annually, will result in a future value of $50,000? Round off the answer to two decimal places.

- Jerard borrowed $20,000 in student loans to help pay for university. His bank offered him an interest rate of 3% per year compounded annually. If he has 10 years to pay back the loan, how much will he have to pay back, in total? What will his monthly payment be?
- Cindy has a credit card debt of $1500. The interest rate for the credit card is 18%, compounded monthly. She can afford to pay only the minimum payment each month, which is 4% of the balance, or $50, whichever is greater, until the balance is paid.
  (a) How long will it take Cindy to pay off the credit card?
  (b) How much will she pay back altogether?
  (c) How much did she pay in interest?
- Kayla's bank has approved a personal loan of $5000 at 6%, compounded monthly, so that she can do some minor home renovations. She wants to finish paying the loan at the end of 5 years. Under those conditions, what will her monthly payment be?
- In three years, Luke wants to buy a new car that costs $15,000. He plans to save $300 per month in a savings account that earns 3% per year, compounded monthly. After three years, his parents agree to pay the remainder of the cost of the car. How much money will his parents give toward the purchase of the car?
- Camille is negotiating with her bank for a mortgage on a house. She has been told that she will need to make a 15% down payment on the purchase price of $140,000. Then the bank will offer her a mortgage loan for the balance at 3%, compounded semi-annually, with a term of 25 years and with monthly mortgage payments.
  (a) How much will each monthly payment be?
  (b) How much interest will Camille end up paying by the time she has paid off the loan in 25 years?
  (c) How much will she pay altogether?
- Create a graphic organizer that lists the pros and cons for four different types of credit, such as, bank credit cards, store credit cards, line of credit, in-store financing, payday credit, and personal loans.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- When discussing the advantages and disadvantages of simple versus compound interest, ask students to explore options that are available to the customer. Ask students if the perspective of the lender and the perspective of the customer should be considered. Ask students to comment on the following points:
  - An advantage for the lender would typically be a disadvantage for the customer.
  - Depending on whether one is investing or borrowing determines whether simple interest or compound interest is more advantageous.

- Ask students for their input and subdivide their responses into borrowing options and investments.

Borrowing	**Investing**
loan credit card mortgage line of credit student loan	savings account chequing account GIC Canada Savings Bond

Discuss with students that Guaranteed Investment Certificates (GICs) and Canada Savings Bonds (CSBs) can have either simple or compound interest.

- Students should solve problems where different options are available and explain why they chose a specific option. Ask students to think about a scenario such as the following:
  - Jamal intends to go to university in five years. His grandmother decides to invest $2000 in a Guaranteed Investment Certificate (GIC) to help with his first-year expenses.

  Ask students to answer the following questions:
  - How much would the GIC be worth in five years if his grandmother chooses a simple interest GIC at 3% annual interest?
  - How much would it be worth if the interest is compounded monthly?
  - Which option is better for the bank?
  - Which option is better for Jamal?

  Remind students that simple interest increases linearly, whereas compound interest increases exponentially. Also, remind them that 3% interest with “monthly compounding” does not mean 3% per month. It means 0.25% per month.

- Ask students to create a word wall to display the different terminology in this unit. They could create a poster that shows their word and a brief definition and/or illustration of it. Below is a list of some suggested words to use.
  - amortization
  - annual future value
  - appreciation investing
  - biweekly
  - borrowing
  - compound interest
  - depreciation balance
  - leasing
  - line of credit investor
  - loan
  - maturity date
  - monthly financing
  - principal lender
  - semi-monthly mortgage
  - simple interest

- Teachers should provide students with examples of a single loan payment at the end of the loan’s maturity date.
  - A farmer making a single lump sum payment on his loan after his crop has been harvested.
  - A payday loan offered by certain financial service providers.

- Ask students to use technology to create an amortization table to illustrate each periodic payment on the loan.

  Students could work through a problem similar to the following:

  – Mark is buying an ATV for the summer. The bank offers him a loan of $7499.99 to pay for his ATV with an interest rate of 4.5% compounding monthly. If Mark makes 36 monthly payments of $223.10, calculate the total interest paid at the end of the loan.

#/Year	Date	Payment	Interest	Principal	Balance
Loan:	12/01/14				7499.99
1/01	01/01/15	223.10	28.12	194.98	7305.01
2/01	02/01/15	223.10	27.39	195.71	7109.30
3/01	03/01/15	223.10	26.66	196.44	6912.86
4/01	04/01/15	223.10	25.92	197.18	6715.68
5/01	05/01/15	223.10	25.18	197.92	6517.76
6/01	06/01/15	223.10	24.44	198.66	6319.10
7/01	07/01/15	223.10	23.70	199.40	6119.70
8/01	08/01/15	223.10	22.95	200.15	5919.55
9/01	09/01/15	223.10	22.20	200.90	5718.65
10/01	10/01/15	223.10	21.44	201.66	5516.99
11/01	11/01/15	223.10	20.69	202.41	5314.58
12/01	12/01/15	223.10	19.93	203.17	5111.41
13/02	01/01/16	223.10	19.17	203.93	4907.48
14/02	02/01/16	223.10	18.40	204.70	4702.78
15/02	03/01/16	223.10	17.64	205.46	4497.32
16/02	04/01/16	223.10	16.86	206.24	4291.08
17/02	05/01/16	223.10	16.09	207.01	4084.07
18/02	06/01/16	223.10	15.32	207.78	3876.29
19/02	07/01/16	223.10	14.54	208.56	3667.73
20/02	08/01/16	223.10	13.75	209.35	3458.38
21/02	09/01/16	223.10	12.97	210.13	3248.25
22/02	10/01/16	223.10	12.18	210.92	3037.33
23/02	11/01/16	223.10	11.39	211.71	2825.62
24/02	12/01/16	223.10	10.60	212.50	2613.12
25/03	01/01/17	223.10	9.80	213.30	2399.82
26/03	02/01/17	223.10	9.00	214.10	2185.72
27/03	03/01/17	223.10	8.20	214.90	1970.82
28/03	04/01/17	223.10	7.39	215.71	1755.11
29/03	05/01/17	223.10	6.58	216.52	1538.59
30/03	06/01/17	223.10	5.77	217.33	1321.26
31/03	07/01/17	223.10	4.95	218.15	1103.11
32/03	08/01/17	223.10	4.14	218.96	884.15
33/03	09/01/17	223.10	3.32	219.78	664.37
34/03	10/01/17	223.10	2.49	220.61	443.76
35/03	11/01/17	223.10	1.66	221.44	222.32
36/03	12/01/17	223.15	0.83	222.32	0
	12/31/17	8031.65	531.66	7499.99	

Ask students to verify the values of the first two or three rows to ensure they understand how to calculate the interest, principal, and present balance. They should then analyze the table to note the pattern in the interest calculation.

- For Pass the Problem, groups of five to six students will complete a finance problem by creating an amortization table. Ask one student to complete the calculations for the first line of an amortization table. The second student will verify the calculations of the previous student, complete the next line and then pass the problem on. The amortization table keeps getting passed through the group until it is completed. The last person in the group is responsible for verifying all calculations by using a technology application.
- Arrange students into four or five groups and provide students with a financial problem similar to the following. Each group must first find the monthly payment and then will be assigned a different task to complete and to report their findings back to the class.

  Sample Tasks:

  - Given a 25 year mortgage of $250,000 at an annual interest rate of 3% compounded biweekly, ask students to complete the following:
    - Find the total cost of the mortgage.
    - Find the total interest paid at the end of the mortgage.
    - Find the amount of principal paid in the first five years.
    - Find the amount of principal paid in year one.
    - Find the amount of principal paid in year 25.
- Investigate Pay Day Loan Stores. Specifically, determine
  - the actual interest rate
  - the pros and cons of the industry
- Pairs of students play the game Finance Tic-Tac-Toe. Each group is given a deck of cards where each card contains a finance problem. A second deck of cards will provide the equation and solution. Cards could be labelled 1–9. Students lay the problem cards face down in a square 3 × 3 grid. They take turns flipping over a card and solving the problem using technology and checking the solution card. If someone solves a problem incorrectly, the other play will receive that square in the grid. However, before continuing on, the group should discuss the solution. The first player with three in a row wins.
- Teachers should have a general discussion with students regarding the advantages and disadvantages of using line of credit, in-store financing options, and credit cards for purchasing. Ask students to reflect on the following questions:
  - Which borrowing method generally has the lowest interest rate?
  - Are you planning on paying the balance off in full at the end each month or carrying forward a balance? How should this factor into your decision?
  - Are there any promotions that you can take advantage of?
  - Are there any “hidden” fees?
  - Should you pay more than the minimum required payment?
  - Which borrowing option is best for large purchases?

- To get a sense of how credit accounts work, students should begin by analyzing information in the two amortization tables in the following example:
  - Brad and Marie decide to order a home gym online. The order totalled $2668 and the shipping cost is $347. They can afford to pay $200 each month. Ask students which credit card they should use.

Brad: credit card charges 13.9%, compounded daily, with an annual fee of $75.

Repayment for Brad's credit card:

#/Year	Date	Payment	Interest	Principal	Balance
Loan:	12/01/13				3015.00
1/01	01/01/14	200.00	37.20	162.80	2927.20
2/01	02/01/14	200.00	35.24	164.78	2762.44
3/01	03/01/14	200.00	30.02	169.98	2592.46
4/01	04/01/14	200.00	31.21	168.79	2423.67
5/01	05/01/14	200.00	28.23	171.77	2251.90
6/01	06/01/14	200.00	27.11	172.89	2079.01
7/01	07/01/14	200.00	24.22	175.78	1903.23
8/01	08/01/14	200.00	22.91	177.09	1726.14
9/01	09/01/14	200.00	20.78	179.22	1546.92
10/01	10/01/14	200.00	18.02	181.98	1364.94
11/01	11/01/14	200.00	16.43	183.57	1181.37
12/01	12/01/14	200.00	13.76	186.24	995.13
Y-T-D 2014	12/31/14	2400.00	305.12	2094.87	
Running	12/31/14	2400.00	305.13	2094.87	
13/02	01/01/15	200.00	11.98	188.02	807.11
14/02	02/01/15	200.00	9.72	190.28	616.83
15/02	03/01/15	200.00	6.70	193.30	423.53
16/02	04/01/15	200.00	5.10	194.90	228.63
17/02	05/01/15	231.29	2.66	228.63	0.00
Y-T-D 2015	12/31/15	1031.29	36.16	995.13	
Running	12/31/15	3431.29	341.29	3090.00	

Marie: credit card charges 19.8%, compounded daily, with no annual fee.

Repayment for Marie's credit card:

#/Year	Date	Payment	Interest	Principal	Balance
Loan:	12/01/13				3015.00
1/01	01/01/14	200.00	51.83	148.17	2866.83
2/01	02/01/14	200.00	49.28	150.72	2716.11
3/01	03/01/14	200.00	42.14	157.86	2558.25
4/01	04/01/14	200.00	43.98	156.02	2402.23
5/01	05/01/14	200.00	39.95	160.05	2242.18
6/01	06/01/14	200.00	38.55	161.45	2080.73
7/01	07/01/14	200.00	34.61	165.39	1915.34
8/01	08/01/14	200.00	32.93	167.07	1748.27
9/01	09/01/14	200.00	30.06	169.94	1578.33
10/01	10/01/14	200.00	26.25	173.75	1404.58
11/01	11/01/14	200.00	24.15	175.85	1228.73
12/01	12/01/14	200.00	20.44	179.56	1049.17
Y-T-D 2014	12/31/14	2400.00	434.17	1965.82	
Running	12/31/14	2400.00	434.17	1965.83	
13/02	01/01/15	200.00	18.04	181.96	867.21
14/02	02/01/15	200.00	14.91	185.09	682.12
15/02	03/01/15	200.00	10.58	189.42	492.70
16/02	04/01/15	200.00	8.47	191.53	301.17
17/02	05/01/15	306.18	5.01	301.17	0.00

Teachers could use the following questions to promote student discussion.

- After the third payment, which credit card option appears to be better? Do you think this will always be the case?
- At the end of the sixth months, which credit card appears to be better? Discuss your findings.
- Overall, which credit card was the better choice and why?

Ask students to comment on the impact of the annual fee on the determination of which card is better.

- Ensure that students are able to enter values into their calculators correctly in order to solve compound interest problems.
- The online game Financial Football from Practical Money Skills for Life can be played individually or in pairs and uses financial questions to allow team movement down the field to score touchdowns. The more questions students answer correctly, the more likely they are to win. (www.practicalmoneyskills.com/games/trainingcamp)
- Invite a guest speaker such as an accountant, banker, mortgage specialist, or financial advisor to generate classroom discussion around borrowing and investing.

Suggested Models and Manipulatives

- amortization tables
- die

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- accelerated biweekly
- annually
- biweekly
- compound interest
- compounded annually
- compounding periods
- fixed interest rate
- future value
- interest
- maturity
- monthly
- present value
- principal
- quarterly
- rate of return
- semi-annually
- semi-monthly
- simple interest
- term
- weekly

Resources/Notes

Internet

- "Financial Football," *Practical Money Skills for Life* (Visa 2015): www.practicalmoneyskills.com/games/trainingcamp

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 1.1, 1.2, 1.3, 1.4, pp. 6–42
  - Section 2.1, pp. 80–96
  - Section 2.2, pp. 98–100
  - Section 2.3, pp. 104–119

SCO FM02 Students will be expected to analyze costs and benefits of renting, leasing, and buying. [CN, PS, R, T]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

FM02.01 Identify and describe examples of assets that appreciate or depreciate.
FM02.02 Compare, using examples, renting, leasing, and buying.
FM02.03 Justify, for a specific set of circumstances, if renting, buying, or leasing would be advantageous.
FM02.04 Solve a problem involving renting, leasing, or buying that requires the manipulation of a formula.
FM02.05 Solve, using technology, a contextual problem that involves cost-and-benefit analysis.

Scope and Sequence

Mathematics 11	**Mathematics 12**
–	**FM02** Students will be expected to analyze costs and benefits of renting, leasing, and buying.

Background

Students should solve problems that involve decisions about whether to buy, rent, or lease in a variety of contexts.

Appreciation and depreciation both deal with asset value over time. Assets such as real estate and bonds usually gain value over time while other assets such as vehicles may decrease in value.

New terminology will include the following:

- **Appreciation —** increase in the value of an asset over time
- **Asset —** an item or a portion of an item owned; also known as property; assets include such items as real estate, investment portfolios, vehicles, art, and gems
- **Depreciation —** the value that an item loses over time (e.g., the average car depreciates about 15% to 20% per year; car depreciation usually slows down after year five)
- **Disposable income —** the amount of income that someone has available to spend after all regular expenses and taxes have been deducted
- **Equity —** the difference between the value of an item and the amount still owing on it; can be thought of as the portion owned (e.g., if a $25,000 down payment is made on a $230,000 home, $205,000 is still owing and $25,000 is the equity or portion owned)
- **Extended warranty —** a service contract between the owner and warranty provider; covers specific maintenance and repairs after the manufacturer's warranty
- **Fixed costs —** costs that do not change from month to month; have to be paid regardless of how much the item is used (e.g., for a car: license, fees, and insurance)

- **Lease —** a type of financing in which you pay for an item for a specified amount of time; at the end of the term the item can be returned to the seller or purchased for a previously set price (e.g., a car)
- **Lessee —** the customer leasing the item from the seller
- **Residual value —** the estimated value of the item at the end of the lease; determined by the seller when the lease is signed (e.g., car, office equipment)
- **Revenue —** income from normal business activities, usually the sale of goods and/or services
- **Variable costs —** costs that change in amount or in how frequently they are paid (e.g., for cars, gas, tires, and maintenance; the distance driven; climate; and owner's driving style affect the variable costs)

When deciding to rent, lease, or buy, completing a cost-and-benefit analysis is essential in determining which option is best. To make an informed decision, ask students to consider

- affordable monthly payments
- amount of disposable income
- the importance of building equity
- interest rates
- initial fees and possible penalties
- amount of down payment
- total end cost including interest
- personal benefits, such as convenience and flexibility
- appreciation and depreciation
- amount of disposable income (i.e., money left over when bills are all paid)
- the importance of building equity (e.g., the portion of a house that is owned once the mortgage is paid off)
- initial fees and possible penalties (e.g., repaying a mortgage early, lawyers' fees, property taxes, etc.)

When deciding whether to buy or lease a car, there are a number of things that must be considered. Some of these considerations are listed below.

Buying Option

- The greatest benefit of buying a car is that it may actually be owned one day. Implied in this benefit is that one day the buyer will be free of car payments. The car will be theirs to sell at any time, and they are not locked into any type of fixed ownership period.
- When buying a car, the insurance premiums on the policy are typically lower than for leasing.
- By owning a car there are no mileage restrictions that typically exist when leasing.
- On the downside, the monthly payment is usually higher on a purchased car than on a leased car. Additionally, dealers usually require a reasonable down payment, so the initial out-of-pocket cost is higher when buying a car.
- Presumably, as a car loan is paid down, the owner has the ability to build equity in the vehicle. Unfortunately, however, this is not always the case. When purchasing a car, the payments reflect the whole cost of the car, usually amortized over a four- to six-year period. Depreciation can take a significant toll on the value of a car, especially in the first couple of years, so the car will lose much of its original value in a short amount of time.

- Carefully check out (or have a qualified mechanic check out) the condition of a used vehicle.

Leasing Option

- Perhaps the greatest benefit of leasing a car is the lower out-of-pocket costs when acquiring and maintaining the car. Leases require little or no down payment, and there are no upfront sales tax payments. Additionally, monthly payments are usually lower, and the lessee gets the pleasure of having a new car every few years.
- With a lease, a car is actually being rented for a fixed number of months (typically 36 to 48 months). Therefore, the lessee pays only for the use, or depreciation, of the car for that period, and is not forced to absorb the full depreciation cost of the vehicle.
- By leasing a car, the lessee will always have a car payment. As long as a car is being leased, it will never really be owned.
- Leasing also provides an alternative when buying a car is not an option, due to not having the required down payment or having a higher monthly payment.
- For business owners, leasing a car may offer tax advantages if the vehicle is used for business purposes.
- Depending on the type of lease, when the lease term is up, the lessee can either hand the keys over to the car dealership and look for another vehicle, or finance the remaining value of the vehicle and go from making lease payments to loan payments, thereby leasing to own.
- Leased vehicles have a mileage allowance. Any kilometres over this allowance will be charged at a fixed rate. The cost of extra kilometres driven is calculated when the vehicle is returned. If the car is driven a great deal during the year, this can cost a significant amount of money.
- Insurers usually charge higher coverage costs for leased vehicles. However, depending on the driver’s age, driving record, and place of residence, the additional cost may be nominal.

When exploring the option of buying a vehicle, students will consider options such as

- paying the full purchase price, plus sales tax
- making a down payment and taking out a loan for the remainder
- taking advantage of special offers from dealerships, such as reduced interest rates or clearance prices on particular models
- buying a new or used vehicle

When considering the possibility of leasing a vehicle, it is important to consider such things as

- lease term
- lease rate
- security deposit
- kilometre allowance
- delivery charge
- option to purchase

When considering leasing to buy, factors to think about are

- the **residual** value (the estimated value of the car at the end of the lease, determined by the car dealership when the lease is signed)
- maintenance of an older vehicle

Students will examine the advantages and disadvantages of buying, leasing, and leasing to buy a vehicle. They will identify the benefits and disadvantages of each and use this information, combined with their own personal factors, such as affordability, to make an informed decision.

Students will solve problems that involve calculating the total cost of purchasing a vehicle after taxes and fees have been applied. Using technology, students determine the monthly payment based on a given interest rate. The financing interest rate is a compound rate and cannot be calculated in the same manner as the tax rate. For this reason, technology should be used to calculate a monthly payment. It is important to understand that a down payment or discount is applied after taxes. It is important to explore the payment schedules available when purchasing a vehicle (e.g., monthly or biweekly) and assess how this affects payments. Biweekly payments are made more frequently, but are less than half of a monthly payment.

To solve problems that involve leasing a vehicle, calculate

- the total monthly payment after taxes are applied
- the first monthly payment, which may include delivery or licensing fees

When determining the total cost of a leased vehicle over the lease term, consider the monthly lease payments and the cost of financing the residual value. If choosing to purchase the vehicle at the end of the lease, the monthly payment will be calculated using a given residual value and interest rate. The total cost of purchasing a vehicle should be compared with the total cost of leasing a vehicle to decide on the most economical option to acquire a vehicle. Students will calculate monthly and annual fixed costs of owning a vehicle when including fees, such as warranty, insurance, and license fees. Students could also explore the relationship between the type of car and the fixed costs of owning that car.

The variable costs of owning a vehicle, including cost of fuel and maintenance, will be considered. Students will calculate annual maintenance costs, given a maintenance schedule with recommended repairs. Discuss the ongoing variable costs associated with maintaining a vehicle after the warranty has expired, particularly major repairs (e.g., transmission replacement) and then consider whether it is more financially viable to continue maintaining an older vehicle without monthly payments or opting to purchase a new vehicle with warranty. Ideas generated through class discussion could be recorded in a chart such as the one shown below.

Old Vehicle		**New Vehicle**	
Advantages	Disadvantages	Advantages	Disadvantages

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- When a person looks for an apartment to rent, what do they consider?
- What are the operating expenses associated with owing a car?

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Colleen participates in a field trip where she skis for the first time. She really enjoys the experience and decides that she will enroll in ski lessons every day for two weeks and then, hopefully, continue skiing for years to come. She needs to decide if she is going to purchase her own skis or rent them. It costs $25 a day to rent skis while it costs $525 to buy them.
  (a) For two weeks, how much would it cost to rent skis?
  (b) For the two-week period, is it more economical to rent or buy the skis?
  (c) After how many days will the rental cost equal the cost of buying?
  (d) Would you recommend Colleen buy or rent the skis? What are some of the things she should consider in making her decision?
- Marie is unsure whether she should buy or lease a new vehicle. Choose two comparable vehicles, each from a different manufacturer. Visit the manufacturers' websites or use advertisements to obtain information on the price. Use online purchasing calculators to answer the following questions:
  (a) Which has the lower purchase price?
  (b) If buying over a 60-month period, which is the better deal?
  (c) Which is better for a 60-month lease?
  (d) Which car and payment option do you think is the better deal? Why? (Note: Be sure to consider vehicle features, interest rates, value after five years, etc.)
- Create a foldable or graphic organizer to compare leasing versus buying a car.
- Describe a situation where renting a house is the best option (e.g., military moves, job uncertainty).
- Determine the monthly payment for each of the following cars.

	Cost	Interest Rate	Term
(a)	$18,000	2.5%	5 years
(b)	$10,000	0.9%	3 years
(c)	$12,500	1.5%	4 years

- Sarah is buying a new van for $25,000 at a car dealership.
  - (a) What is the cost of the van, including HST?
  - (b) The dealership is offering 0.9% financing annually for up to 60 months. Sarah decides to finance the van for 60 months. What will it cost her, in total, for the van?
  - (c) Sarah has $5000 for a down payment. How much money will she have to finance?
  - (d) What will Sarah's monthly payment be?
- John wants to lease a new car with a price of $15,500.
  - (a) His monthly payment is $254.17 plus HST. What will be his total monthly payment?
  - (b) John has to pay a license fee of $155 plus the first month's payment. How much must he pay before he can take the car?
  - (c) John leases the car for 36 months. At the end of the lease, he decides to buy the car. The interest rate is 3%, and he will take out the loan for two years. How much will John pay in total for the car if the residual value of the car is $10,000? Assume simple interest.
  - (d) If John had bought the car initially, he could have had an interest rate of 0.9% for five years. How much would he have saved if he had bought the car initially?
- Janelle wants to buy a used car that she sees on a dealer's lot. She has $2500 for a down payment.

  The details are as follows:
  - Payments are $4200 per year for three years.
  - The annual insurance premium is $1955.
  - A two-year/100 000 km extended warranty is available for $1000.
  - (a) Calculate Janelle's biweekly fixed cost.
  - (b) If Janelle keeps her car for two more years after it is paid off, calculate her biweekly fixed cost during this two-year period.
  - (c) Janelle decides to buy the extended warranty. How much money, in total, will she have paid on the car during the first five years?
- Zack has just bought a car, and expects that he will drive about 3000 km per month. The car's fuel consumption is 5.5 L/100 km. The table shows common maintenance and repair cost.

Maintenance/Repair	**Schedule**	**Cost**
Change oil and filter	Every 4 months	$35
Rotate tires	Every 4 months	$25
Replace air filter	Every 12 months	$30
Replace windshield wipers	Every 6 months	$40
Replace front brakes	Every 50 000 km	$400
Replace back brakes	Every 100 000 km	$400
Replace tires	Every 4 years	$1200
Replace timing belt	Every 150 000 km	$1050

  - (a) How much will Zack spend on gas each month? (Assume that gas is $1.20/L.)
  - (b) How much will Zack spend on maintenance in the first year?
  - (c) Zack intends to keep the car for six years. How much can he expect to spend over that time in repairs and maintenance?

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Ask students to think about situations where assets can appreciate or depreciate, and what factors contribute to this. Houses usually appreciate over time, for example, but there are economic circumstances that may cause them to depreciate. Likewise, vehicles can appreciate or depreciate. Students could discuss why they should factor this in when they are deciding to buy a new car versus a used car.
- The following example is a good starting point to present to students.
  - Sarah is going to university in the fall and her parents are trying to decide whether to buy a house or rent an apartment for the 5 years she will be there. The house they are considering buying costs $200,000 and requires a down payment of $10,000. The bank will provide a 20-year mortgage, for the remainder of the cost, at 3% compounded semi-annually with payments every month. The house they are considering renting is $1400 per month and requires an initial damage deposit of $700.

  Ask students to answer the following questions:

  (a) What is the monthly payment for the mortgage?

  (b) What is the total amount spent in the first five years if you purchase the house?

  (c) What is the monthly payment for renting?

  (d) What is the total cost for renting for five years?

  (e) In the first five years, how much has been paid on the principal of the mortgage?

  (f) If we only consider monthly costs, which option would be best?

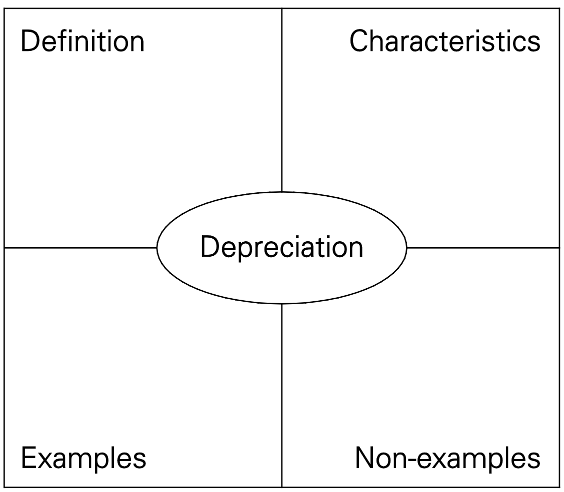
  (g) What other factors (e.g., equity, maintenance, initial fees, penalties) should a person consider when making this decision?
- Teachers should discuss the factors involved when deciding to rent, lease, or buy a home. The difference between renting and leasing a home is the length of time a property is being rented.

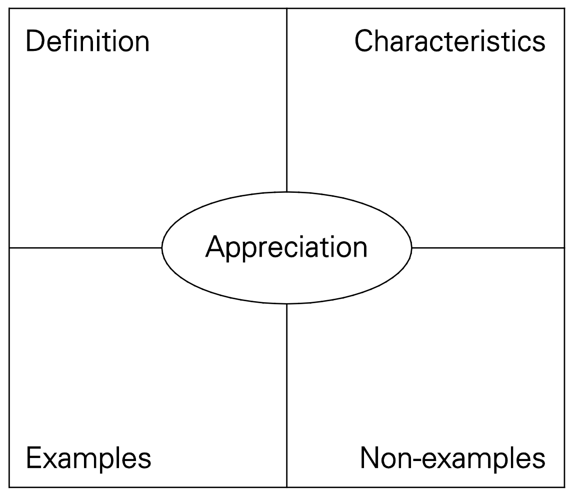
<table>
<tr><th>Buy</th><th>Rent</th><th>Lease</th></tr>
<tr><td>▪ build equity
▪ long-term investment
▪ renovate the house
▪ maintenance costs</td><td>▪ can be short term
▪ can be contract free</td><td>▪ long-term investment
▪ includes a contract
▪ must pay in full if the person decides to move before the end of the contract</td></tr>
<tr><td></td><td colspan="2">▪ do not own
▪ cannot change the property
▪ rent can change after 12 months
▪ owner’s rules
▪ no maintenance costs</td></tr>
</table>

- Discuss the factors involved when deciding to lease or buy a car.

Buy	**Lease**
▪ build equity ▪ long-term investment ▪ warranty ▪ unlimited mileage	▪ do not own ▪ maintenance costs ▪ generally lower payments ▪ warranty ▪ option to buy at the end of the lease ▪ inspection upon return ▪ limited mileage

- Ask students to complete the following Frayer models.





Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- appreciation
- asset
- depreciation
- disposable income
- equity
- lease

Resources/Notes

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Section 2.4, pp. 120–133

SCO FM03 Students will be expected to analyze an investment portfolio in terms of interest rate, rate of return, and total return. [ME, PS, R, T]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

FM03.01 Determine and compare the strengths and weaknesses of two or more portfolios.
FM03.02 Determine, using technology, the total value of an investment when there are regular contributions to the principal.
FM03.03 Graph and compare the total value of an investment with and without regular contributions.
FM03.04 Apply the Rule of 72 to solve investment problems, and explain the limitations of the rule.
FM03.05 Determine, using technology, possible investment strategies to achieve a financial goal.
FM03.06 Explain the advantages and disadvantages of long-term and short-term investment options.
FM03.07 Explain, using examples, why smaller investments over a longer term may be better than larger investments over a shorter term.
FM03.08 Solve an investment problem.

Scope and Sequence

Mathematics 11	Mathematics 12
–	**FM03** Students will be expected to analyze an investment portfolio in terms of: interest rate, rate of return, and total return.

Background

Students should solve problems that involve decisions about investments portfolios.

The focus for this outcome is on analyzing, comparing, and designing **investment portfolios** to meet specific financial goals.

An investment portfolio is comprised of all the different investments that an individual or organization holds. Investments can include **shares**, **bonds**, or **investment certificates**.

When buying **shares** (also called stocks or equities), the investor becomes a part owner in a company. This gives the investor the right to a portion of the company's earnings and may provide entitlement to vote at the shareholder meetings. Compared to other types of investments, shares can be riskier but can potentially offer higher returns. The value of the shares are dependent on the success of the company.

When buying a **bond**, the investor is lending money to a government or company for a certain period of time. In return, the investor receives a fixed rate of interest and money back at the end of the term. Company bonds offer better rates of return than investments such as **guaranteed investment**

certificates because if the company fails, the investor may not get back all the money originally paid, so there is more risk involved. Government bonds such as Canada Savings Bonds are more secure.

Mutual funds are a mix of stocks and bonds.

Investment certificates are offered at banks at a higher rate of interest than a chequing account, and are easier to access than stocks or bonds.

New terminology will include the following:

Bank of Canada prime rate — a value set by Canada's central bank, which other financial institutions use to set their interest rates
Line of credit — a pre-approved loan that offers immediate access to funds, up to a pre-defined limit, with a minimum monthly payment based on accumulated interest; a secure line of credit has a lower interest rate because it is guaranteed against the client's assets, usually property
Portfolio — one or more investments held by an individual investor or by a financial organization
Rule of 72 — a simple formula for estimating the doubling time of an investment; 72 is divided by the annual interest rate as a percent to estimate the doubling time of an investment in years; the Rule of 72 is most accurate when the interest is compounded annually

The graphing calculator can be used to assist students when comparing the value of investing a lump sum over time versus continuing to contribute to an investment.

One lump sum compounding over time.	**Continual regular contributions over time.**
When you were born, your grandparents deposited $5000 in a special account for your 21st birthday. The interest was compounded monthly at 5%. How much will it be worth on your 21st birthday? N = (21)(12) = 252	When you were born, your grandparents deposited $2000 in a special account for your 21st birthday. The interest was compounded monthly at 5%. What if, every month they also added $20. How much will it be worth on your 21st birthday?
N = 252 I% = 5 PV = 5000 PMT = 0 FV = 0 P/Y = 12 C/Y = 12 PMT: END BEGIN N = 252 I% = 5 PV = 5000 PMT = 0 FV = –14257.12055 P/Y = 12 C/Y = 12 PMT: END BEGIN	N = 252 I% = 5 PV = 2000 PMT = 20 FV = 0 P/Y = 12 C/Y = 12 PMT: END BEGIN N = 252 I% = 5 PV = 2000 PMT = 20 FV = –14589.68395 P/Y = 12 C/Y = 12 PMT: END BEGIN

Students can use the Rule of 72 to quickly approximate doubling time for an investment. This provides an opportunity to use mental mathematics. Students can use the Rule of 72, for example, to estimate how long it will take for an investment to double at an interest rate of 1.95% compounded annually. They can round the interest rate to 2% and use the formula to determine that half of 72 is 36. They should conclude that an investment purchased at a rate of 1.95% would take over 36 years to double.

As students explore various investment options they should conclude that in most cases, ultra conservative portfolios will see slower, steady growth. This is the tradeoff for keeping money stable and secure. For more growth potential, a more aggressive asset mix, with a higher level of risk should be selected. Losses are more likely when more risk is taken when investing, and it is important to ensure that there is enough time and money to recover from those losses. Registered advisers are available to determine the right asset mix for a person's situation.

The US Securities and Exchange Commission's website entitled, "Tips for Teaching Students about Saving and Investing," is a valuable teaching tool to assist with this outcome.
(www.sec.gov/investor/students/tips.htm)

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- If you won $500,000, what would you do with the money?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Thomas has two years before he goes off to community college. He has figured out that it will cost a total of about $10,000 for college. He invests in a GIC that pays 6% interest per year. He deposits $360 per month for two years. Using a TVM solver, determine if Thomas will have enough money to go to college or if he will have to find another way to supplement his education.
- Ron and Ann have just won $100,000 in a lottery. They have a young daughter who will be two in August. Ron suggests that they buy a GIC at 2.3% with the $100,000 over a 16-year period. If they cash in the GIC after 16 years, will they have enough interest to pay for their daughter's first year at university? First-year tuition is expected to grow 175% from the present $3500 in 16 years.
- Alphonse has $2000 he wishes to save. Under what conditions would you recommend that he invest in
  (a) a 10-year GIC earning 4.5% compounded semi-annually
  (b) a savings account that earns 2% compounded weekly
  (c) a five-year bond that earns 5% compounded quarterly

- Use the Rule of 72 and mental mathematics to estimate how long it would take an investment to double at an interest rate of 3.1% compounded annually.
- Use the Rule of 72 to estimate the doubling time for $1250 invested at 6% compounded annually. Which is the better investment? Explain.
  (a) 4.5% compounded annually
  (b) 4.45% compounded quarterly
- Which of the charts would be recommended for an individual who was less than five years from retirement?

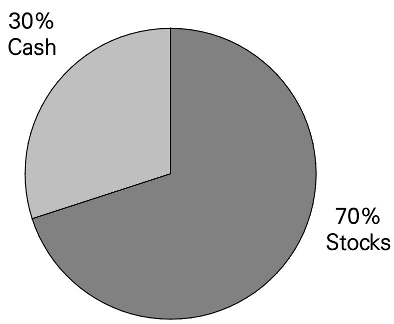


Chart A

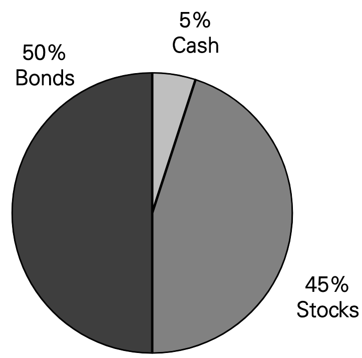


Chart B

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Go to the Canadian Securities Association website for an online resource for youth money management
  — “Make it Count: An Instructor’s Resource for Youth Money Management” (www.makeitcountonline.ca/csa/instructors/)
- The Canadian Securities Administrators site offers a wealth of information (in English and French) on investing, including free downloadable PDF brochures on investing: *Investing Basics: Getting Started; Investments at a Glance;* and *Understanding Mutual Funds.* (www.securities-administrators.ca)
- Periodically, provide a copy of a newspaper that carries a World Markets section (e.g., *The Globe and Mail*) and have students report on what they read.
- Go to the GetSmarteraboutMoney website (www.getsmarteraboutmoney.ca), to access lesson plans, videos for students, and reference materials that support this curriculum.
- Invite a local financial planner to speak to the class and discuss how financial planners make their money and what role they play in helping individuals manage their finances.
- Using a TI-83 or TI-84 graphing calculator, have students explore the TVM solver application for various interest rates, amortization periods, principal amounts, etc.
- Present the story of Joan, Michel, and Miriam from the Investor Education Fund website to the class and discuss as a group. The story is found at www.getsmarteraboutmoney.ca/managing-your-money/planning/investing-basics/Pages/the-power-of-asset-mix-joan-michel-and-miriams-stories.aspx.

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- Bank of Canada prime rate
- CSB
- GIC
- line of credit
- portfolio
- rate of return
- Rule of 72
- TFSA

Resources/Notes

Internet

- "Ballpark E$timate," American *Savings Education Council [ASEC]* (American Savings Education Council [ASEC] 2015): www.choosetosave.org/ballpark (The Ballpark Estimate is a single-page worksheet that helps individuals quickly calculate how much they will need to save each year towards retirement. Be sure to hand out the Ballpark Estimate for students to take home to their parents.)
- "Get the Facts: The SEC's Roadmap to Saving and Investing," *US Securities and Exchange Commission* (US Securities and Exchange Commission 2015): www.sec.gov/investor/pubs/roadmap.htm (A basic primer from to get you on the road to saving and investing.)
- "GetSmarterAboutMoney.ca," *Investor Education Fund* (Investor Education Fund 2015): www.getsmarteraboutmoney.ca.
- "Make It Count: An Instructor's Resource for Youth Money Management," *Canadian Securities Association* (Canadian Securities Association 2015): www.makeitcountonline.ca/csa/instructors
- "Reach Your Goals by Saving [poster]," *American Savings Education Council [ASEC]* (American Savings Education Council [ASEC] 2015): www.ebri.org/pdf/big**poster**.pdf (Created by the American Savings Education Council, this bright, colourful poster reinforces the message that students *can* save if they put their minds to it and that those savings can add up over a lifetime.)
- "Savings Links: The AIE Savings Calculator," *The Investor's Clearing House* (Alliance for Investor Education 2015): www.investoreducation.org/cindex2.cfm (This online, interactive tool shows how small amounts saved today can add up to big money over a lifetime. Using real-life examples—such as CDs, fast food, jeans, and sneakers—the calculator tells students how much they will accumulate by retirement if they save money instead of spending it. The calculator assumes an eight percent annual return and retirement at age 65. The calculator and other helpful information on saving and investing are available on this website.)
- "Test Your Money $marts," *US Securities and Exchange Commission* (US Securities and Exchange Commission 2015): www.sec.gov/investor/tools/quiz.htm (The SEC's financial quiz tests the top ten things students should know about money.)
  - "Test Your Money $marts: Facts on Saving and Investing Campaign [quiz]: www.sec.gov/pdf/monyquiz.pdf
  - "Answers to Test Your Money $marts: Facts on Saving and Investing Campaign [quiz]: www.sec.gov/pdf/quizansr.pdf

- "The Power of Asset Mix: Joan, Michel, and Miriam's Stories," *GetSmarterAboutMoney.ca.* (Investor Education Fund 2015): www.getsmarteraboutmoney.ca/managing-your-money/planning/investing-basics/Pages/the-power-of-asset-mix-joan-michel-and-miriams-stories.aspx
- "Tips for Teaching Students about Saving and Investing," *US Securities and Exchange Commission* (US Securities and Exchange Commission 2015): www.sec.gov/investor/students/tips.htm
- *Canadian Securities* Administrators (Canadian Securities Administrators 2015): www.securities-administrators.ca (PDF brochures: *Investing Basics: Getting Started; Investments at a Glance;* and *Understanding Mutual Funds*)

Print

- *2014 Consumer Action Handbook: Be a Smarter Consumer* (Federal Citizen Information Center 2014) (A helpful calendar tool from the American Financial Services Association Education Foundation to help you organize your finances and manage your money. PDF available at http://publications.usa.gov/USAPubs.php?PubID=5131&PHPSESSID=cv4pqfn6gr4aqulh84gfafbr85.)
- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Section 1.3, pp. 20–33
  - Sections 1.5 and 1.6, pp. 46–67
  - Section 2.1, pp. 80–96

Logical Reasoning
20–25 hours

GCO: Students will be expected to develop logical reasoning.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO LR01 Students will be expected to analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies. [CN, ME, PS, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

(It is intended that this outcome be integrated throughout the course by using games and puzzles such as chess, sudoku, Nim, logic puzzles, magic squares, Kakuro, and cribbage.)

LR01.01 Determine, explain and verify a strategy to solve a puzzle or to win a game; for example,

- guess and check
- look for a pattern
- make a systematic list
- draw or model
- eliminate possibilities
- simplify the original problem
- work backward
- develop alternative approaches

LR01.02 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.

LR01.03 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

Scope and Sequence

Mathematics 11	Mathematics 12
LR02 Students will be expected to analyze puzzles and games that involve spatial reasoning, using problem-solving strategies.	**LR01** Students will be expected to analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.

Background

This outcome should be developed throughout the course and not in isolation. It is suggested that students engage in at least one activity related to this outcome every week or two as time permits.

In Mathematics 11, the focus was on spatial puzzles and games. The focus in Mathematics 12 is on analyzing numerical and logical reasoning and problem-solving strategies. Students should be able to explain and verify a strategy to solve a puzzle or win a game. Mathematical or logical concepts, typical conventions, and rules from previous courses can be applied, placing the emphasis on reasoning.

It is important to note that some games primarily involve logical reasoning (such as chess and sudoku) while others also involve numerical reasoning (like Kakuro and Slitherlink). Both types of games should be equally examined.

Games provide opportunities for building self-concept, enhancing reasoning and decision making, and developing positive attitudes towards mathematics through reducing the fear of failure and error. In comparison to more formal activities, greater learning can occur through games due to increased interaction between students, opportunities to explore intuitive ideas, and problem-solving strategies.

Students' thinking often becomes apparent through the actions and decisions they make during a game, so teachers have the opportunity to formatively assess learning in a non-threatening situation. Problem-solving strategies will vary depending on the puzzle or game. The emphasis of this outcome should NOT be on the successful completion of any puzzle or game but rather on analyzing numerical and logical reasoning and problem-solving strategies.

Some students will explain their strategy by working backwards, looking for a pattern, using guess and check, or by eliminating possibilities, while others will plot their moves by trying to anticipate their opponents' moves. As students play games and analyze strategies, they explore mathematical ideas and compare different strategies for efficiency.

A variety of puzzles and games, such as board games, online puzzles and games, appropriate selections for gaming systems, and pencil-and-paper games should be used. It is not intended that the activities be taught in a block of time, but rather explored periodically during the year.

Timing and integration of this outcome should be included in teacher planning throughout the course. Therefore, students can be exposed to three or four games at different times, whether it be at the beginning or end of each unit, or a set game day. Students could engage in a game when they are finished other work. As students work through the different games and puzzles, they will begin to develop effective strategies for solving the puzzle or game.

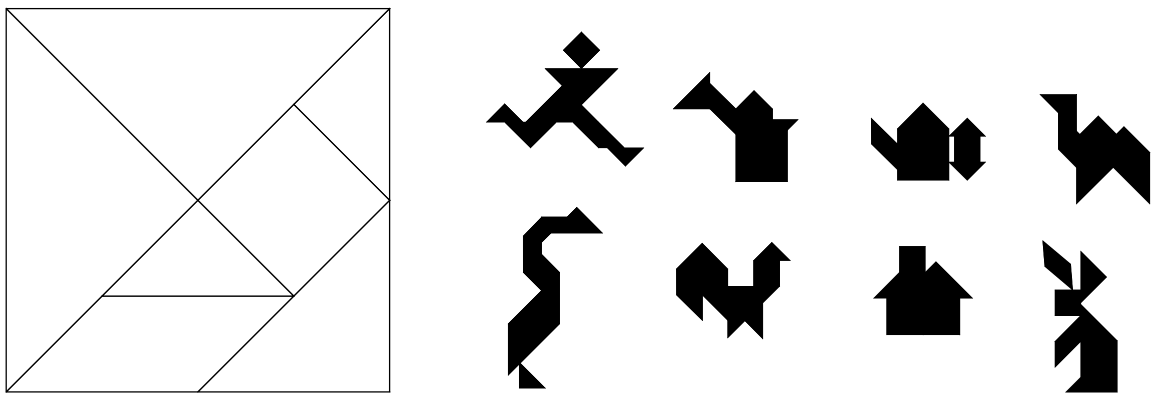
Assessment, Teaching, and Learning

Assessment Strategies

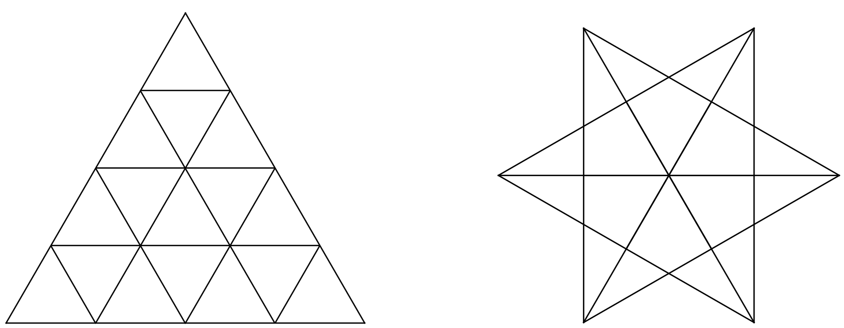
Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Using the seven tangram pieces shown below, create one of the following images.



- How many triangles are there in the diagrams?



- Arrange three quarters and two dimes in a straight row: quarter, dime, quarter, dime, quarter. Your task: In three moves, rearrange the coins so that the three quarters are together and the two dimes are together with no empty space in between each coin. At the end of each move, the coins are always in a line as in the original configuration. Each move consists of interchanging two adjacent coins at one time.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Solve the following puzzles:

(a)

	2	4	
1			3
4			2
	1	3	

(b)

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

(c)

80×		3	5–		2÷
	11+		1–		
9×	2	3–		30×	
		11+		2÷	
6	8×		13+		8+
10×				1	

- During a recent police investigation, the police chief interviewed five local villains to try to identify who stole Ms. Goody's cake from the bake sale. Each villain made one true and one false statement. Using the summary of the statements shown below, determine who stole the cake.

  Arnold: It wasn't Edward. It was Brian.
  Brian: It wasn't Charlie. It wasn't Edward.
  Charlie: It was Edward. It wasn't Arnold.
  Derek: It was Charlie. It was Brian.
  Edward: It was Derek. It wasn't Arnold.

- What puzzles and games did you find were the most fun to play? Why?
- Using a game or puzzle of your choice, write a description of the game/puzzle, the rules of play, and helpful hints.
- Play "Fast Figuring."

  Using the number cards from an ordinary deck, five cards are dealt to each player.
  One more card is turned up to reveal the target number. Players race to use their five cards and any four operations (+, –, ×, ÷) to form a statement that results in the target number.
  The first player to do so wins a point.
  If, after three minutes, no one can find a solution, the players show their hands for checking, then the cards are shuffled and the play continues.
- Play "Roll Six."

  Each player rolls six dice and uses five of the numbers together with any four operations to make the sixth number. Points are scored for successful equations.
- Working with a partner, pick a game of a puzzle that you feel comfortable with.
  (a) Explain the rules of the game in their own words to other students.
  (b) Give a brief demonstration.
  (c) Explain the strategy you tried in solving the puzzle or playing the game.
  (d) What general advice would you give to other students trying to solve the puzzle or play the game?
  (e) What did you do when you were having difficulty?

- Find a game and puzzle that involves numerical or logical reasoning and that has not been covered in class. This may involve bringing games or puzzles from home or searching the Internet to find games, puzzles, or online applications of interest. Answer the following questions:
  - (a) What is the puzzle/game and where did you find it?
  - (b) Describe the puzzle/game. Why did you select this puzzle/game?
  - (c) What is the object of the game and the rules of play?
  - (d) What strategy was used to solve the puzzle or win the game?
- Many logic puzzles are based on statements that contain clues to the solution of the problem. These clues may be positive or negative (i.e., they may tell you part of the answer or tell you what the answer is not). Why can it be just as important to know what can be eliminated as it is to find the actual correct answer? Give an example.
- Given a 5 × 5 grid, arrange the numbers in order, from 1 to 25. The first five numbers, from 1 to 5, must be ordered on the first row, the next 5 numbers on the second row and so on.
  - (a) Choose any number in the first row and circle it. Then cross out all the numbers in the same column below the circled number.
  - (b) Circle any one of the remaining numbers in the second row. Then cross out all the numbers in the same column above and below the circled number.
  - (c) Repeat this process for the third and fourth row.
  - (d) Circle the remaining number in the fifth row and cross out all the numbers in the same column above the circled number. What is the sum of the circled numbers?
  - (e) Repeat steps (a) to (d) by circling a different number in the first row. How does the sum of the circled numbers compare with your result above?
  - (f) What is the sum of the circled numbers on a 3 × 3 grid? 4 × 4 grid?
  - (g) Choose a grid of your choice and explain why the sum of the circled numbers is always the same.
- In a magic square, the sum of the numbers in each row, column, and diagonal is the same; in this case, 15.

	4	9	2	15	
	3	5	7	15	
	8	1	6	15	
15	15	15	15		15

  Rearrange the numbers so that the sums for each row, column, and diagonal are all different, and none of the sums is 15.
- Select any three-digit number whose digits are not all the same. Arrange the digits in decreasing order, and then rearrange them in increasing order and subtract the two three-digit numbers (Kaprekar's constant). With the resulting three-digit solution, repeat the process. If the difference consists of two digits, use a 0 if necessary. Consider the following example using the digits 2, 7, and 9:

972	963	954	954
– 279	– 369	– 459	– 459
693	594	495	495

This process will lead to the number 495 again.

(a) Working with a partner, apply the Kaprekar constant to a two-digit number in which the digits are not the same (i.e., 9 is written as 09). Compare your results, and discuss what you notice.

(b) Repeat the process for a four-digit number. Compare your results, and write a conjecture about the resulting number.

- Four friends met on Saturday morning for breakfast. Each friend ordered a different drink and breakfast meal during their visit, and when it was time to leave, each got a different drink to go. Using the following information, determine the first name of each friend, the drink, and meal each ordered for breakfast, and the drink each ordered to go.
  - Brenda had waffles, but not an espresso.
  - The friend who ordered the pancakes also ordered decaf coffee to go but didn't have cranberry juice.
  - The woman who ordered the omelet had water to drink, but she wasn't Amy.
  - The two friends who ordered juice were Emily and the friend who ordered an egg sandwich.
  - The friend who ordered a cappuccino didn't order orange juice.
  - Melony ordered a hot tea to go.

  Draw an organized list similar to the following:

	cranberry juice	milk	orange juice	water	egg sandwich	omelet	pancakes	waffles	cappuccino	decaf coffee	espresso	hot tea
Amy												
Brenda												
Emily												
Melony												
cappuccino												
decaf coffee												
espresso												
hot tea												
egg sandwich												
omelet												
pancakes												
waffles												

  Place an X in a box when you can eliminate that possibility. Place an O in a box that you know is correct. Use the completed table to answer the question.

- Use the following clues to solve the puzzle:
  - Amanda, Jahzara, Luk, and Sarah each have different coloured cars. One car is red, one is blue, one is white, and the other is black.

    Clue 1: Amanda's car is not red or white.

    Clue 2: Jahzara's car is not blue or white.

    Clue 3: Luk's car is not black or blue.

Clue 4: Sarah's car is red.

Determine which person has which car.

- Three containers can hold 19 L, 13 L, and 7 L of water, respectively.

  The 19-L container is empty. The other two are full. How can you can measure 8 L of water using no other container and no other water supply?

- Alanis is playing a game of sudoku. Explain how you could determine two of the values that are not filled in.

6	1							
	4	8		5	3	1		2
		3		8			4	5
4				9		5		6
	5	7		3		2		
3			5		8		1	
	3	9				6	2	1
7	2			6	5	3		
8		4	3		2		5	9

- Sanjai placed the circled numbers in the sudoku shown below. Identify and correct his errors.

4		1	2	9			7	5
2			3			8		
	7			8				6
			1		3		6	2
1		5				4		3
7	3	(7)	6		8			
6	(4)			2			3	
		7			1			4
8	9			6	5	1		7

- Explain why the blocks shown in bold need to be the numbers indicated.

2	1–		1
3+		7+	**4**
7+	1–		3
	4	3+ **1**	

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- The following are some tips for using games in the mathematics class.
  - Use games for specific purposes.
  - Keep the number of players to between two to four so that turns come around quickly.
  - Communicate to students the purpose of the game.
  - Engage students in post-game discussions.
  - Students need time to play and enjoy the game before analysis begins.
  - As students play a game, it is important to pose questions and engage them in discussions about the strategies they are using. They can then discuss the game, determine the winning strategies, and explain these strategies through demonstration, oral explanation, or in writing.
  - Before assigning games to individual students, try games together as a group, as instructions to games are not always clearly understood.
- When introducing games, students will need to understand the rules and procedures of the game. Consider the following:
  - Introduce the game to one group of students while others are completing individual work. Then divide the whole class into groups, putting a student from the group learning the game into each of the other groups to teach them the game.
  - Choose students to play the game as a demonstration, possibly with assistance in decision making from the whole class.
  - Divide the class into groups. Play the game with the whole class, with each group acting as a single player.
- When working with puzzles and games, teachers could set up learning stations. Consider the following tips when creating the stations.
  - Depending on the nature of the game, some stations may require multiple games, while another station may involve one game that requires more time to play.
  - Divide students into small groups. At regular intervals, have students rotate to the next station.
- As students play games or solve puzzles, ask probing questions and listen to their responses. Record the different strategies, and use these strategies to begin a class discussion. Possible discussion starters include the following:
  - Thumbs up if you liked the game, thumbs sideways if it was okay, and thumbs down if you did not like it. What did you like about it? Why?
  - What did you notice while playing the game?
  - Did you make any choices while playing?
  - Did anyone figure out a way to quickly find a solution?
- Find simple versions of the games, and increase the difficulty as students gain experience.
- Have students look for patterns, and then develop a strategy to fit these patterns.
- Have students develop a game for classmates to play.

- Change a rule or parameter to a well-known or familiar game and explain how it affects the outcome of the game. For example, playing checkers and being allowed to move a piece back as well as forward.
- There may be situations where students are able to play the game and solve problems but are unable to determine a winning strategy. Teachers could participate with the group and think through the strategies out loud so the group can hear the reasoning for selected moves. Ask the groups' opinions about moves in the game and facilitate discussions around each of the other players' moves and strategies.
- Have students find a game online and critique the quality of the game.
- Do not confine the class to paper-and-pencil games and puzzles. Try to have at least a few options that are more hands-on, such as chess, Rubik's cubes, Chinese checkers, and backgammon. Another option is to use online flash games that require strategies.
- Have students consider various games, and determine which is better—guess and check or eliminating possibilities?
- Have students create a strategy journal organized into sections for individual or group, and number or logic games and puzzles. They should use this journal to track strategies and solutions used throughout the course.
- To create a game, students could use the rules of an existing game, but use different materials or add extra materials. They could also use the idea for a game and change the rules. Another option is to use a board game and add mathematics tasks to it. Rather than writing tasks directly onto the boards, they can place coloured stickers on certain spaces and make up colour-coded cards with questions. A game such as Snakes and Ladders, for example, can be modified to Operations Snakes and Ladders. The board can be used with two dice. On each turn, to determine the number of spaces to move, the player has the option of multiplying, dividing, adding, or subtracting the two numbers, with a maximum answer of twenty.
- The following guiding questions could be used to help students evaluate their games.
  - Can the game be completed in a short time?
  - Is there an element of chance built in?
  - Are there strategies that can be developed to improve the likelihood of winning?

Suggested Models and Manipulatives

- Various board games and puzzles (a short list is provided below to illustrate the types of games and puzzles that are primarily spatial, numerical, or logical).

Spatial	Numerical	Logical
Othello tangrams pentominoes Free Flow Blokus	KenKen Kakuro cribbage	sudoku chess checkers backgammon

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- conjecture
- elimination
- systematic list
- logic
- strategy

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Integrated throughout text and course

Internet

There are many Internet sites that provide interactive games and puzzles for use in addressing this outcome.

SCO LR02 Students will be expected to solve problems that involve the application of set theory. [CN, PS, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

LR02.01 Provide examples of the empty set, disjoint sets, subsets, and universal sets in context, and explain the reasoning.
LR02.02 Organize information such as collected data and number properties using graphic organizers, and explain the reasoning.
LR02.03 Explain what a specified region in a Venn diagram represents, using connecting words (and, or, not) or set notation.
LR02.04 Determine the elements in the complement, the intersection, or the union of two sets.
LR02.05 Explain how set theory is used in applications such as Internet searches, database queries, data analysis, games and puzzles.
LR02.06 Identify and correct errors in a given solution to a problem that involves sets.
LR02.07 Solve a contextual problem that involves sets, and record the solution, using set notation.

Scope and Sequence

Mathematics 11	**Mathematics 12**
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry.	**LR02** Students will be expected to solve problems that involve the application of set theory.

Background

In this unit, students organize information into sets and subsets. They use Venn and Euler diagrams to illustrate relationships between sets and subsets and use set notation to describe sets.

Students will determine the number of elements in each region of a Venn diagram and then determine the number of elements in the union of two or three sets.

Sets are a mathematical way to represent a collection or a group of objects, called elements. Students often deal with a set of books, for example, or a collection of hockey cards.

A set, *A*, of elements 11, 12, 13, 14, and 15 can be represented by

- listing the elements; $A = \{11, 12, 13, 14,\ 15\}$
- using words or a sentence; $A = \{\text{all integers greater than 10 and less than 16}\}$
- using set builder notation; $A = \{x | 10 < x < 16, x \in Z\}$

The number of elements in a set *A* is denoted by *n*(*A*), *n*(*A*) = 5.

In Mathematics 10, students worked with domain and range using set notation and interval notation (RF01).

Students should be introduced to the following types of sets: universal set, subset, empty set, disjoint set, and complement of a set.

They will understand and be able to explain statements such as, The sum of the number of elements in a set, *A*, and its complement,A', is equal to the number of elements in the universal set, *U* and $n(A) + n(A') = n(U)$.

Students will examine the relationships between sets and subsets. They will differentiate between finite and infinite sets, and disjoint and overlapping sets. Students will determine if two sets intersect, and when they do, identify the number of elements in each region and what each region represents.

Students should understand that each element in a universal set appears only once in a Venn diagram. If an element occurs in more than one set, it is placed in the area of the Venn diagram where the sets overlap.

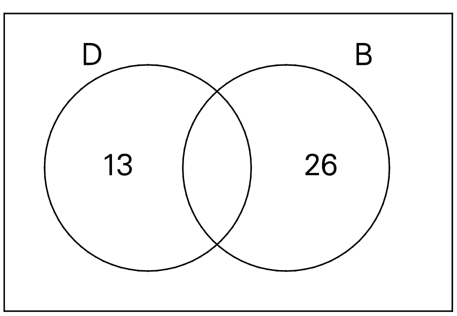
The intersection of two sets, denoted $A \cap B$, consists of all the elements that are common to both sets. It is represented by the region of overlap on a Venn diagram. Intersection is indicated by the word *and*.

The union of two sets, denoted $A \cup B$, consists of all the elements that are in at least one of sets *A* and *B*. This union is represented by the combined region of these sets on a Venn diagram and is indicated by the word *or*.

When two sets *A* and *B* are disjoint, the number of elements in *A* or *B*, denoted$n(A \cup B)$, is $n(A \cup B) = n(A) + n(B)$.

If two sets, *A* and *B*, contain common elements, the number of elements in *A* or *B*, denoted$n(A \cup B)$, is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

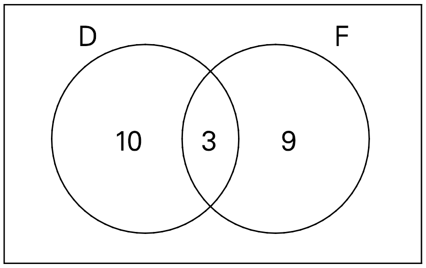
For example, in a standard deck of 52 cards, the sets *D* (all diamonds) and *B* (all black cards) are disjoint since there is no overlap.



$$n(D \cup B) = n(D) + n(B) \text{ since } n(D \cap B) = 0$$

Specifically, $n(D \cup B) = 13 + 26 = 39$.

In contrast, the sets *D* (all diamonds) and *F* (all face cards) have some cards in common, so they are not disjoint sets.



$$n(D \cup F) = n(D) + n(F) - n(D \cap F)$$

Specifically, $n(D \cup F) = 13 + 12 - 3 = 22$

A method will be developed and then formalized for determining the number of elements in the union of two sets, $n(A \cup B)$.

When calculating the number of elements in the union of two sets, students should conclude that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. This result is known as the principle of inclusion and exclusion. Students are asked to determine the value of $n(A \cap B)$ if the sets are disjoint. This mathematical principle informs students how to keep track of what to add and what to subtract in problems. It is important to give students a choice to either use a Venn diagram or the formula to solve problems involving union and intersections.

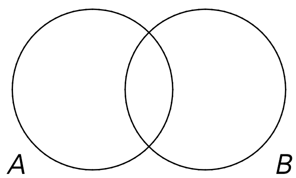
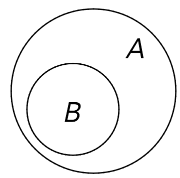
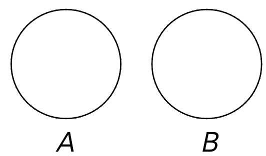
Students will use sets and set notation to solve problems and will discover practical applications of set theory, such as conducting Internet searches, solving puzzles, and solving problems that involve three sets.

It is important to note that this is not meant to be a study in abstract algebra. As described by the performance indicators, the goal of this outcome is to have students be proficient in organizing data based on characteristics and understanding relationships between elements. The various notations should be the vehicle for communicating the connections in a mathematically valid way, but should be not the main focus of this outcome.

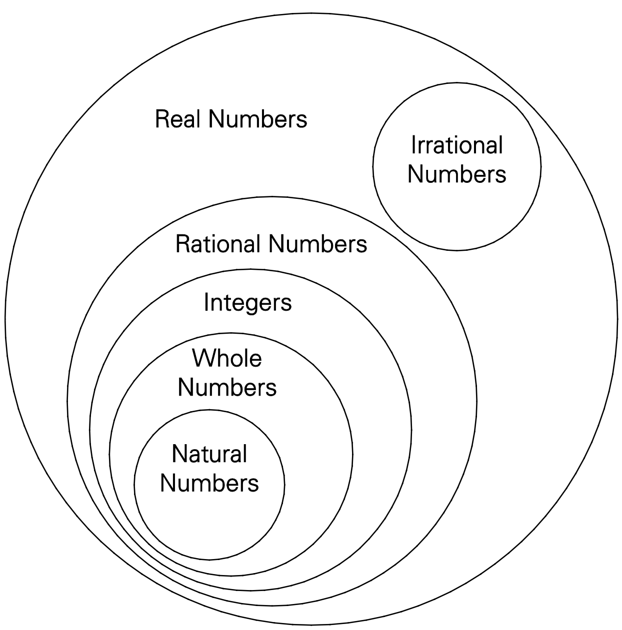
New terminology will include the following:

- **Element:** An item or object that is part of a set (e.g., 5 is an element in the set of prime numbers).
- **Empty set:** A set that contains no elements (e.g., the set of dogs with six legs). An empty set is represented as { } or ∅.
- **Subset:** A set in which all the elements of the set are also the elements of another larger set, represented as$A \subset B$.
- **Universal set:** A set that contains everything, including the set itself, represented as *U*.
- **Complement:** The set of all elements that are not in a defined set, represented as $\overline{A}, A', or\ A^c$.If a set is defined and has a certain number of elements, the complement set contains all elements that are not in the set.
- **Union:** The set of all elements of two or more sets. The union of set *A* and set *B* would be represented by $A \cup B$.
- **Intersection:** The set of all elements common to two or more sets. The intersection of set *A* and set *B* would be represented as $A \cap B$.

- **Disjoint sets** or **mutually exclusive sets:** Two sets that share no elements. The intersection produces a null set, represented as $A \cap B = \emptyset$.
- **Euler Diagrams:** Sets are represented by either intersecting circles or circles that do not intersect.
- **Venn Diagrams:** Sets are represented by intersecting circles, the intersection can be empty. Venn diagrams are a subset of Euler diagrams.

Venn Diagram (circles that intersect). *These are also Euler Diagrams.*  **Note:** Venn diagrams can be used for sets that are disjoint. The common region will be empty in this case.	Euler Diagram that is not a Venn Diagram (circles that do not intersect).  ***Or*** 

In Mathematics 10, students were introduced to Euler diagrams to organize the various subsets of the real number system (AN02).



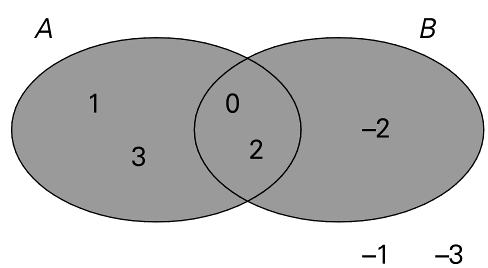
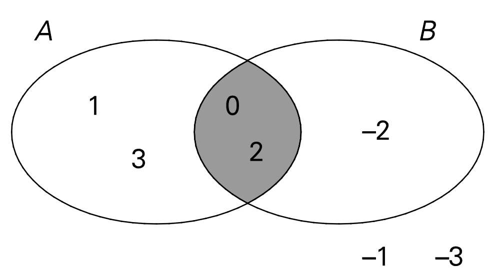
In this unit, students continue to organize information into sets and subsets using Euler and Venn diagrams and explore what the different regions of the Venn diagram represent.

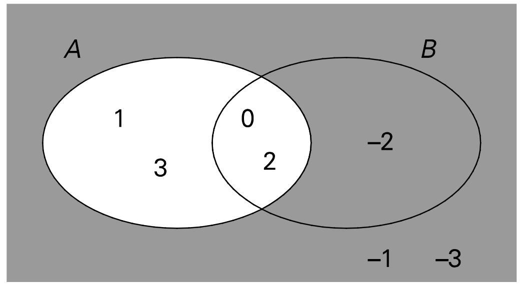
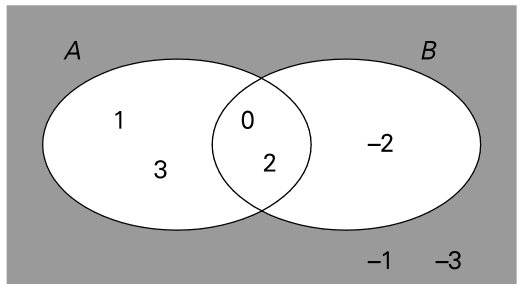
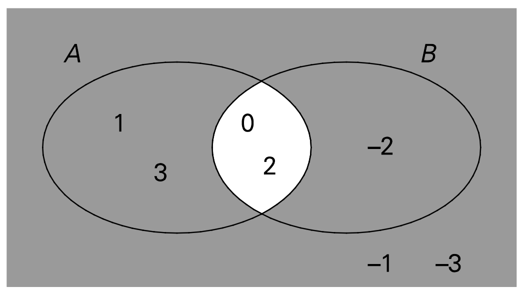
Students should be provided with a table, such as the one shown below, where the set notation and its associated definition is provided.

Notation	**Definition**
$\in$	Element of …
$\notin$	Not an element of …
$\subset$	Subset of …
$\not\subset$	Not a subset of …
$\subseteq$	A subset and equal to …
$\cup$	Union "OR"
$\cap$	Intersection "AND"
$A', \overline{A}, A^c$	Complement of *A* "NOT"
$\emptyset$	Empty set

The following chart will assist in seeing the link between set notation and Venn diagrams.

- *Universal set:* All integers from –3 to +3.
- *Set A:* Non-negative integers between –3 and +3.
- *Set B:* Integers divisible by 2 that are between –3 and +3.

Set Notation	**Meaning**	**Venn Diagram**	**Answer**
$A \cup B$ (*A* union *B*)	any element that is in either of the sets		{–2, 0, 1, 2, 3}
$A \cap B$ (*A* intersect *B*)	only elements that are in both *A* and *B*		{0, 2}

Set Notation	Meaning	Venn Diagram	Answer
A' (*A* complement or not *A*)	all elements in the universal set outside of *A*		{–3, –2, –1}
$(A \cup B)'$ not (*A* union *B*)	elements outside *A* and *B*		{–3, –1}
$(A \cap B)'$ not (*A* intersect *B*)	elements outside of the overlap of *A* and *B*		{–3, –2, –1, 1, 3}

Given a contextual problem, students will be expected to draw a Venn diagram and write the elements that correspond to the notation.

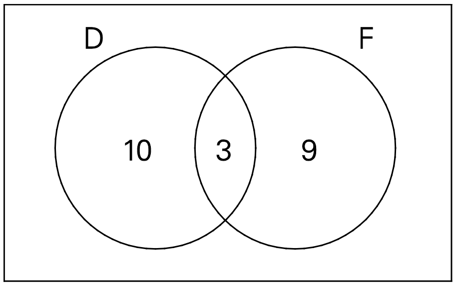
It is important to give students a choice to use either a Venn diagram or the formula to solve problems involving union and intersections.

Example:

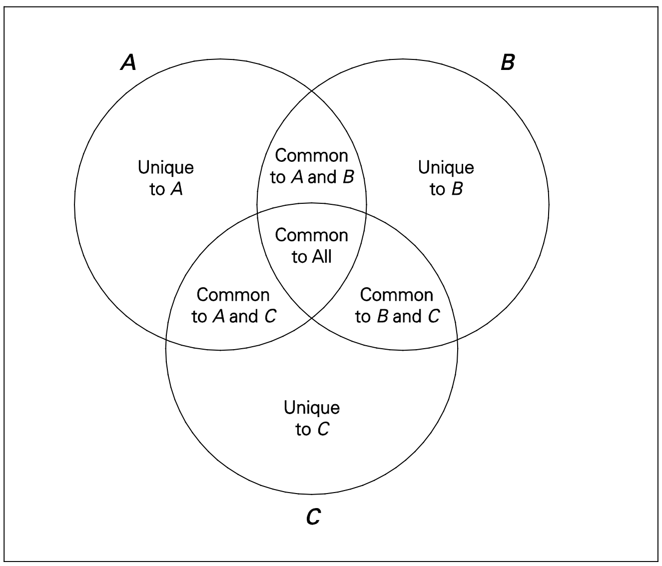
- Given Set *D* (all diamonds in a standard deck) and Set *F* (all face cards in a deck), students could compute$n(D \cup F)$ using the formula

$$n(D \cup F) = n(D) + n(F) - n(D \cap F)$$
$$n(D \cup F) = 13 + 12 - 3 = 22$$

or using a Venn diagram as shown below.



Students should analyze three intersecting sets.



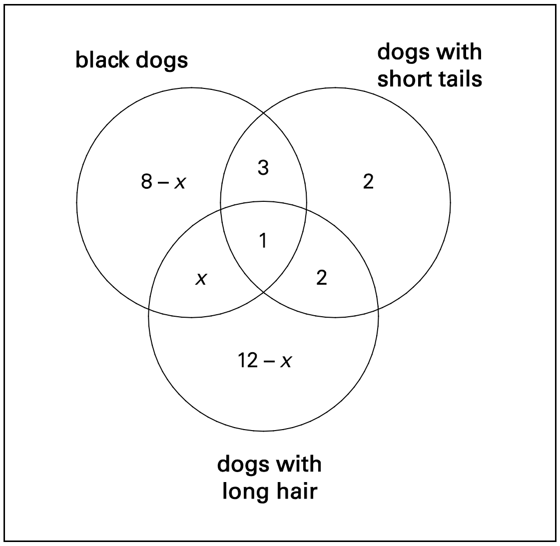
Students should observe that there is one region in which all three sets overlap, $A \cup B \cup C$. There are three regions in which exactly two of the three sets overlap, $A \cup B$; $A \cup C$; and $B \cup C$. When working with three intersecting sets, students continue to use the principle of inclusion and exclusion to determine the number of elements in different sets.

A formula can be used to determine $n(A \cup B \cup C)$. Due to the complexity of the formula, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$, students are more likely to make errors, hence a Venn diagram would be a more efficient strategy for students.

There are problems involving sets where students should use an equation to solve the problem. They should use the clues or information to define the sets, identify regions of the Venn diagram, and identify the region of interest using a variable.

Example:

There are 25 dogs at the dog show. Twelve of the dogs are black, 8 of the dogs have short tails, and 15 of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Three of the dogs are black with short tails and do not have long hair. Two of the dogs have short tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?



We do not know how many dogs are black and have long hair so let that region be represented by x.

We are then able to determine two other regions to be $12 - x$ and $8 - x$.

This gives us the equation $(8 - x) + (x) + (4) + (4) + (12 - x) = 25$.

Solving the equation, $(8 - x) + (x) + (4) + (4) + (12 - x) = 25$, students should determine there are three dogs that are black with long hair.

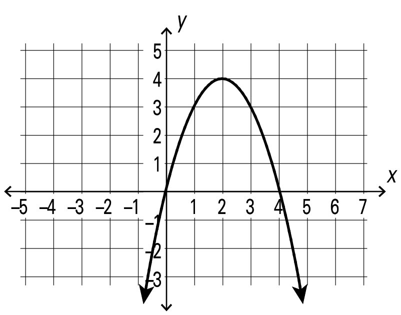
Assessment, Teaching, and Learning

Assessment Strategies

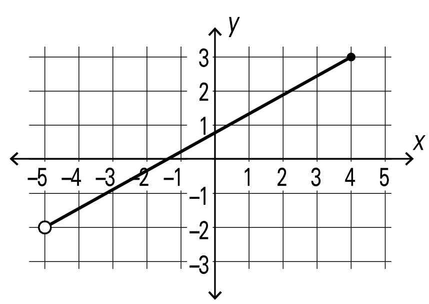
Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

Use set notation to describe the domain and range of the data sets illustrated below:



Graph A



Graph B

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Explain whether a set can be considered a subset of itself.

- Mary created the sets *P* {1, 3, 4} and *Q* {2, 3, 4, 5, 6}. John stated that $P \subset Q$ since the elements 3 and 4 are in both sets. Do you agree or disagree with John? Explain the reasoning.
- Consider the following sets:
  - *U* is the universal set of playing cards in a standard 52-card deck
  - *S* is the set of all 13 spades
  - *B* is the set of all 26 black cards (spades and clubs)
  - *D* is the set of all 13 diamonds

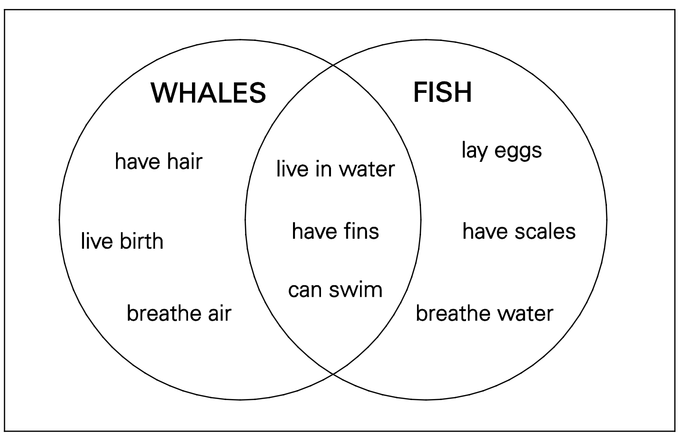
  (a) Which of these sets are subsets of other sets?

  (b) List the disjoint sets, if there are any.

  (c) Represent the sets using a Venn diagram.
- Create your own disjoint sets and subsets. Draw a Venn diagram for each situation. Consider the following example:

  C = Set of consonants in the alphabet

  V = Set of vowels in the alphabet
- Explain the different regions of the Venn diagram shown below, using the terms **sets**, **subsets**, and **disjoint**.



- Use a Venn diagram to represent the set of squares and the set of quadrilaterals.

  (a) Why is the set of squares a subset of the set of quadrilaterals?

  (b) Are there other subsets?

  (c) What does the region outside of the quadrilateral circle represent?
- *R* is the set of positive odd numbers less than 10. *S* is the set of multiples of 3 between 4 and 20. *T* is the set of prime numbers less than 12.

  (a) List the elements of

  (i) $R \cup S$

  (ii) $R \cap S$

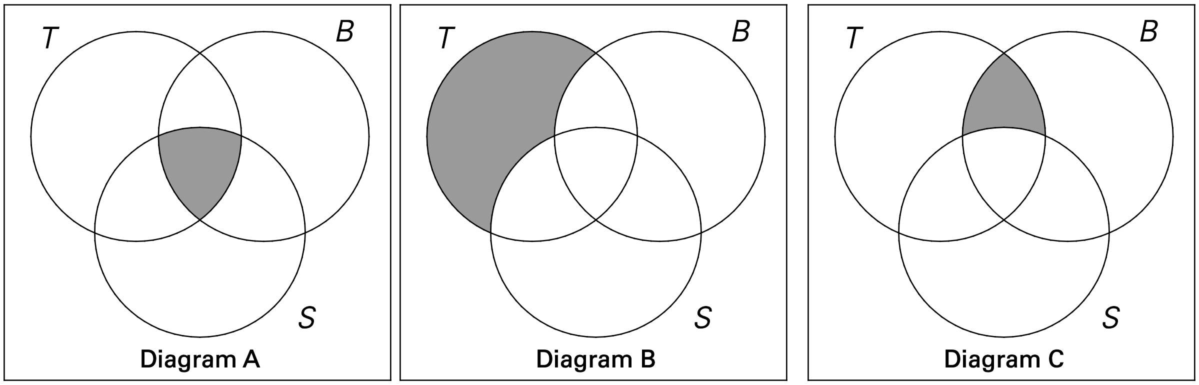
  (b) What does it mean to write$x \in (R \cap T)$? List all possible values of x.

  (c) Is it true that $S \cap T$ is the empty set? Explain your answer.
- The diagrams below represent the activities chosen by youth club members. They can choose to play tennis (*T*), baseball (*B*), or go swimming (*S*). Decide which diagram has the shading that represents the following descriptions:

(a) Those who participate in all three sports.

(b) Those who play tennis and baseball, but do not swim.

(c) Those who play only tennis.



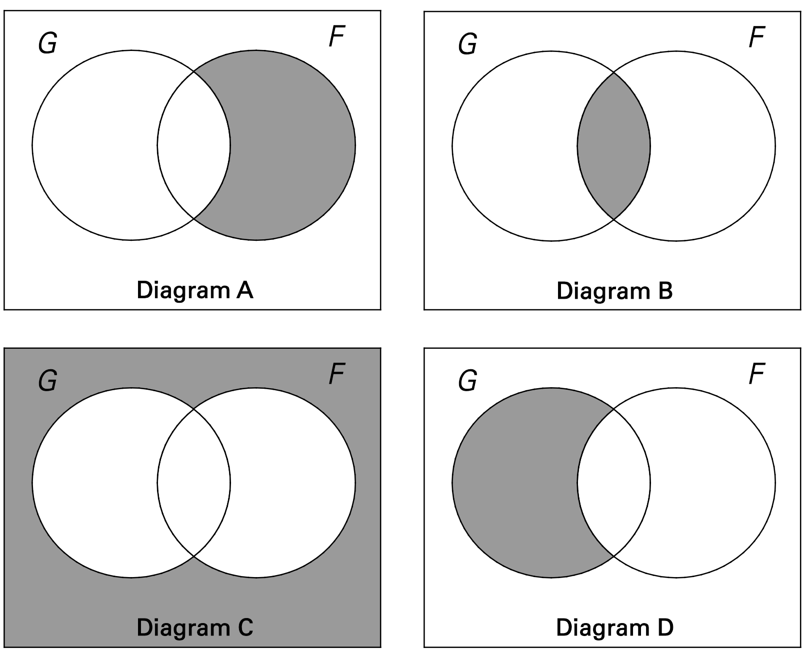
- The diagrams below represent a class of children. *G* is the set of girls and *F* is the set of children who like fencing. Decide which diagram has the shading that represents

  (a) girls who like fencing

  (b) girls who dislike fencing

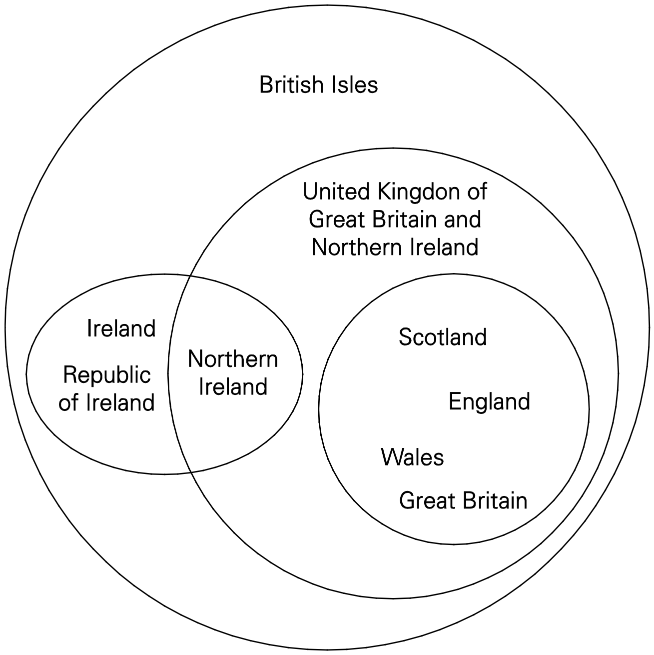
  (c) boys who like fencing

  (d) boys who dislike fencing



- Of 26 people in a class, 18 students sing in the chorus, 12 play in the jazz band, and 5 students are in both the chorus and the band. Create a Venn diagram and then determine how many students neither sing in the chorus nor play in the jazz band.

- Based on the diagram shown, which of the following statements are true?

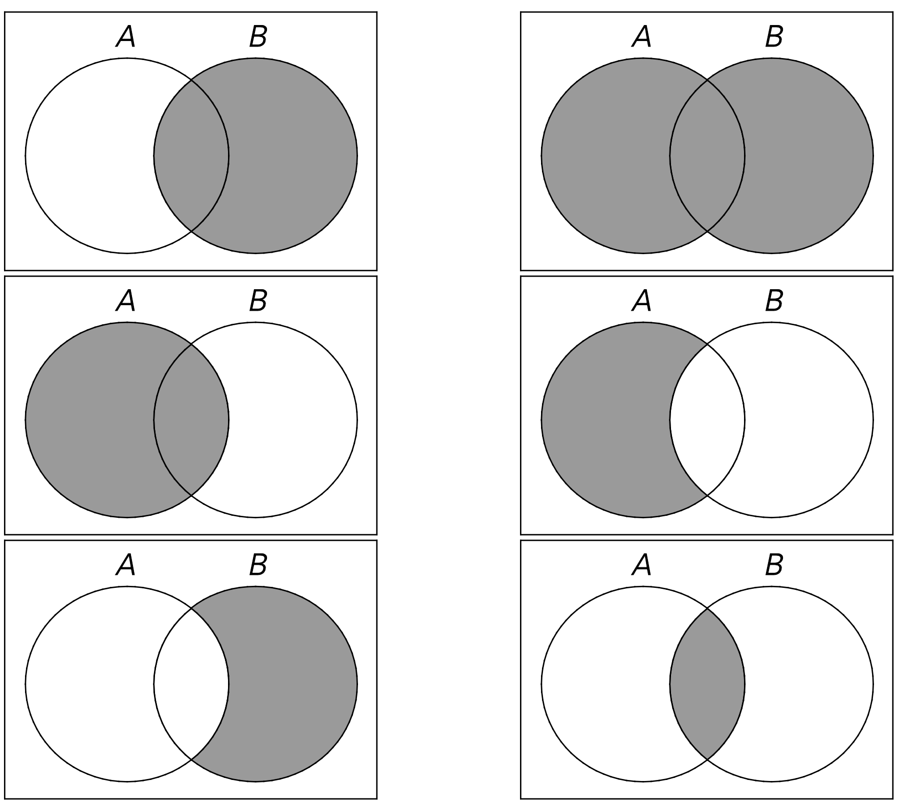


(a) Scotland is part of Great Britain, the United Kingdom, and the British Isles.

(b) Northern Ireland is part of Great Britain, the United Kingdom, and the British Isles.

(c) Ireland is part of the United Kingdom.

- Use appropriate set notation such as $A \cup B$ or $A \cap B$ to describe each of the regions shown below.



- A group of 30 students are surveyed to find out which of the three sports, soccer (*S*), basketball (*B*), or volleyball (*V*), they play. The results are as follows:
  - three children do not play any of these sports
  - two children play all three sports
  - six play volleyball and basketball
  - three play soccer and basketball
  - six play soccer and volleyball
  - 16 play basketball
  - 12 play volleyball

  (a) How many students play soccer only?

  (b) How many students play soccer but not basketball?

  (c) How many students play volleyball but not basketball?
- There are 36 students who study science.
  - 14 study physics, 18 study chemistry, 24 study biology
  - five study physics and chemistry, eight study physics and biology
  - 10 study biology and chemistry, three study all three subjects

  (a) Determine the number of students who study physics and biology only.

  (b) Determine the number of students who study at least two subjects.

  (c) Determine the number of students who study biology only.
- A survey of a machine shop reveals the following information about its employees:
  - 44 employees can run a lathe
  - 49 employees can run the milling machine
  - 56 employees can operate a punch press
  - 27 employees can run a lathe and a milling machine
  - 19 employees can run a milling machine and operate a punch press
  - 24 employees can run a lathe and operate a press punch
  - 10 employees can operate all three machines
  - nine employees cannot operate any of the three machines

  Determine the number of people employed at the machine shop.
- A firm manufactures three types of shampoo: Shine, Bubble, and Glory. 1000 families were surveyed, and the results were posted in an advertisement as follows:
  - 843 use Shine
  - 673 use Bubble
  - 585 use Glory
  - 600 use both Shine and Bubble
  - 423 use both Shine and Glory
  - 322 use both Bubble and Glory
  - 265 use all three types

Identify where the error occurs in the survey results as reported in the advertisement. **Note:** It may be helpful to illustrate the information using a Venn diagram.

- When performing a Web search, you are defining a set as the collection of those websites that have some common feature in which you are interested. There are three logical operators, *and*, *or*, and *and not* that help in narrowing your search to a manageable number of sites that might be most useful for a particular project.
  (a) What do you know about making effective searches for information on the Internet? How is this related to set theory?
  (b) In groups of two, conduct an Internet search on a topic of your choice. Record each successive search, including the number of hits. Challenge: obtain 100 hits or fewer.
- 40 members in a sports club were surveyed:
  - two play all three sports
  - 23 play ball hockey
  - 24 play tennis
  - 18 play golf
  - 14 play tennis and ball hockey
  - eight play tennis and golf
  - one member makes the refreshments and does not play any sport

  Determine the number of people who play both ball hockey, and golf.
- In a survey of 55 people, the following results were recorded:
  - 13 people like Hawaiian pizza
  - 19 people like pepperoni pizza
  - 26 people like cheese pizza
  - 15 people do not like pizza
  - five people like Hawaiian pizza and pepperoni pizza, but not cheese pizza
  - two people likes all types of pizza
  - two people like Hawaiian pizza and cheese pizza, but not pepperoni pizza

  Determine how many people like only cheese pizza.

Planning for Instruction

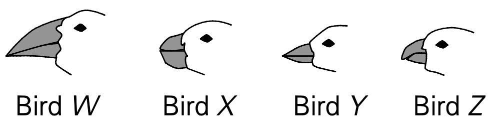
SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Have permanent (reusable) examples of Euler and Venn Diagrams on the wall in the classroom or a set of reusable diagrams for student use.
- It may be beneficial to start this section with a discussion about the number systems (natural, whole, integer, etc.) as these would be familiar and serve as a good example of using an Euler diagram.

- Give students a set of playing cards and have them examine and sort the cards based on their set properties. They should then create graphic organizers and use set notation to describe the relationships.
- Have groups interview individual students on various characteristics (on either/or items, such as owning a dog or having a sister) and create a Venn diagram representing the various sets. Then have a different group analyze the intersections and unions of the sets.
- Cross-curricular strategies may include having students analyze dichotomous keys (e.g., from biology) and turning them into an Euler diagram. For example,

  The dichotomous key shown below can be used to identify birds *W*, *X*, *Y* and *Z*.



Dichotomous Key to Representative Birds		
1.	(a) The beak is relatively long and slender	*Certhidea*
	(b) The beak is relatively stout and heavy...	go to 2
2.	(a) The bottom surface of the lower beak is flat and straight	*Geospiza*
	(b) The bottom surface of the lower beak is curved.....................................	go to 3
3.	(a) The lower edge of the upper beak has a distinct bend	*Camarhynchus*
	(b) The lower edge of the upper beak is mostly flat	*Platyspiza*

  Bird *X* is most likely

  – Certhidea
  – Geospiza
  – Camarhynchus
  – Platyspiza

- Provide students with a written description, such as Set M consists of the multiples of 3 from 1 to 100.

  Ask students to first rewrite the set as a list and then progress to an algebraic expression.

  M = {3, 6, 9, …, 93, 96, 99}

  M = 3*x*, $\{x \mid 1 \leq x \leq 33, x \in N\}$

  To promote discussion, ask students the following questions:

  – Is one form more efficient than the other?
  – What does $1 \leq x \leq 33$ represent?
  – Why is it important to state that *x* belongs to the natural numbers?

- Ask students the following questions about some specific sets, such as those indicated below, to initiate discussion around the types of sets:

  Set A: all numbers that multiply to 12

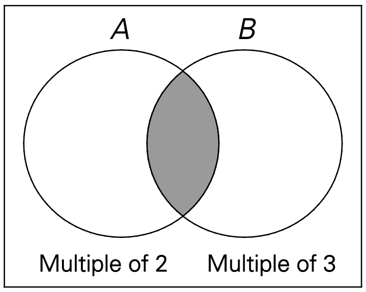
  Set B: all whole numbers that add to 7

  – Is it possible to explicitly list all the elements of these sets?
  – Which set is finite and which set is infinite?
  – Is one set a subset of the other?

- Set *A* or Set *B* could be defined as the universal set for this problem. Are there elements of the universal set that do not belong to the subset?
- What is the complement set of *B*, defined as B'?

- Introduce students to examples of the empty set, such as the set of months with 32 days or the set of squares with five sides. Ask them to create their own empty sets and share their responses with the class.
- Provide students with a set such as the following:

  S = {4, 5, 6, 8, 9, 11, 15, 17, 20, 24, 30, 32} and the categories for its associated Venn diagram.



  As students place the numbers in the appropriate location, they should consider

  - the universal set and how many elements are in this universal set
  - the elements of each subset
  - why the two circles overlap
  - the common elements in Set *A* and Set *B*
  - why some numbers are not placed in either circle

  Ask students to add another circle to the Venn diagram to represent multiples of 4. They should think about where they should place the circle and whether the three circles overlap.

- Discuss with students that there are situations where two (or more) sets may have no elements in common, known as disjoint sets. Ask students to draw a Venn diagram, for example, for the multiples of 3 and the factors of 8. Inform students that the two sets are defined as mutually exclusive.
- Ask students to participate in the activity Find Your Partner. Half of the students should be given a card with a description on it and the other half should be given a card with a Venn diagram. Students need to move around the classroom to match the description with the correct Venn diagram. They should then present their findings to the class.
- Provide students with the following sets and ask them to predict the number of elements in the union of Sets *A* and *B*.

  Set *A* has elements {2, 3, 6, 8, 9}

  Set *B* has elements {4, 5, 6, 7, 9}

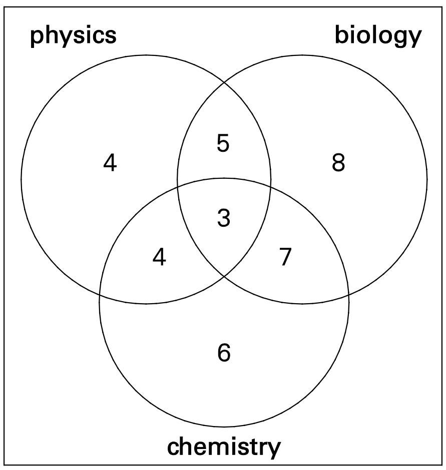
  Using an algebraic approach, students may determine that $n(A) = 5$ and $n(B) = 5$ and initially make the assumption that there are ten elements in the union of Sets *A* and *B*. Using a visual representation, such as a Venn diagram, should help them understand that $n(A \cup B) = 8$.

  Use the following prompts to help students develop a formula for $n(A \cup B)$.

  - Which elements were added twice when you added $n(A)$ and $n(B)$?

- How can you compensate for this over-counting?
- Predict a formula for $n(A \cup B)$ using $n(A)$, $n(B)$ and $n(A \cap B)$.
- When will $n(A \cup B) = n(A) + n(B)$?

- A Venn diagram enables students to visualize the resulting set and easily account for the repetition of common elements. When students start filling out a three-set Venn diagram, encourage them to start by completing the centre, then the two set intersections, before filling the remainder of each set.
- A common student error occurs when students do not take into account the overlapping region of the three sets. If eight people studied physics and biology, they sometimes write 8 instead of 5 forgetting that three people studied all of physics, biology, and chemistry. Students may draw an incorrect Venn diagram, such as the following:



- When completing a Venn diagram, ask students to identify how the sets overlap as well as the region that satisfies neither of the sets. When these regions are completed, it may become easier for students to identify what the totals should be for each circle. Remind students that the numbers in all the regions total the number in the universal set.

Suggested Models and Manipulatives

- standard decks of cards
- Venn and Euler diagrams

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- complement
- disjoint set
- element
- empty set
- Euler diagram
- finite set
- infinite set
- intersection
- mutually exclusive
- principle of inclusion and exclusion
- set
- subset

- union
- universal set
- Venn diagram

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 3.1, 3.2, and 3.3, pp. 146–175
  - Section 3.4, pp. 179–194

SCO LR03 Students will be expected to solve problems that involve conditional statements. [C, CN, PS, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

LR03.01 Analyze an “if-then” statement, make a conclusion, and explain the reasoning.
LR03.02 Make and justify a decision, using “what if?” questions, in contexts such as probability, finance, sports, games, or puzzles, with or without technology.
LR03.03 Determine the converse, inverse, and contrapositive of an “if-then” statement; determine its veracity; and, if it is false, provide a counterexample.
LR03.04 Demonstrate, using examples, that the veracity of any statement does not imply the veracity of its converse or inverse.
LR03.05 Demonstrate, using examples, that the veracity of any statement does imply the veracity of its contrapositive.
LR03.06 Identify and describe contexts in which a biconditional statement can be justified.
LR03.07 Analyze and summarize, using a graphic organizer such as a truth table or Venn diagram, the possible results of given logical arguments that involve biconditional, converse, inverse, or contrapositive statements.

Scope and Sequence

Mathematics 11	**Mathematics 12**
LR01 Students will be expected to analyse and prove conjectures, using inductive and deductive reasoning, to solve problems.	**LR03** Students will be expected to solve problems that involve conditional statements.

Background

In Mathematics 11, students analyzed and proved conjectures using inductive and deductive reasoning (LR01). In this course, students will use various organizational tools to consider the veracity of conditionals statements, converse statements, inverse statements, and contrapositive statements.

A conditional statement consists of a hypothesis, *p*, and a conclusion, *q*. Different ways to write a conditional statement include

- If *p*, then *q*
- *p* implies *q*
- $p \to q$ or $p \Rightarrow q$

Think of it this way:

- Your aunt tells you that if you help her organize her office, she will make you a chocolate cake.

Fact 1: You help her organize her office.

Fact 2: You get a chocolate cake.

When is your aunt's statement true?

- If you help your aunt organize her office (Fact 1 true) and she bakes you a chocolate cake (Fact 2 true), then her statement is *true*.
- If you help your aunt organize her office (Fact 1 true) and she does not make you a chocolate cake (Fact 2 false) then her statement is *false*.
- If you do not help your aunt organize her office (Fact 1 false), we cannot judge the truth of your aunt's statement. She did not tell you what would happen if you did *not* help her organize her office. Since we cannot accuse her of making a false statement, we assign *true* to the statement regardless of whether the second fact is true or false.

Therefore your aunt's statement, If you help me organize my office, then I will make you a chocolate cake, will be true in all cases except one: when you help her organize her office and she does *not* make you a chocolate cake.

In general, conditional statements are *false* only when the first condition (if) is true and the second condition (then) is false. All other cases are *true*.

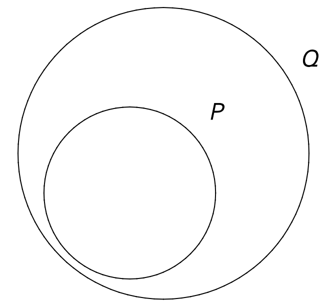
Symbols and truth tables can be used to represent concepts in logic. The use of these variables, symbols, and tables creates a shorthand method for discussing logical sentences.

p: You help your aunt organize her office.
q: Your aunt makes you a chocolate cake.

p	***q***	$p \Rightarrow q$	**Interpretation**	**Veracity of statement:** *You help your aunt organize, and she will make you cake.*
T	T	T	You help your aunt organize; she makes you a cake.	True
T	F	F	You help your aunt organize; she does not make you a cake.	False
F	T	T	You do not help your aunt organize; she makes you a cake.	Cannot be sure, so assume true.
F	F	T	You do not help your aunt organize; she does not make you a cake.	Cannot be sure, so assume true.

Note that a conditional statement is false only when the hypothesis is true and the conclusion is false. Otherwise, the conditional statement is true, even if the hypothesis is false.

Alternatively, a conditional statement that is true can be represented using an Euler diagram with the inner oval representing the hypothesis and the outer oval representing the conclusion, as shown to the right. In other words, IF it is true that if I help my aunt organize her office, *p*, then she will make me a cake, *q*, then helping her organize her office is a subset of my aunt making me a cake. In other words, the statement "*p* implies *q*," means that *p* is a subset of *q*.



Only one counterexample is needed to show that a conditional statement is false.

For example, we know that the conditional statement, If I draw a red card from a standard deck, then I will have drawn a heart, is false since I could have drawn the 5 of diamonds.

To write the **converse of a conditional statement**, switch the hypothesis and the conclusion. In other words, the converse of "If *p*, then *q*," is "If *q*, then *p*."

The statement: If I help my aunt organize her office, then she will make me a chocolate cake.
The converse: If my aunt makes me a chocolate cake, then I will help my aunt organize her office.

It is important to remember that the converse does *not* necessarily have the same truth value as the original conditional statement. This is because we are verifying the truth of a different statement.

Truth Table			**Interpretation of Truth Table**	**Veracity of statement:** *Your aunt makes you a cake, you help her clean.*
p	*q*	$q \Rightarrow p$	*p*: I help clean; *q*: my aunt makes cake	$q \Rightarrow p$
T	T	T	My aunt makes cake; I help clean.	True
T	F	T	My aunt does not make cake; I do help clean.	Cannot be sure, so assume true.
F	T	F	My aunt makes cake; I do not help clean.	False
F	F	T	My aunt does not make cake; I do not help clean.	Cannot be sure, so assume true.

If a conditional statement and its converse are both true, then they can be combined to create a **biconditional statement** using the phrase, "if and only if" or "iff."

It is true that if a triangle is equilateral, then all its angles measure 60°. It is also true, if all angles in a triangle measure 60°, then the triangle is equilateral.

Therefore, this biconditional statements can be written as, All angles in a triangle measure 60° iff the triangle is equilateral. Or A triangle is equilateral iff all its angles measure 60°.

The **inverse of a conditional statement** is formed by negating the hypothesis and negating the conclusion of the original statement. In other words the inverse of "If *p*, then *q*," is "If not *p*, then not *q*." Is this correct? Or symbolically, "if $\sim p$ then$\sim q$" or "if $\neg\, p \Rightarrow \neg\, q$."

The statement: If I help my aunt organize her office, then she will make me a chocolate cake.
The inverse: If I do not help my aunt organize her office, then she will not make me a chocolate cake.

The symbol $\neg P$ or $\sim P$ means *not P*. In other words, if *P* means, I help my aunt organize her office, then $\sim P$ means I do not help my aunt organize her office.

Truth Table			**Interpretation of Truth Table**	**Veracity of statement:** *You do not help your aunt clean, she does not make you a cake.*
$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$	*p*: I help clean. *q*: My aunt makes cake. $\neg p$: I do not help clean. $\neg q$: My aunt does not make cake.	$\neg p \Rightarrow \neg q$
T	T	T	I do not help clean; my aunt does not make cake.	True
T	F	F	I do help clean; my aunt does not make cake.	False
F	T	T	I do help clean; my aunt does make cake.	Cannot be sure, so assume true.
F	F	T	I help clean; my aunt makes cake.	Cannot be sure, so assume true.

The **contrapositive of a conditional statement** is formed by negating both the hypothesis and the conclusion, and then interchanging the resulting negations. In other words the contrapositive does *both* the jobs of the inverse and the converse.

The statement: If I help my aunt clean her office, then she will make me a chocolate cake.
The contrapositive: If my aunt does not make me a chocolate cake, then I will not help her clean her office.

Truth Table			**Interpretation of Truth Table**	**Veracity of statement:** *Your aunt does not make you a cake, you do not help her clean.*
$\neg p$	$\neg q$	$\neg q \Rightarrow \neg p$	*p*: I help clean. *q*: My aunt makes cake. $\neg p$: I do not help clean. $\neg q$: My aunt does not make cake.	$\neg q \Rightarrow \neg p$
F	F	T	My aunt does make cake; I do help clean.	Cannot be sure, so assume true.
F	T	F	My aunt does not make cake; I do help clean.	False
T	F	T	My aunt does make cake; I do not help clean.	Cannot be sure, so assume true.
T	T	T	My aunt does not make cake; I do not help clean.	True

An important fact to remember about the contrapositive, is that it always has the *same* truth value as the original conditional statement.

- If the original statement is true, the contrapositive is true.
- If the original statement is false, the contrapositive is false.
- The original statement and the contrapositive are said to be logically equivalent.

Note: It is important to note that this is not meant to be a study in logic or truth tables. As described by the achievement indicators, the goal of this outcome is to have students be determining the veracity of various statements and justifying their analysis. The various notations should be the vehicle for communicating the connections in a mathematically valid way, but should not be the main focus of this outcome.

For example, students should be able to consider the statement, If I draw a heart from a standard deck of cards, then it is red, as a true statement.

- **The converse:** If I draw a red card from a standard deck, then it is a heart is false. Students would give a counterexample such as the Ace of diamonds.
- **The inverse:** If I do not draw a heart from a standard deck of cards, then it is not red is false. Students would give a counter example such as the Jack of diamonds.
- **The contrapositive:** If I do not draw a red card from a standard deck, then it is not a heart is true.
- **The biconditional statement:** I draw a heart from a standard deck of cards, iff it is red is false since both the conditional statement and its converse are not true.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Are the following statements sometimes true, always true, or never true?
  (a) If I draw a red card from a deck of cards, it is a diamond.
  (b) If I draw a diamond from a deck, it is a red card.
- Anya states, "If the two diagonals of a quadrilateral are perpendicular, then the quadrilateral is a square." Is she correct? Explain.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Write the converse of the following statement: If a triangle has three equal angles, then it is an equilateral triangle.

- Determine if the following statements are valid biconditional statements:
  (a) A polygon is a square if and only if it has four right angles and four equal sides.
  (b) An animal is a spaniel iff it is a dog.
- Conditional statement: If you clean your room, then you can go to the movies.
  - The hypothesis: You clean your room.
  - The conclusion: You can go to the movies.

  (a) Provide the inverse statement.
  (b) Provide the converse statement.
  (c) Provide the contrapositive statement.
- Consider the following statement: If two diameters in a circle are perpendicular, then the diameters divide a circle into four equal parts.
  (a) Write its converse statement.
  (b) If the converse is true, write an iff statement.
- Given the following facts:
  - *p*: It is raining.
  - *q*: The highway is wet.

  (a) What does $p \Rightarrow q$ mean? Is it true? Explain.
  (b) What does $q \Rightarrow p$ mean? Is it true? Explain.
  (c) What does $\sim p \Rightarrow \sim q$ mean? Is it true? Explain.
  (d) What does $\sim q \Rightarrow \sim p$ mean? Is it true? Explain.
- When testing to see if an electrical circuit is working, we plug in a lamp and see if it lights.
  - *p*: Electrical circuit is working.
  - *q*: Lamp lights up.

  Complete the following truth tables for this situation, and explain what each table illustrates in terms of the specific situation of the electrical circuit working and the lamp lighting up.

		Veracity of the following statements:	
		Conditional	**Converse**
p	*q*	$p \Rightarrow q$	$q \Rightarrow p$
T	T		
T	F		
F	T		
F	F		

		Veracity of the following statements:	
		Inverse	**Contrapositive**
$\sim p$	$\sim q$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	**T**		
T	**F**		
F	**T**		
F	**F**		

- Let *R* represent the hypothesis, It is raining, and *C* represent the conclusion, The picnic will be cancelled.
  (a) State this hypothesis and conclusion as a conditional statement.
  (b) Create a truth table to illustrate this conditional statement.
  (c) Create a Venn diagram to illustrate this conditional statement. Is this necessary?
  (d) Is the converse of the conditional statement true? Explain.
  (e) Is the inverse of the conditional statement true? Explain.
- If a child has a peanut allergy, then they will get sick if they eat peanuts.
  (a) Write the converse, inverse, and contrapositive of this conditional statement.
  (b) Which of these statements is true?
  (c) Is this statement biconditional? Explain.
- Given the following hypothesis and conclusion:
  - *The hypothesis:* You eat all your vegetables.
  - *The conclusion:* You will be healthy.

  (a) State as a conditional statement. Is it true? Justify.
  (b) State the converse of this statement. Is it true? Justify.
  (c) State the inverse of this statement. Is it true? Justify.
  (d) State the contrapositive of this statement. Is it true? Justify.
- Given the following hypothesis and conclusion:
  - *The hypothesis:* A quadrilateral has diagonals that are the same length.
  - *The conclusion*: Its sides are the same length.

  (a) State this hypothesis and conclusion as a conditional statement. Is it true? Justify.
  (b) State the converse of this statement. Is it true? Justify.
  (c) State the inverse of this statement. Is it true? Justify.
  (d) State the contrapositive of this statement. Is it true? Justify.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Vary the subject matter of the statements; it is important to use both mathematical and non-mathematical statements to help students make connections to other subject areas.
- Have permanent, reusable graphic organizers on the wall that can be used often while teaching and referred to during individual or group work.
- In groups of four, give each student a cue card. Each student begins by writing a conditional statement on the card. It is then passed to the person on their left, who analyzes the original statement as to whether it is true or false. The card is passed again, and the third person to receive it writes the converse (regardless of the outcome at step two). The fourth person analyzes the converse as to whether it is true or false, as the second person did for the original statement. The activity is completed with the original owner of the card writing a biconditional statement, if both the original statement and the converse were determined to be true.
- When students are creating truth tables for a conditional statement, its converse, its inverse, or its contrapositive, have them interpret the truth tale in terms of a specific context.
- When playing the games and puzzles associated with outcome LR01, have students write the conditional statements that result from various moves (e.g., when playing chess, have students come up with conditional statements, such as, If I move my pawn here, then it can be taken by my opponent's bishop.)

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- biconditional
- conclusion
- conditional statement
- contrapositive
- converse
- counterexample
- hypothesis
- inverse
- truth table

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Section 3.5, pp. 195–206
  - Section 3.6, pp. 208–216

Internet

- *Geometry: Related Conditionals* (Oswego City School District Exam Prep Center 2012): www.regentsprep.org/regents/math/geometry/GP2/indexGP2.htm (Practice for conditional, inverse, converse, and counterpositve statements.)

Probability
15–20 hours

GCO: Students will be expected to develop critical-thinking skills related to uncertainty.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO P01 Students will be expected to interpret and assess the validity of odds and probability statements. [C, CN, ME]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

P01.01 Provide examples of statements of probability and odds found in fields such as media, biology, sports, medicine, sociology, and psychology.
P01.02 Explain, using examples, the relationship between odds (part-part) and probability (part--whole).
P01.03 Express odds as a probability and vice versa.
P01.04 Determine the probability of, or the odds for and against, an outcome in a situation.
P01.05 Explain, using examples, how decisions may be based on probability or odds and on subjective judgments.
P01.06 Solve a contextual problem that involves odds or probability.

Scope and Sequence

Mathematics 11	Mathematics 12
–	**P01** Students will be expected to interpret and assess the validity of odds and probability statements.

Background

Students had exposure to calculating probabilities in middle school and in Mathematics 9. However, this is the first time that probability will be explored in depth as its own topic. Students will determine probability and odds in various situations, and will interpret and assess probability and odds statements. Outcomes P01, P02, P03, and P04 should be considered collectively when planning.

Odds compares the number of times a favourable outcome will occur to the number of times an unfavorable outcome will occur (part:part).

Probability compares the number of times a favourable outcome will occur to the number of times all outcomes will occur (part:whole).

For example, the odds that a randomly chosen day of the week is Sunday are 1 to 6 (1:6). However, the probability of it being Sunday is 1 in 7, or $\frac{1}{7}$, and the outcome of interest is indicated first.

Although it is not correct, often the words **odds** and **probability** are used interchangeably in the media.

It is important to be consistent in vocabulary used, with odds using *to*, and probability using *in*.

Students will be required to express both the odds for an outcome and the odds against an outcome, and be clear on which they are describing.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- What is the probability of tossing a coin and having heads turn up?
- What is the probability of tossing a six-sided die and getting a 2?
- Describe an event that would have a probability of $\frac{1}{4}$ or 25% of happening.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Express the following in the requested form:
  - (a) There is a 0.25 probability of drawing a heart from a standard deck of cards. What are the odds of this happening?
  - (b) The odds against rolling a number less than 3 on a die are 2:1. What is the probability of rolling a number less than 3?
- A bag of buttons has 3 yellow, 4 red, and 5 green buttons. What is the probability of drawing a button at random that is
  - (a) green?
  - (b) not red?
  - (c) yellow or red?
- Create a digital or physical collage of probabilities or odds found in an online news site, a magazine, or newspaper.
- Calculate the odds (or probability) of selecting an "e" from the word Iceberg.
- Identify the following statements as odds or probability:
  - (a) The chances of rolling a 1 on a fair six-sided die is $\frac{1}{6}$.
  - (b) The chances of drawing a 4 from a standard 52-card deck is 1:12.
- The odds of winning a contest are 5:9. What is the probability of winning the contest?
- The probability of you passing the next mathematics test is 75%. What are the odds of you passing?
- A jar contains three red marbles and some green marbles. The odds are 3:1 that a randomly chosen marble is green. How many green marbles are in the jar?

- Some people use, or choose not to use, probability and odds to make daily decisions. The probability of winning a lottery is extremely low, but millions of people make the subjective judgement that it is still worth the expense. You may have heard the phrase, "Someone has to win." Discuss the merits of this statement.
- A test is usually conducted to determine blood glucose levels to help assist in diagnosing Type 2 diabetes. There is a 10% chance that a person will have diabetes given a high blood glucose level. 85% of the time, individuals with diabetes will register a high blood glucose level, and 99% of the time people without diabetes will show normal blood glucose levels. Determine, based on the information given, if this an effective test. Explain your reasoning.
- Which of the following "odds for" and "probability" statements describe the same event? Explain your reasoning.

	Odds for	**Probability**
a	1:3	$\frac{1}{3}$
b	4:5	$\frac{4}{9}$
c	4:6	$\frac{2}{5}$

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Students should be able to work comfortably with probabilities and odds interchangeably and in various forms (fractions, decimals, ratios, etc.). Start by having students compare like forms, such as fractions with fractions. As they become more familiar with the idea, present problems with different expressions and forms, such as fractions and percentages.
- Use ordering (e.g., least likely to most likely) as a way of familiarizing students with the various expressions of odds and probability.
- Incorporate discussion about interpretations. For example, ask, Does this mean that the most likely event will happen, or that the least likely event will not happen?"
- The outcome focuses on how valid odds and probability statements are. Make it a common part of teaching to ask, Is this valid? This should incorporate discussion about the source of the statement, and if that source is reliable and reputable.
- Incorporate discussion about the nature of subjective judgment. For example, ask, Why do people hear that there is a 60% chance of rain, yet still decide to go to the beach?
- Give students information in the form of odds or probabilities, and have them interpret the information in a written description. For example, give them the odds on a series of boxing matches and have them come up with a "story" to go with them.
- Statements of probability and odds are referenced frequently in various media and texts, such as newspapers, magazines, websites, and television. Ask students to find statements about probability and odds, and bring them to class. Statements may relate to political polls, games of chance, sports, and social statistics. This activity promotes student discussion of the similarities and differences between these two concepts using interesting contexts found in the media.

- When there is uncertainty about the occurrence of an event, students can attempt to describe the chances of it happening with probability.

  For example, ask students to consider the following questions:
  - Will the Halifax Mooseheads or the Cape Breton Screaming Eagles be more likely to win the game?
  - What is the chance of drawing a king from a deck of cards?
  - What is the possibility of it raining today?
  - What are the chances of getting tails in one toss of the coin?
- Students may need guidance to be able to distinguish between odds for and odds against. If a student is trying to calculate the odds of getting a heart in a deck of cards, for example, they will have to consider the probability that they will draw a heart from the deck. Since the probability of choosing a heart is $\frac{1}{4}$, the odds in favour would be 1:3. Students could be asked questions, such as the following:
  - How would this ratio change to determine the odds against a student choosing a heart?
  - How are the odds in favour related to the odds against?
- Provide students with a newspaper, magazine editorial, or an opinion piece. Ask them to identify examples of probability or odds present in the article. Ask them to describe how the author uses probability to support the argument being put forth.
- Provide students with a variety of examples where they are asked to express odds as a probability and vice versa. Examples can be found or created based on school statistics (e.g., the number of biology students expected this year in grade 12) or sports (e.g., the chance of a team winning a championship). All odds and probability calculations begin with two of three values: total possibilities, favourable outcomes, and non-favourable outcomes.
- For an example such as, the odds in favour of drawing an ace in a standard deck of cards is 4:48 (or 1:12), ask students the following questions:
  - What does this ratio represent?
  - What does the sum of the parts of the ratio represent?
  - What are the odds against selecting an ace?
  - What is the probability of selecting an ace? Not selecting an ace?
  - How does the probability of selecting an ace and the probability of its complement relate to the odds in favour of selecting an ace?
  - Why is it important to know if the ratio represents odds against or odds in favour when using odds to determine the probability?
- Students should conclude that if the odds in favour of an event is a:b, then the total number of possibilities is represented by $a + b$. Hence, the probability of an event occurring is $\frac{a}{a+b}$. The odds against an event, however, is b:a. Students have the option to find the probability of an event not happening using the complement, $1 - \frac{a}{a+b}$, or the ratio $\frac{b}{a+b}$.
- Students can conduct in-class surveys and calculate the odds and probability based on the results. A question on the survey could be, How many students use Twitter daily? Other student interests may include music, celebrities, or school teams.

SUGGESTED MODELS AND MANIPULATIVES

- cards
- coins
- dice

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- experimental probability
- fair game
- odds against
- odds in favour
- theoretical probability

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 5.1 and 5.2, pp. 302–312

SCO P02 Students will be expected to solve problems that involve the probability of mutually exclusive and non-mutually exclusive events. [CN, PS, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

P02.01 Classify events as mutually exclusive or non-mutually exclusive, and explain the reasoning.
P02.02 Determine if two events are complementary, and explain the reasoning.
P02.03 Represent, using set notation or graphic organizers, mutually exclusive (including complementary) and non-mutually exclusive events.
P02.04 Solve a contextual problem that involves the probability of mutually exclusive or non-mutually exclusive events.
P02.05 Solve a contextual problem that involves the probability of complementary events.
P02.06 Create and solve a problem that involves mutually exclusive or non-mutually exclusive events.

Scope and Sequence

Mathematics 11	**Mathematics 12**
S02 Students will be expected to interpret statistical data, using confidence intervals, confidence levels, margin of error.	**P02** Students will be expected to solve problems that involve the probability of mutually exclusive and non-mutually exclusive events.

Background

The use of set theory (LR02) is fundamental to the study of probability theory. For this reason, the logical reasoning outcome should be addressed prior to this probability outcome. Students should continue to use Venn diagrams for illustrating unions and intersections of sets. They will apply the principle of inclusion and exclusion to determine the probability of non-mutually exclusive events and will use the probabilities to make decisions.

Mutually exclusive events are events that cannot happen concurrently. For example, a coin cannot be flipped and get “head” and “tail” at the same time — these are mutually exclusive. The intersection of two mutually exclusive events in terms of their set spaces is an empty set (this can be discussed in conjunction with outcome LR02). In a Venn diagram, the intersecting region would be empty.

Non-mutually exclusive events are events in which at least one outcome is in common. For example, if event A is rolling an even number on a six-sided die, and event B is rolling a number less than 3 on a six-sided die, then A and B are non-mutually exclusive, because in the case of rolling a 2, both outcomes would be satisfied.

Two events are said to be **complementary events** if one or the other must happen. For example, getting “heads” on a coin flip and getting “tails” on a coin flip are complementary because one or the other must happen, and to not get one result is to get the other.

The notation for complementary events is shown as, If *A* is an event, then$\overline{A}$, A', or A^C would all represent its complement.

If *P(A)* represents the probability of event *A*, then$P(\overline{A})$ would represent the probability of the complement of event *A*. In all cases,$P(A) + P(\overline{A}) = 1$ or $P(A) = 1 - P(\overline{A})$.

Any event, even those with more than two outcomes, can be stated in a complementary manner. For example, if event *R* is selecting a red bead, the complement of$R$, $\overline{R}$, would be not selecting a red bead, regardless of how many additional colours of beads are available for selection.

Students are familiar with classifying events that are mutually exclusive (disjoint sets) and events that are non-mutually exclusive. They worked with Euler diagrams and represented mutually exclusive sets as sets that did not intersect (i.e., $n(A \cap B) = 0$).

Students are also familiar with an event and its complement. They know that the outcomes of an event and the outcomes of the complement make up the entire sample space.

Students will now solve probability problems that involve non-mutually exclusive events and mutually exclusive events. It is important for students not only to apply the formula for the probability of two non-mutually events, but to understand how it is developed.

Students should understand that in order to determine the probability of events, they have to think about the following questions:

- Are the two sets intersecting or disjoint?
- How many elements are in each set?
- How many elements are in the universal set *S*?

Using the principle of inclusion and exclusion, guide students through the following probability formula for non-mutually exclusive events:

For the set of all playing cards from a standard deck *D:* All diamonds *F:* All face cards	**Development of Formula for probability** $P(D \cup F) = P(D) + P(F) - P(D \cap F)$
$n(Diamonds \cup Face\ cards) = 13 + 12 - 3$ $= 22$	$n(D \cup F) = n(D) + n(F) - n(D \cap F)$
$P(Diamonds \cup Face\ cards) = \frac{13 + 12 - 3}{52}$ $= \frac{22}{52}$	$P(D \cup F) = \frac{n(D) + n(F) - n(D \cap F)}{n(S)}$
$P(Diamonds \cup Face\ cards) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$ $= \frac{22}{52}$	$P(D \cup F) = \frac{n(D)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(D \cap F)}{n(S)}$ $P(D \cup F) = P(D) + P(F) - P(D \cap F)$

Students will understand that when events are mutually exclusive (i.e., Sets are disjoint, such as when Set *D* represents all diamonds and *B* represents all black cards), there is no overlap: $P(D \cap B) = 0$ and, thus, $P(D \cup B) = P(D) + P(B)$.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

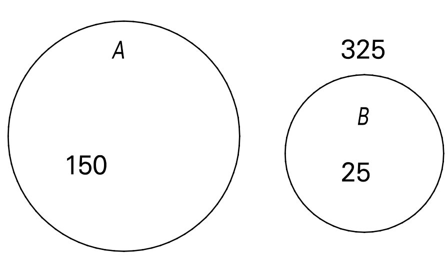
- What is the probability of rolling an even number when rolling a single six-sided die?
- How many face cards are there in a standard 52-card deck?
- What is the probability of selecting a face card from a standard 52-card deck?
- A bag contains four red, three green, and five yellow balls. What is the probability of selecting a green ball?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

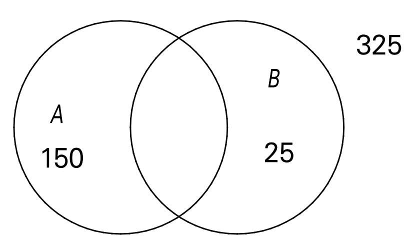
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Classify the events in each experiment as either mutually exclusive or non-mutually exclusive:
  (a) The experiment is rolling a die. The first event is rolling an even number, and the second event is rolling a prime number.
  (b) The experiment is playing a game of hockey. The first event is that your team scores a goal, and the second event is that your team wins the game.
  (c) The experiment is selecting a gift. The first event is that the gift is edible and the second event is that the gift is an iPhone.
- Using the school population as the total, ask students to determine the complement of people doing Mathematics 12 in your school this year.
- Determine the $P(A \cup B)$ using the Euler or Venn diagram, below. The sample space has 500 outcomes.

Euler Diagram



Venn Diagram



The probability that D'Arcy will make the hockey team is $\frac{2}{3}$. The probability that she will make the swimming team is $\frac{3}{4}$. If the probability of D'Arcy making both teams is $\frac{1}{2}$, determine the probability that she will make

(a) at least one of the teams

(b) neither team

- The probability that the Toronto Maple Leafs will win their next game is 0.5. The probability that the Montreal Canadiens will win their next game is 0.7. The probability that they will both win is 0.35. Determine the probability that one or the other will win (assume they do not play each other). Create a poster to show their solution that includes a Venn diagram.
- Survey the class to determine who brings a pencil to class, a pen to class, or both. Use a Venn diagram to illustrate the survey results. What is the probability that the teacher will randomly select a student who
  (a) has both a pen and a pencil
  (b) has a pen only
  (c) has neither a pen nor a pencil

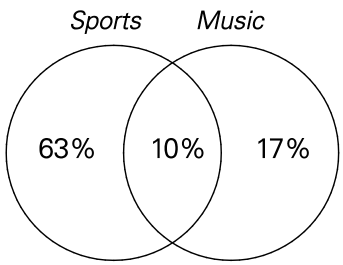
Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Ask students to participate in the Find Your Partner activity. Half of the students should be given a card with an event and the other half should be given a card with the complement on it. Students need to move around the classroom to match the event with the correct complement. They could then present their findings to the class.
- Ask students to create their own examples that involve determining the probability of two mutually exclusive or non-mutually exclusive events. They could exchange problems with a classmate and practice solving various problems.
- Use numerical examples with students to discuss why the sum of the probability of an event and its complement must equal one. If the probability that a student picks the ace of diamonds, for example, from a standard deck of cards is $\frac{1}{52}$, then the probability that the student will not pick the ace of diamonds is $\frac{51}{52}$ (i.e., $\frac{1}{52} + \frac{51}{52} = 1$). Solving the equation $P(A) + P(A') = 1$ for $P(A')$, students should write that $P(A') = 1 - P(A)$.
- Encourage students to check the reasonableness of their answers. Provide students with the following example of the results of a class survey.
  - 63% of students play sports
  - 27% of students play a musical instrument
  - 20% play neither sports nor a musical instrument

  If the events were mutually exclusive, the probabilities of these events and the sum of the complement should total 100%. Students should understand that 63% + 27% + 20% = 110%. This implies 10% of the students play sports and play a musical instrument. The events, therefore, are not mutually exclusive. This could be displayed in a Venn diagram as illustrated below.



- Visualization of events as sets will help some students understand the difference between mutually exclusive and non-mutually exclusive pairs of events. It will be useful to have permanent, reusable Venn diagrams of both mutually exclusive and non-mutually exclusive events (see outcome LR02).
- Have students use manipulatives such as cards or multi-link cubes during class time to simulate events as a way of helping them discover the differences between mutually exclusive and non-mutually exclusive.
- Incorporate set notation into your teaching as the material is presented, as a way of helping students meet the achievement indicators and to make connections with other GCOs (see outcome LR02).

SUGGESTED MODELS AND MANIPULATIVES

- multi-link cubes
- standard deck of playing cards

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- complementary events
- mutually exclusive events
- non-mutually exclusive events
- principle of inclusion and exclusion

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Section 5.4, pp. 328–343

SCO P03 Students will be expected to solve problems that involve the probability of two events. [CN, PS, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

P03.01 Compare, using examples, dependent and independent events.
P03.02 Determine the probability of an event, given the occurrence of a previous event.
P03.03 Determine the probability of two dependent or two independent events.
P03.04 Create and solve a contextual problem that involves determining the probability of dependent or independent events.

Scope and Sequence

Mathematics 11	**Mathematics 12**
–	**P03** Students will be expected to solve problems that involve the probability of two events.

Background

In Mathematics 7 and Mathematics 8, students calculated the probability of two independent events (7SP06, 8SP02). They should now distinguish between events that are dependent and independent and use this information when determining the probabilities of two events.

To determine whether two events are independent, students will determine whether one event will affect the probable outcome of the other event. If one event does not affect the other event, the events are **independent**.

Example:

You have a bag of 12 coloured marbles with three yellow, four green, and five red. To find the probability of drawing a red marble, replacing it, then drawing a yellow marble, you would determine the probability of each event and then multiply them together:

$$P(red, yellow) = \left(\frac{5}{12}\right)\left(\frac{3}{12}\right) = \frac{15}{144} = \frac{5}{48}$$

The number of possible marbles to choose from has not changed because the first marble drawn was replaced.

If one event does affect the other event, then the events are **dependent**, and students will use conditional probability to calculate the probability of both events occurring.

Example:

You are taking one crayon from the box at a time. There are 12 crayons: three yellow, four green, and five red. To determine the probability of picking a red crayon first, and then picking a yellow crayon after that, you would determine the probability of each event and multiply them together:

$$P(red, yellow) = \left(\frac{5}{12}\right)\left(\frac{3}{11}\right) = \frac{15}{132} = \frac{5}{44}$$

The number of possible crayons to choose from on the second draw has been reduced because the red crayon was not replaced.

Students should analyze different events and judge whether the outcome of the first event has an effect on the probability of the second event occurring. Students can use concrete materials, such as a deck of cards, a spinner, or a bag of coins, to show the difference between independent and dependent events.

Students will be provided with examples to prompt discussion about whether events *A* and *B* are independent or dependent.

- *Event A:* Drawing a queen from a standard deck of cards

  Event B: Drawing a king from the remaining cards in the same deck
- *Event A:* Rolling a 5 on a die

  Event B: Rolling a 3 on the same die

Through activities, students discover that when events are independent of each other, the probability of event *B* does not depend on the probability of event *A* occurring. In such cases, *P*(*A* and *B*) = *P*(*A*) × *P*(*B*).

When events are dependent, the probability of event *B* does depend on the probability of event *A* occurring. In such cases *P*(*A* and *B*) = *P*(*A*) × *P*(*A*|*B*).

In this unit, students will be introduced to conditional probability and the notation *P*(*B*|*A*). Students determined the probability of two independent events by multiplying their individual probabilities. They should now determine the probability of two dependent events in a similar way. Ask students to calculate the probability of drawing a card from a deck given two chances, with or without replacement. Discuss with them that if a card is not replaced, the events are dependent. Students should multiply the probability of *A* and the probability of *B* after *A* occurs. Introduce them to the term **conditional probability**, the notation *P*(*B*|*A*) and the formula *P*(*A* and *B*) = *P*(*A*) × *P*(*B*|*A*). Ask students to rewrite *P*(*A* and *B*) using set notation and manipulate the formula to solve for *P*(*B*|*A*):

$$P(A \cap B) = P(A) \times P(A|B)$$
$$\frac{P(A \cap B)}{P(A)} = P(A|B)$$

For example, in a card game, suppose a player needs to draw two cards of the same suit in order to win. Of the 52 cards, there are 13 cards in each suit. Suppose first the player draws a heart. Now the player wishes to draw a second heart. Since one heart has already been chosen, there are now 12 hearts remaining in a deck of 51 cards. So the conditional probability *P*(Draw second heart | First card a heart) = $\frac{12}{51}$.

Consider the following example to illustrate conditional probability.

- Suppose an individual applying to a college determines there is an 80% chance of being accepted, and knows that dormitory housing will only be provided for 60% of all of the accepted students. The chance of the student being accepted *and* receiving dormitory housing is defined by

  P(Accepted **and** Dormitory Housing)
  = *P*(Dormitory Housing | Accepted) × *P*(Accepted)
  = (0.60)(0.80) = 0.48.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

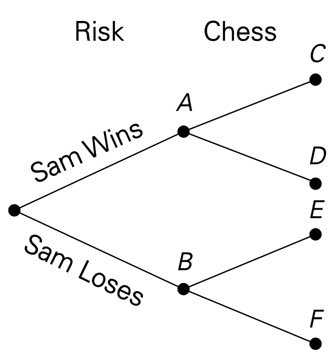
- When drawing a card from a standard deck, what is the probability that it will be
  (a) red
  (b) a club
  (c) an ace

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Classify the following events as either independent or dependent and explain why.
  (a) The experiment is rolling a die and flipping a coin. The first is rolling a six, and the second event is obtaining tails.
  (b) The experiment is rolling a pair of dice. The first event is rolling an odd number on one die, and the second event is rolling an even number on the other die.
  (c) The experiment is dealing five cards from a standard deck. The first event is that the first card dealt is a spade, the second event is that the second card is a spade, the third event is that the third card is a spade, and so on.
  (d) The experiment is to sample two members of a family, a mother and her child. The first event is that the mother has blond hair, and the second event is that the child has blond hair.
- Determine the probability of selecting two diamonds from a standard deck of cards.
  (a) With replacement.
  (b) Without replacement.

- Cards are drawn from a standard deck of 52 cards *without* replacement. Calculate the probability of obtaining
  - (a) a king, then another king
  - (b) a club, then a heart
  - (c) a black card, then a heart, then a diamond
- Cards are drawn from a standard deck of 52 cards *with* replacement. Calculate the probability of obtaining
  - (a) a king, then another king
  - (b) a club, then a heart
  - (c) a black card, then a heart, then a diamond
- Marisa encounters two traffic lights on her way to school. There is a 55% chance that she will encounter a red light at the first light, and a 40% chance she will encounter a red light on the second light. The two traffic lights operate on separate timers. Determine the probability that both lights will be red on her way to school.
- A jar contains black marbles and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?
- Sam has played a lot of games with his cousin, Rae. Sam wins at Risk $\frac{3}{4}$ times. His cousin wins chess $\frac{4}{5}$ times when she plays against Sam.
  - (a) Complete the following tree diagram.



  - (b) Calculate the probability of Sam winning exactly one of the games the next time Sam and Rae play Risk and chess.
- A number is selected, at random, from the set $\{1,2,3,4,5,6,7,8\}$. Find:
  - (a) *P*(odd)
  - (b) *P*(prime | odd)
- A box contains three blue marbles, five red marbles, and four white marbles. If one marble is drawn at random, find:
  - (a) *P*(blue | not white)
  - (b) *P*(not red | not white)
- A number is selected randomly from a container containing all the integers from 10 to 50. Find:
  - (a) *P*(even | greater than 40)

(b) P(greater than 40 | even)

(c) P(prime | between 20 and 40)

- A coin is tossed. If it shows heads, a marble is drawn from Box 1, which contains one white and one black marble. If it lands tails, a marble is drawn from Box 2, which contains two white and one black marble. Find:

  (a) P(black | coin fell heads)

  (b) P(white | coin fell tails)

- A hockey team has jerseys in three different colours. There are four green, six white, and five orange jerseys in the hockey bag. Todd and Birk are given a jersey at random. Students were asked to write an expression representing the probability that both jerseys are the same colour. Which student correctly identified the probability and why?

Tony	$\left(\frac{2}{4}\right)\left(\frac{2}{6}\right)\left(\frac{2}{5}\right)$
Sam	$\left(\frac{2}{4}\right)+\left(\frac{2}{6}\right)+\left(\frac{2}{5}\right)$
Lesley	$\left(\frac{4}{15}\right)\left(\frac{3}{14}\right)+\left(\frac{6}{15}\right)\left(\frac{5}{14}\right)+\left(\frac{5}{15}\right)\left(\frac{4}{14}\right)$
Dana	$\left(\frac{4}{15}\right)\left(\frac{4}{15}\right)+\left(\frac{6}{15}\right)\left(\frac{6}{15}\right)+\left(\frac{5}{15}\right)\left(\frac{5}{15}\right)$

- Determine whether the situation describes a series of dependent events or independent events.

  (a) A deck of cards is cut, and the top card, which is the queen of clubs, is given to the first player. The deck of cards is cut a second time, and this time the 5 of clubs is the top card and is given to the second player.

  (b) A deck of cards is cut, and a red card shows. The deck of cards is cut a second time and a card with an odd number shows.

  (c) I vote for the Liberal candidate. You vote for an NDP candidate.

- There are 300 numbered parking spaces in a parking garage. Parking spaces are randomly assigned by a parking officer as cars enter. Joey, Alex, and Katie pull into the garage, in that order, and are the first to arrive that morning. If Joey is assigned space #56, what is the probability that Alex, then Katie, are also assigned one of the first 100 spaces?
- Micah and Callie go with their parents to a frozen yogurt shop. If there are 29 flavours to choose from, what is the probability that Micah chooses chocolate and Callie chooses banana-nut?
- A bag contains 10 blue marbles and 6 red marbles. What is the probability of randomly selecting two red marbles in a row, provided that the marbles are not replaced after being selected?

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- It is beneficial to discuss the probability of independent events before moving on to problems that involve dependent events. Ask students to use a tree diagram, for example, to model a problem and verify that the probability of two independent events, *A* and *B*, is the product of their individual probabilities. Ask students to determine, for example, the probability of rolling a 3 on a die and tossing a head on a coin. Prompt discussion using the following questions:
  (a) Does the outcome of the first event affect the outcome of the second event?
  (b) Using the tree diagram, what is the probability of rolling a 3 and tossing heads [*P*(3 and *H*)]?
  (c) What is the probability of rolling a 3 on a die [*P*(3)]? What is the probability of tossing heads on a coin [*P*(*H*)]?
  (d) What is the value of *P*(3) × *P*(*H*)?
  (e) What did you notice about the value *P*(3) × *P*(*H*) and the value from the tree diagram [*P*(3 and *H*)]?
- Once students understand that to determine the probability of two independent events they multiply the individual probabilities of each event, they now determine the probability of two dependent events in a similar way. Ask students to calculate the probability of drawing a card from a deck given two chances, with or without replacement. Discuss with them that if a card is not replaced, the events are dependent.
- Be very careful to identify which event depends upon the other. In general *P*(*A*|*B*) is not equal to *P*(*B*|*A*). That is the probability of *A* given the event *B* is not the same as the probability of *B* given the event *A*.

  To illustrate this, consider the following:

  P(roll sum less than 6 | roll a 3) = $P(A|B) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{4}{36}}{\frac{10}{36}} = \left(\frac{4}{36}\right)\left(\frac{36}{10}\right) = \frac{4}{10} = \frac{2}{5}$. Therefore, when two dice are rolled, the probability of rolling a 3, given that a sum of less than 6 has been rolled was $\frac{4}{10}$.

  P(roll of a 3 | roll sum less than 6) = $P(A|B) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{4}{36}}{\frac{11}{36}} = \left(\frac{4}{36}\right)\left(\frac{36}{11}\right) = \frac{4}{11}$. Therefore, when two dice are rolled, the probability of rolling a sum less than 6, given that a 3 has been rolled was $\frac{4}{11}$.
- Depending on the questions posed, encourage students to create a variety of probabilities using techniques learned throughout the unit.
- Students should be given the opportunity to share their work with others and to question and assess the work of their peers.

SUGGESTED MODELS AND MANIPULATIVES

- cards
- coins
- number cubes
- spinners
- tiles in two different colours

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- dependent events
- independent events
- probability

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  – Sections 5.5 and 5.6, pp. 344–363

SCO P04 Students are expected to solve problems that involve the fundamental counting principle. [PS, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

P04.01 Represent and solve counting problems, using a graphic organizer.
P04.02 Generalize the fundamental counting principle, using inductive reasoning.
P04.03 Identify and explain assumptions made in solving a counting problem.
P04.04 Solve a contextual counting problem, using the fundamental counting principle, and explain the reasoning.

Scope and Sequence

Mathematics 11	**Mathematics 12**
–	**P04** Students are expected to solve problems that involve the fundamental counting principle.

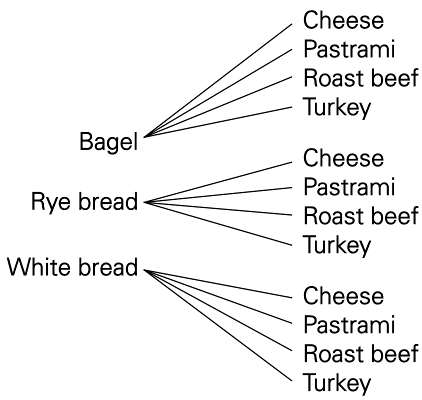
Background

For this outcome, students will use graphic organizers, such as tree diagrams, to visualize and calculate sample space. Tree diagrams were used in Mathematics 7 (7SP05 and 7SP06) and Mathematics 8 (8SP02) to determine the number of possible outcomes in probability problems. From patterns observed, students will formulate an understanding of the *fundamental counting principle*.

This **fundamental counting principle**, also known as the **multiplication principle**, states that if one event has m possible outcomes and a second independent event has n possible outcomes, there will be a total of $m \times n$ possible outcomes for these two outcomes occurring together. This enables finding the number of outcomes without listing and counting each one. If one event has three possible outcomes and a second independent event has four possible outcomes, there will be a total of 12 possible outcomes for these two outcomes occurring together.

Example:

If a person is to choose a sandwich for lunch and had three different bread choices and four different meat choices, they would have 12 possible sandwich choices as illustrated in the tree diagram below.



The fundamental counting principle is a means to find the number of ways of performing two or more operations together. Students should develop an understanding of how the principle works, when it can be used, and the advantages and disadvantages it has over methods that involve direct counting, when used to solve counting problems. In P05 and P06, the fundamental counting principle will be applied in work with permutations and combinations.

It is important for students to understand when to use the fundamental counting principle and when to use the principle of inclusion and exclusion. It is also important for students to distinguish between the words *and* and *or*. To help make this comparison, ask students to calculate the number of possibilities when choosing a sandwich and a side and a beverage compared to choosing a sandwich or a side or a beverage.

When students are asked what effect the words *and* and *or* have on their answers, they should conclude that

- when choosing a sandwich and a side and a beverage, the word *and* indicates these three selections (operations) are performed together, so the number of ways of doing each individual selection are multiplied
- when choosing a sandwich or a side or a beverage, the word *or* indicates these selections are being done separately, so the number of ways of doing each individual selection are added

In summary, students should understand that

- *and* implies multiplication and the use of the fundamental counting principle
- *or* implies addition and the use of the principle of inclusion and exclusion, depending on whether the tasks are mutually exclusive or not

Problems involving the fundamental counting principle sometimes contain restrictions. When arranging items, for example, a particular position must be occupied by a particular item. Students should be exposed to examples such as the following where restrictions exist:

- In how many ways can a teacher seat five boys and three girls in a row of eight seats if a girl must be seated at each end of the row?

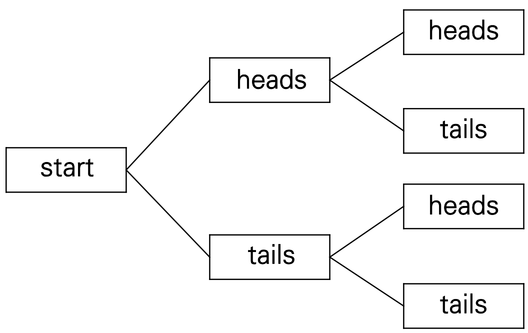
Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- The school cafeteria restaurant offers a lunch combo for $6 where a person can order one sandwich (chicken, turkey, grilled cheese), one side (fruit, yogurt, soup), and one drink (juice, milk). Draw a tree diagram or create a table to determine the possible lunch combos.
- Write a statement that you can conclude based on the tree diagram shown below.



Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- A restaurant offers "select-your-own" sundaes choosing one item from each of three categories:

Ice Cream	**Sauce**	**Extras**
vanilla strawberry chocolate	chocolate caramel mint	cherries peanuts

  (a) Using a tree diagram, list all possible desserts that can be ordered.

  (b) Would you expect the choices of a dessert to be equally likely for most customers? Why or why not?

  (c) If the probability of selecting chocolate ice cream is 40%, and vanilla is 10%, chocolate sauce is 70%, and cherries 20%, what is the dessert with the highest probability of being selected?

- A certain model car can be ordered with one of three engine sizes, with or without air conditioning, and with automatic or manual transmission.

  (a) Show, by means of a tree diagram, all the possible ways this model car can be ordered.

(b) Suppose you want the car with the smallest engine, air conditioning, and manual transmission. A car agency tells you there is only one of the cars on hand. What is the probability that it has the features you want, if you assume the outcomes to be equally likely?

- In a restaurant there are four kinds of soup, 12 different entrees, six desserts, and three kinds of drinks.

  (a) How many different four-course meals does a patron have to choose from?

  (b) If four of the 12 entrees are chicken and two of the desserts involve cherries, what is the probability that someone will order wonton soup, a chicken dinner, a cherry dessert, and milk?

- How many variations of frozen yogurt sundaes can be made if the choices are vanilla and chocolate, the toppings are nuts or fruit pieces, and the sauces are dark chocolate, strawberry, or whipped cream? Each sundae must contain one type of frozen yogurt, one topping, and one sauce.

- In Nova Scotia, a license plate consists of a letter-letter-letter-digit-digit-digit arrangement such as CXT 132.

  (a) How many license plate arrangements are possible?

  (b) How many license plate arrangements are possible if no letter or digit can be repeated?

  (c) How many license plate arrangements are possible if vowels (a, e, i, o, u) are not allowed?

- Canadian postal codes consist of a letter-digit-letter-digit-letter-digit arrangement.

  (a) How many codes are possible, and how does this compare with the number of license plates in Nova Scotia?

  (b) In Nova Scotia, all postal codes begin with the letter B. How many postal codes are possible?

- Greg is trying to select a new cell phone based on the following categories:

  Brands: Ace, Best, Cutest

  Colour: Lime, Magenta, Navy, Orange

  Plans: Text, Unlimited Calling

  With the aid of a tree diagram, a table, and/or an organized list, explain why it makes sense to multiply the options from each category to determine the number of ways of selecting Greg's new cell phone.

- Research the format and restrictions on a license plate in a province or country of your choice (i.e., some letters are not used, such as I, O, and/or Q to avoid confusion with the digits 1 and 0). Determine the number of plates possible, and present your findings to the class.

- How many three-digit numbers can you make using the digits 1, 2, 3, 4, and 5? Repetition of digits is not allowed.

- How does the application of the fundamental counting principle change if repetition of the digits is allowed? Determine how many three-digit numbers can be formed that include repetitions.

- How many ways can you order the letters MUSIC if it must start with a consonant and end with a vowel?

- Create and solve your own problems that require the use of the fundamental counting principle. Exchange problems and compare their solutions.

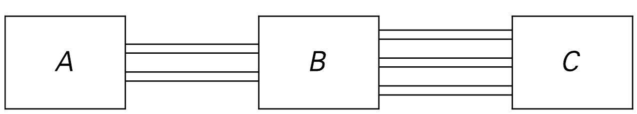
Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Begin instruction with having students construct tree diagrams to represent different outfits (tops, pants, socks, shoes) or different meal combinations served in a restaurant (appetizers, main courses, desserts, beverages), and then count the total number of different arrangements/combinations in each situation.
- The following example could be used to activate students' prior knowledge.
  - The school cafeteria restaurant offers a lunch combo for $6 where a person can order one sandwich (chicken, turkey, grilled cheese), one side (fruit, yogurt, soup) and one drink (juice, milk). Ask students to draw a tree diagram or create a table to determine the possible lunch combos.

  Modifying this example to include a fourth category or adding more options in one of the categories can be used to illustrate the limitations on the practicality of using graphic organizers for counting problems and offers a good introduction to the fundamental counting principle. When the sample space is very large, the task of listing and counting all the outcomes in a given situation is unrealistic. The fundamental counting principle enables students to find the number of outcomes without listing and counting each one.
- Use examples that are relevant to the student.
- Ask students to investigate the configuration for each of the provincial license plates. Could you tell from the configuration which province(s) likely have the largest populations?
- In groups of two or three students, ask students to discuss the advantages and disadvantages of using graphic organizers to determine the sample space compared to using the fundamental counting principle. Teachers should observe the groups and ask questions such as the following:
  - Which solution do you prefer?
  - Why do you prefer that solution?
- Ask students to write a journal entry summarizing any shortcuts they observe as to how they can calculate the total number of outcomes. Verify the results obtained by applying the fundamental counting principle to each situation.
- A visual that may help students understand the meaning of *and* as well as *or* is a maze

  Imagine you begin in room *A*, move to *B*, **and** then to *C*. How many paths are there? (2 × 3 = 6)

  Imagine you begin in room *B* and move to *A* **or** to *C*. How many paths are there? (2 + 3 = 5)
- Provide students with examples, such as the following, to help them differentiate between the fundamental counting principle and the principle of inclusion and exclusion (LR02).
  - How many possible outcomes exist if we first flip the coin and then roll the die?
  - How many possible outcomes exist if you either flip the coin or roll the die?

- Determine the number of ways that, on a single die, the result could be odd or greater than four?
- A buffet offers five different salads, 10 different entrees, eight different desserts, and six different beverages. In how many different ways can you choose a salad, an entree, a dessert, and a beverage?

Students should understand the mathematical meaning behind the words *and* and *or* as well as the strategy they will use to solve problems that involve these words.

- Students should be exposed to examples such as the following where restrictions exist:
  - In how many ways can a teacher seat five boys and three girls in a row of eight seats if a girl must be seated at each end of the row?

  Teachers should ask the following questions to help guide students as they work through the example:
  - Are there any restrictions for seating girls and boys?
  - Why should you fill the girls' seats first?
  - How many choices are there for seat 1 if a girl must sit in that seat?
  - How many girls remain to sit in seat 8?
  - How many choices of boys and girls remain to sit in each of seats 2 through 7?
  - Which mathematical operation should you use to determine the total number of arrangements?

  Students should conclude, by the fundamental counting principle, that the teacher can arrange the girls and boys in $3 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 = 4320$ ways.

SUGGESTED MODELS AND MANIPULATIVES

- cards
- counters or coins
- multi-link cubes

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- disjoint sets
- fundamental counting principle
- mutually exclusive
- principle of inclusion and exclusion

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Section 4.1, pp. 228–237
  - Section 4.7, pp. 283–290

SCO P05 Students will be expected to solve problems that involve permutations. [ME, PS, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

(It is intended that circular permutations not be included.)

P05.01 Represent the number of arrangements of *n* elements taken *n* at a time, using factorial notation.
P05.02 Determine, with or without technology, the value of a factorial.
P05.03 Simplify a numeric or algebraic fraction containing factorials in both the numerator and denominator.
P05.04 Solve an equation that involves factorials.
P05.05 Determine the number of permutations of *n* elements taken *r* at a time.
P05.06 Determine the number of permutations of *n* elements taken *n* at a time where some elements are not distinct.
P05.07 Explain, using examples, the effect on the total number of permutations of *n* elements when two or more elements are identical.
P05.08 Generalize strategies for determining the number of permutations of *n* elements taken *r* at a time.
P05.09 Solve a contextual problem that involves probability and permutations.

Scope and Sequence

Mathematics 11	Mathematics 12
–	**P05** Students are expected to solve problems that involve permutations.

Background

This outcome should be taught in conjunction with P06, which involves combinations.

Students will be introduced to factorial notation and how this relates to the concept of permutations. A formula will be developed and applied in problem-solving situations, including those that involve permutations with conditions. Students should first be introduced to permutations of *n* different elements taken *n* at a time. They will then move to permutations of *n* different elements taken *r* at a time.

As students create tree diagrams and determine the number of possible outcomes using the fundamental counting principle (P06), they should understand and use *n*! (*n* factorial) to represent the number of ways to arrange *n* distinct objects.

4! is the product of all positive integers less than or equal to 4. Thus $4! = 4 \times 3 \times 2 \times 1 = 24$

In general, $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ where $n \in N$.

Students should observe that $n! = n(n-1)!$ Ask students to use the value of 5!, for example, to determine 6!.

Students should simplify expressions where the product can be expressed as a single factorial or the expression can be written as the product of two binomials. For example, students are expected to simplify expressions similar to $\frac{8!}{6!}$ and $\frac{n!}{(n-2)!}$.

Students should also solve algebraic equations that involve factorial notation. In some cases, extraneous roots will occur due to the nature of the definition of factorial notation. Limit questions of this type to those where the resulting equation is the product of two binomials resulting in a quadratic equation. Students should solve and then verify their solutions. Students are not expected to solve an equation, such as $\frac{n!}{(n-3)!} = 1716$, where the resulting equation $n(n-1)(n-2) = 1716$ is cubic. They are only familiar with solving quadratic equations.

When order matters, factorials can be used to find the number of possible arrangements or permutations for a given number of people or objects.

For example, 3! will give the number of permutations for three people standing in a line. There are three people to choose from for the first position, two people left to choose from for the second position in the line, and only one person to choose for the third position, or (3)(2)(1) = 3! possible arrangements.

Providing students with several examples where they determine the number of ways to arrange *n* distinct objects makes the link to factorials clear. For example, ask students to determine the number of ways to arrange or permute a group of five people in a line by filling in the table provided.

1st person	**2nd person**	**3rd person**	**4th person**	**5th person**
5 options	4 options remaining	3 options remaining	2 options remaining	1 option remaining

By the fundamental counting principle, there are 5 × 4 × 3 × 2 × 1 or 120 ways or 5! ways. This should allow them to make the conjecture that the number of permutations is *n*!.

Students should also work with problems where only some of the objects are used in each arrangement (i.e., arranging a subset of items). Students can use the fundamental counting principle to develop an understanding of a permutation of *n* elements taken *r* at a time ${}_nP_r = \frac{n!}{(n-r)!}, 0 \leq r \leq n$. This formula should not just be provided to students. It will make more sense if students can see how it is derived from the work they have already completed with factorials.

In applying the permutation formula, students will encounter 0!. It is important that 0!, including 0! = 1, be discussed at this point (see instructional strategies).

Emphasize to students that in the notation, ${}_nP_r$, *n* is the number of elements in a set and *r* is the number of elements to be selected at any given time. With this understanding well in hand, students will understand that $n \geq r$, $n \in N, r \in W$, and that $r \geq 0$.

Rather than giving students the formula for permutations, it is expected that students will develop the formula through consideration of various situations.

For example, if first, second, and third prizes are to be awarded to a group of eight individuals, there is a choice of eight people for the first prize, seven people for the second prize, and six people for the third prize. Therefore, there are (8)(7)(6) = 336 ways to award three prizes to eight people (assuming the prizes are different), or more formally stated, there are 336 permutations of eight objects taken three at a time. This is expressed as ${}_8P_3 = (8)(7)(6)$. It is equivalent to ${}_8P_3 = \frac{(8)(7)(6)(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)} = \frac{8!}{5!} = \frac{8!}{(8-3)!}$.

The general form is ${}_nP_r$ for *n* objects taken *r* at a time, so ${}_nP_r = \frac{n!}{(n-r)!}$.

When working with the permutation formula, it is important for students to use factorial notation and that they be able to simplify the resulting quotient of factorials. When using the formula to evaluate ${}_6P_2$, for example, encourage students to use mental mathematics to simplify ${}_6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} =$ $6 \times 5 = 30$.

When *n* and *r* are large, however, using the formula in conjunction with a calculator would be more manageable.

Another situation that should be understood is when some objects of a set are the same. In this case, there are fewer permutations because some arrangements are identical.

If a set of three marbles consists of two identical green marbles and one blue marble, students may initially write the number of arrangements as 3! They may not understand that the set {*G*1, *G*2, *B*} is identical to {*G*2, *G*1, *B*}. This configuration is counted as two different arrangements instead of one, therefore, it must be removed from the total count.

A set of *n* objects containing *a* identical objects of one kind and *b* identical objects of another kind can be arranged in $\frac{n!}{a!b!}$ ways.

Dividing *n*! by *a*! and *b*! eliminates arrangements that are the same and that would otherwise be counted multiple times.

For example, for the word BANANA, there is one B, two Ns, and three As, so the total number of permutations of the letters for the word BANANA will be $P = \frac{6!}{1!2!3!} = 60$.

To be successful, it is important for students to read the problem carefully and decide whether they should apply the fundamental counting principle or use the permutation formula. Ask students to work through a mixture of problems. They should be asking themselves the following questions:

- Does the fundamental counting principle apply?
- Is it possible to use the formula ${}_nP_r$?

Permutation problems sometimes involve conditions. In certain situations, objects may be arranged in a line where two or more objects must be placed together, or certain objects(s) must be placed in certain positions. For example, How many arrangements of the word FAMILY exist if A and L must always be together? When certain items are to be kept together, students should treat the joined items as if they were only one object. Therefore, there are five groups in total, and they can be arranged in 5! ways. However, the letters AL can be arranged in 2! ways, and so the total arrangements would be 5! × 2! = 240.

Students should work through counting problems that include words such as, "at most" or "at least," such as the following:

- To open the garage door of Mary's house, she uses a keypad containing the digits 0 through 9. The password must be at least a four-digit code to a maximum of six digits, and each digit can only be used once in the code. How many different codes are possible?

Students may find this difficult and need to be guided as they calculate the number of permutations possible for 10 numbers, where each arrangement uses 4, 5, or 6 of those objects. Students should understand there are three cases to consider, ${}_{10}P_4 + {}_{10}P_5 + {}_{10}P_6$.

Students will also learn to use permutations to solve probability questions, first determining the number of permutations of *n* objects taken *r* at a time, ${}_nP_r$, and then relating that to the probability of one particular case.

For example, to determine the probability that *A*, *K*, and *Q* of spades will be dealt in that order as the first three cards from a well-shuffled deck of 52 cards, the number of permutations for 52 cards taken three at a time, would be determined as ${}_{52}P_3 = \frac{52!}{(52-3)!} = \frac{52!}{49!} = (52)(51)(49) = 132\ 600$. The probability of one particular sequence is only 1 out the total number of 132 600 possibilities or $P(A, K, Q) = \frac{1}{132\ 600}$.

Knowledge of permutations should be applied to solve equations of the form ${}_nP_r = k$. Students should solve equations where they simplify the expression so that it no longer contains factorials. Limit examples where they are only required to work with equations where they solve for *n*. They will not be expected to solve the equation for *r*. For example, ${}_nP_2 = 30$ can be solved as follows:

${}_nP_2 = 30$ $\frac{n!}{(n-2)!} = 30$ $(n)(n-1) = 30$	$\frac{n!}{(n-2)!} = 30$ $(n)(n-1) = 30$ $n^2 - n - 30 = 0$
Students can reason that consecutive integers with a product of 30 are needed. Since 6(6 – 1) = 30, *n* = 6.	Alternatively, they can solve the quadratic equation $n^2 - n - 30 = 0$. This equation has roots 6 and –5. Since *n* must be a natural number, we know that *n* = 6.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve the following equations:
  (a) (n)(n – 1) = 30
  (b) $\frac{(n-1)(n-2)}{n-2} = 30$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Arrange the following in order of magnitude.
  (a) 6!; 11!; $\frac{15!}{12!}$; 3! × 4
  (b) $\frac{9!}{3!}$; $\frac{10!}{8!}$; $\frac{86!}{53!}$; $6! - 5!$

  Pick three of the above expressions and create a problem in which these symbols would be used in the solution.
- Write each as a ratio of factorials.
  (a) 7 × 6 × 5 × 4
  (b) 35 × 34 × 33 × 32 × 6 × 5 × 4 × 3 × 2 × 1
  (c) 14 × 13 × 8 × 7 × 6 × 3 × 2 × 1
  (d) 25 × 24 × 23 × 5 × 4 × 3 × 2 × 1
- Sheila is attempting to simplify $\frac{640!}{638!4!}$. She wrote the following steps:

$$\frac{640!}{638!\,4!}$$
$$= \frac{640 \times 639 \times 638!}{638!\,4!}$$
$$= \frac{640 \times 639}{4!}$$

  = 160 × 639

  = 102 240

  (a) Explain the strategy that was used in the second step of the solution.
  (b) Identify and explain any errors in Shelia's work.
  (c) Write the correct solution.
- Solve the following equations for *n*:
  (a) $\frac{(n+2)!}{(n+1)!} = 10$
  (b) $\frac{(n+5)!}{(n+3)!} = 56$
  (c) $\frac{n!}{(n-2)!} = 182$
  (d) $\frac{2(n+3)!}{(n+1)!} = 180$

- Create a question that has an answer of 8!. Exchange your problem with a partner to discuss the similarities and differences.
- In how many different ways can a set of five distinct books be arranged on a shelf?
- In how many different orders can 15 different people stand in a line?
- Consider the word COMPUTER and the ways you can arrange its letters using each letter only once.
  (a) One possible permutation is PUTMEROC. Write five other possible permutations.
  (b) Use factorial notation to represent the total number of permutations possible. Write a written explanation to explain why your expression makes sense.
- The electronic lock to a house has six buttons. To open the lock, a four-button combination has to be entered in sequence and can be tried only once before the lock freezes.
  (a) If none of the buttons are repeated, what is the probability of randomly entering the correct combination?
  (b) If you are allowed to repeat buttons, what is the probability of randomly entering the correct combination?
- Arrange a set of four different coloured blocks into as many different configurations of groups of two blocks as you can. Explicitly state any assumptions you are making, and then calculate the number of configurations to verify your results.
- There are 10 movies playing at a theatre. In how many ways can you see two of them on consecutive evenings?
- A soccer league has 12 teams, and each team plays every other team twice; once at home, and once away. How many games are scheduled?
- Determine the number of ways four different graduation scholarships can be awarded to 30 students under each of the following conditions:
  (a) No student may receive more than one scholarship.
  (b) Any student may receive any number of scholarships.
- Nine people try out for nine positions on a baseball team. Each position is filled by selecting players at random.
  (a) In how many ways could the nine positions be filled?
  (b) What is the probability that one particular person (e.g., Sebastien) will be the pitcher?
  (c) What is the probability that one of three people (e.g., George, Sandy, or Errol) will be first baseman?
  (d) What is the probability that one of three people (e.g., Chantal, Ashley, or Sandy) will be first baseman, and that one of two other people (e.g., Sam or Marg) will be pitcher?
- The code for a lock consists of three numbers selected from 0, 1, 2, 3, with no repeats. For example, the code 1-2-1 would not be allowed but 3-0-2 would be allowed. Using the permutation formula, determine the number of possible codes.
- How many two-digit numbers can be formed using the digits 1, 2, 3, 4, 5, and 6 if repetition is allowed?
- How many distinct arrangements of three letters can be formed using the letters of the word LOCKERS?
- How many three-letter arrangements can be formed using the letters M, A, and T allowing for repetition of the letters?
- Jean, Kyle, Colin, and Lori are to be arranged in a line from left to right.

(a) How many ways can they be arranged?

(b) How many ways can they be arranged if Jean and Lori cannot be side by side?

(c) How many ways can they be arranged if Kyle and Colin must be side by side?

(d) How many ways can they be arranged if Jean must be at one end of the line?

- If there are nine different nuts (four brazil nuts, three almonds, and two macadamia nuts), in how many different orders can you eat all of them?
- How many ways can the letters in ACTION be ordered if all the vowels must be kept together?
- How many ways can the letters MASK be formed?
- Determine the number of arrangements (of any number of letters) that can be formed from the word MASK? (Hint: Arrangement can be two letters, three letters, or four letters.)
- Show that you can form 120 distinct five-letter arrangements from GREAT but only 60 distinct five-letter arrangements from GREET.
- How many distinct arrangements can be formed using all the letters of STATISTICS?
- Find the total number of arrangements of the word SILK and the total number of arrangements of the word SILL. How do your answers compare? Explain why this relationship exists.
- Mary has a set of posters to arrange on her bedroom wall. She can fit only two posters side by side. If there are 72 ways to choose and arrange two posters, how many posters does she have in total?
- Solve the following equations for *n*.

  (a) ${}_{n}P_{2} = 42$

  (b) ${}_{n+1}P_{2} = 20$

  (c) ${}_{n-1}P_{2} = 12$
- Raffle tickets for a new bicycle are given out at a fundraiser. Each number is three digits, and the first digit cannot be zero.

  (a) What is the probability of ticket number 514 winning the bicycle? What assumption did you make?

  (b) What is the probability that a ticket with three as a second digit will win the bicycle?

- A code consists of three letters chosen from A to Z and three digits chosen from 0 to 9, with no repetition of letters or numbers. Students should explain why the total number of possible codes can be found using the expression ${}_{26}P_{3} \times {}_{10}P_{3}$.

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Many tasks in this unit can be solved several ways: using a graphic organizer such as a tree diagram, listing all the arrangements (provided the numbers are reasonable), using the fundamental counting principle, and using a formula involving permutations. It is important for teachers to allow students the opportunity to choose their method (within reason).
- In preparation for working with formulas involving permutations, students should be given opportunities to simplify factorial expressions. Ask students to simplify an expression such as $\frac{9!}{6!}$.

Initially, they may rewrite each factorial in its expanded form and then cancel common factors to obtain the final answer. Ask students to think about a more efficient way to rewrite the numerator so they do not have to expand the denominator. Students sometimes mistakenly reduce $\frac{9!}{6!}$ to $\frac{6!}{2!}$. Encourage them to write out the expanded form so they can identify their mistake.

- When students solve the equation $\frac{n!}{(n-2)!} = 20$, for example, ask them if both solutions *n* = 5 and *n* = –4 are correct and to explain why or why not. Since factorial notation is only valid for natural numbers, students should understand that *n* = –4 is an extraneous root.
- When working with permutations, it is important for students to make the connection to the fundamental counting principle and to factorial notation. In an example where there were three letters (*A*, *B*, and *C*) to arrange, for the first letter, students should notice there are three options since any of the three letters can be selected. Each time a letter is placed, there is one fewer letter to choose from the group. As they write the product of the numbers (3 × 2 × 1 = 6), they should observe that this process is similar to the fundamental counting principle. Introduce this product in abbreviated form as 3!.
- Using their prior knowledge, allow students to work in groups to solve a simple problem similar to the following: Adam, Marie, and Brian line up at a banking machine. In how many different ways could they order themselves?

  Using a systematic list, students might come up with six different ways.

  Repeat with a similar-type question, asking students to look for patterns that involve the use of factorials. Have them generalize the formula to apply for *n* objects selected *r* at a time.
- Students should solve problems where arrangements are created with and without repetition. Consider the arrangement of a five-digit password if only the digits 0–9 can be used. Ask students the following:
  - Is the order of the digits in a password important? Explain.
  - How many arrangements are possible if repetition is allowed? Do the number of choices stay the same or are they reduced?
  - How many arrangements are possible if repetition is not allowed? Do the number of choices stay the same or are they reduced?
  - In which case is there a greater number of permutations possible?
- When students are comfortable determining the number of permutations, introduce the use of permutations to determine wanted probabilities.
- To explain 0!

  Method 1

  Think about it as determining the number of ways there are to count an empty set. Since there is nothing to count, ask students how many ways it is possible to count nothing? They are likely to understand that the answer would be 1. Using this reasoning allows students to understand why both 1! *and* 0! equal 1.

  Method 2

  Ask students how the formula changes if all of the objects are used in the arrangement. Using substitution where *n* = *r*, results in $_nP_n$ which is *n*!. If the number of permutations of six people arranged in a line is 6!, ask students to illustrate this using the permutation formula. They should be able to explain that this example is a permutation of a set of six objects from a set of six. Therefore, applying the formula $_nP_r = \frac{n!}{(n-r)!}$ becomes $_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!}$. The only value of 0! that makes sense is 0! = 1! since 36 divided by 1 equals 36.

Suggested Models and Manipulatives

- colour counters

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- factorial notation
- permutations

Resources/Notes

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 4.2 and 4.3, pp. 238–257
  - Section 4.4, pp. 260–269
  - Section 4.7, pp. 283–290
  - Section 5.3, pp. 313–324

SCO P06 Students will be expected to solve problems that involve combinations. [ME, PS, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

P06.01 Explain, using examples, why order is or is not important when solving problems that involve permutations or combinations.
P06.02 Determine the number of combinations of *n* elements taken *r* at a time.
P06.03 Generalize strategies for determining the number of combinations of *n* elements taken *r* at a time.
P06.04 Solve a contextual problem that involves combinations and probability.

Scope and Sequence

Mathematics 11	Mathematics 12
–	**P06** Students will be expected to solve problems that involve combinations.

Background

For this outcome, students will investigate combinations, where the order of the selection is not important.

It is important that students be able to identify when the order is important and when it is not important.

In this outcome, students are introduced to the formula for finding combinations and compare this formula to the formula for finding permutations.

For example, if three people are chosen to be given door prizes and the first one is given $20, the second one chosen, $15, and the third one selected gets $10, then the order in which they were chosen is important, and this would be a permutation. However, if the three people chosen each got $15, then the order would not matter, and we would use a combination.

Rather than simply giving students the combination formula, have them develop it by considering how many of the permutations that are possible would be considered to be the same.

For example, if a committee of three people is selected from a group of five individuals, it is not important in what order they are named; the committee still includes the same three people.

The five people are represented by the letters *A*, *B*, *C*, *D*, and *E*. If *A*, *B*, and *C* are selected to be on the committee, the number of permutations of choosing *A*, *B*, and *C* from the larger set of five letters, ${}_5P_3$, will include all 3! permutations of *ABC* (that is, *ABC*, *ACB*, *BCA*, *BAC*, *CAB*, and *CBA*). To eliminate counting each of these permutation as unique, the total number of possible permutations must be divided by 3!. It is clear to students that the number of combinations must be smaller than the number of permutations and with a bit of explanation they will understand that ${}_nC_r = \frac{{}_nP_r}{r!}$.

The number of combinations, denoted as ${}_nC_r$ or $\binom{n}{r}$ or *n* choose *r*, would simplify to ${}_nC_r = \frac{{}_nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$.

Therefore, ${}_5C_3$ or $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!} = 10$, so there are 10 possible ways to select three people from a group of five when order is not important.

Similar to permutations, teachers should provide combination problems where students may have to use the fundamental counting principle or problems that involve conditions. A good application-type problem involves asking students to form subcommittees. In the case of selecting two groups from the same larger group, the number of possible combinations will combine both groups.

For example, if a committee of four people and a committee of three people are selected from a group of 10 people, and no person is assigned to both committees, the combinations for each committee will be determined first. For the first committee of four from 10 people, ${}_{10}C_4$or $\binom{10}{4} = \frac{10!}{4!6!} = 210$, so there are 210 ways to form the committee. There are just six people left, so for the second committee of three from six people, ${}_6C_3$or $\binom{6}{3} = \frac{6!}{3!3!} = 20$, and there are 20 ways to form this committee. Combining the two committees, there are 210 × 20 = 4200 ways to form both committees from 10 people. Selecting the smaller committee first will yield the same result.

Combinations are sometimes used along with other counting techniques. Students should be comfortable using technology to determine combinations, permutations, and factorials.

For example, a 17-member student council at the high school consists of nine girls and eight boys, and one of the committees has four council members; two must be girls and two must be boys. There are ${}_9C_2 = \frac{9!}{2!7!} = 36$ ways of selecting the two girls and ${}_8C_2 = \frac{8!}{2!6!} = 28$ ways of selecting the two boys. Because the committee must include two girls and two boys, there are 36 × 28 = 1008 ways of forming the committee. If the four committee members are selected at random, there are ${}_{17}C_4 = \frac{17!}{4!13!} = 2380$ possible combinations. Therefore, the probability that the committee will consist of two boys and two girls is approximately 42.4%. $P(2B, 2G) = \frac{{}_8C_2 \times {}_9C_2}{{}_{17}C_4} = \frac{1008}{2380} \doteq 0.424$.

Once students have solved many problems that involve permutations and combinations independently, they should progress to a mixture of problems where they must make a decision as to which concept applies.

When reading a problem, students should be asking themselves questions such as, Does order matter in this problem? If yes, then they know to solve using permutations; otherwise they will use combinations.

- For a permutations problem, the next question could be, Are the objects identical or distinguishable?

- For a combination problem, the next logical question to follow would be, Are there multiple tasks (calculations) required? If yes, then does the fundamental counting principle apply?

By developing a systematic approach, students should gain more confidence when faced with novel problem situations.

Students are expected to solve a variety of equations in the form ${}_nC_r = k$ for *n*. Ask students to solve an equation such as ${}_nC_2 = 15$. They should be able to easily verify that the value of *n* must be greater than or equal to *r* in ${}_nC_r$, having explained the reasoning when working with permutations. If they understand that the notation ${}_nC_r$ means choosing *r* elements from a set of *n* elements, it naturally flows that *r* cannot be larger than *n* and that *r* must be a natural number.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve: $n^2 - n = 30$.
- List all the possible arrangements of *ABC*.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Why is the number of combinations of six letters less than the number of permutations of six letters?
- There are three black marbles and two white marbles in a box. Without looking in the box, choose two of the five marbles. How many ways can
  - (a) two marbles that are the same colour be selected
  - (b) each marble be a different colour
- If a coin is flipped five times, what is the probability of flipping four heads and one tail?
- A scratch-and-win game has nine prize boxes. You are allowed to scratch three boxes and if the three pictures show identical objects, you win the object.
  - (a) If the ticket has three pictures of a bicycle, and the other six pictures are of six other things, what is the probability that you will win the bicycle?
  - (b) If the ticket has four pictures of a coffee, and the other five pictures show five different things, what is the probability that you will win a coffee?
  - (c) If the ticket has three pictures of a coffee, three pictures of a sandwich, and three pictures of a bicycle, what is the probability that you will win one of these three things?

- At a local frozen yogurt shop, you can order a sundae with a choice of toppings. There are three different sauces to choose from (chocolate, strawberry, raspberry) and four different dry toppings (peanuts, almonds, granola, walnuts). When selecting one sauce and one dry topping, how many different sundaes could you order?
- A volleyball team has 12 players. How many ways can the coach choose the starting line-up of six players?
- If a committee of eight people is to be formed from a pool of 13 people, but Mitchell and Lisa must be on the committee, how many different committees can be formed?
- The student council decides to form a sub-committee of five members to plan their Christmas concert. There are a total of 11 student members; five males and six females.
  - (a) How many different sub-committees are possible of exactly three females?
  - (b) How many different ways can the sub-committee consist of at least three females?
  - (c) How many different ways can the sub-committee consist of at least one female?
- Create a display or foldable in which you list or draw all of the possible combinations for a scenario and verify the answer using a combination formula (i.e., a pizza shop offers 5 toppings).
- From a standard deck of 52 cards, how many five-card hands have
  - (a) at least one red card
  - (b) at least one face card (J, K, or Q)
  - (c) at most, three aces
  - (d) no spades
- A volleyball coach decides to use a starting line-up of one setter, two middle hitters, two power hitters, and one right-side hitter. She chooses 14 players for the team: three setters, four middle hitters, four power hitters, three right-side hitters. How many possible starting line-ups are there?
- Stores sell “combination locks.” Is this correct mathematical terminology?
- Create a foldable or flowchart outlining a series of questions or problem-solving strategies for problems involving permutations or combinations.
- Solve for *x*: ${}_{x+2}C_2 = 21$.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- To emphasize the important difference between the order mattering and the order not mattering, have each student in the class write their name on a piece of paper and then place their name in a hat. Draw three names.
  - For the first draw where the order does not matter, give the first student a granola bar, the second a granola bar, and the third a granola bar (or some other item—all items must be the same).
  - For the second draw where order does matter, give the first student a pencil, the second a granola bar, and the third a loonie.
- To distinguish between a permutation and a combination, students should be given a situation for each where the number of possibilities can be determined with simple counting methods. They have been introduced to permutations as an arrangement of objects in which order matters. Combinations, however, is a grouping of objects where order does not matter. It is important for students to highlight the differences between permutations and combinations. Ask students to identify problems, such as the following, as a permutation or a combination.
  - *My fruit salad is a combination of apples, grapes, and strawberries*. Students should understand that whether it is a combination of strawberries, grapes, and apples or grapes, apples, and strawberries, it is the same fruit salad.
  - *The code to the safe was 4-7-2*. In this case, students should understand that the order 7-2-4 or 2-4-7 would not work. It has to be exactly 4-7-2.
- Starting with an arrangement involving a small number will give students a visual representation of the possibilities.

  For example, an assignment consists of three questions (A, B, C) and students are required to attempt two.
  - Calculate the number of permutations for choosing two of the three questions.
  - Write the number of arrangements to verify your answer. (AB, BA, BC, CB, AC, CA)
  - How many ways can two questions be arranged?
  - Is the order in which questions are chosen important?
  - Why is it necessary to divide ${}_3P_2$ by 2! to determine the number of combinations?

  Discuss with students that the order in which questions are chosen is not important, therefore, each group of two permutations is just one combination. The number of combinations of *n* elements taken *r* at a time is represented by ${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$.
- Group work is effective here. Have groups of students use systematic lists and to look for patterns as they solve problems. Having worked previously with permutations and the formula involving factorials, now extend the formula. Have students work towards generalizing the formula to apply for *n* objects selected *r* at a time.

- Ask students to participate in a Quiz-Quiz-Trade activity. Provide students with cards, each having a scenario that is either a permutation or combination. Student 1 must read a card to student 2 who then decides whether it is a permutation or combination, explaining why they think so. They then switch roles, after which students will trade cards and find another partner.
- When students are comfortable determining the number of combinations, introduce the use of combinations to determine wanted probabilities.

SUGGESTED MODELS AND MANIPULATIVES

- colour counters
- playing cards

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- combination

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 4.5, 4.6, and 4.7, pp. 271–290
  - Section 5.3, pp. 313–324

Internet

- "TI Math," *Texas Instruments* (Texas Instruments Incorporated 2015): http://education.ti.com/calculators/timath/ (Permutations and combinations activities using TI graphing calculators.)
- "Pure Math 30: Explained!" *Barry Mabillard* (Barry Mabillard 2009): www.bmlc.ca/PureMath30.html (Provides examples and a series of problem sets related to combinations.)
- "Delta State University," Delta State University: www.deltastate.edu (Contains a permutations/combinations group activity.)

Relations and Functions
35–40 hours

GCO: Students will be expected to develop algebraic and graphical reasoning through the study of relations.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO RF01 Students will be expected to represent data, using polynomial functions (of degree ≤ 3), to solve problems. [C, CN, PS, T, V]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF01.01 Describe, orally and in written form, the characteristics of polynomial functions by analyzing their graphs.
RF01.02 Describe, orally and in written form, the characteristics of polynomial functions by analyzing their equations.
RF01.03 Match equations in a given set to their corresponding graphs.
RF01.04 Graph data and determine the polynomial function that best approximates the data.
RF01.05 Interpret the graph of a polynomial function that models a situation, and explain the reasoning.
RF01.06 Solve, using technology, a contextual problem that involves data that is best represented by graphs of polynomial functions, and explain the reasoning.

Scope and Sequence

Mathematics 11	Mathematics 12
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, axis of symmetry.	**RF01** Students will be expected to represent data, using polynomial functions (of degree ≤ 3), to solve problems.

Background

In Mathematics 10, students related linear relations expressed in slope-intercept form, general form, and slope-point form to their graphs (RF06). In Mathematics 11, they examined quadratic functions expressed in standard form: $f(x) = ax^2 + bx + c$, vertex form: $f(x) = a(x - h)^2 + k$, and factored form: $f(x) = a(x - r)(x - s)$ (RF02). In this unit, students work with cubic functions in the form $f(x) = ax^3 + bx^2 + cx + d$.

A polynomial function has the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, where the coefficients are real numbers and the exponents are natural numbers. In other words, a polynomial is the sum of one or more monomials with real coefficients and non-negative integer exponents.

The degree of the polynomial function is the highest value for *n* where a_0 is not equal to 0.

- A degree 0 polynomial function, such as $f(x) = 3$, is also called a linear function or a constant function.
- A degree 1 polynomial function, such as $f(x) = 2x + 3$, is also called a linear function or an oblique linear function.
- A degree 2 polynomial, function such as $f(x) = 4x^2 - 5x + 3$, is also called a quadratic function.

- A degree 3 polynomial, function such as $f(x) = 5x^3 - 4x^2 + 2x - 7$, is also called a cubic function.

Polynomial functions of only one term are called **monomials or power functions**. A power function has the form $f(x) = ax^n$.

The approach to graphing in this outcome is similar to that for quadratics in Mathematics 11. The primary goal is to understand the characteristics of the graphs and their relationship to applications.

During the intermediate grades, students were encouraged to translate between different representations of a relationship. For example, from tables, they should be able to describe relationships and make equations and graphs; from graphs of situations, they should be able to describe the situations in words and with equations; from a written description, they should be able to sketch a graph or make a table.

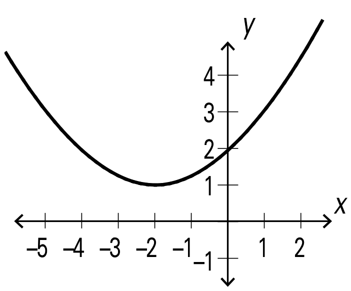
Students have analyzed and applied linear functions and should understand that a linear function represents a constant growth rate (M10). Students have also worked with quadratic functions and their properties: vertex, intercepts, domain, and range (M11).

Students should, through a variety of experiences with functions, recognize the elements in a real-world problem that suggest a particular model. For example, area, accelerated motion, and trajectory suggest quadratic functions.

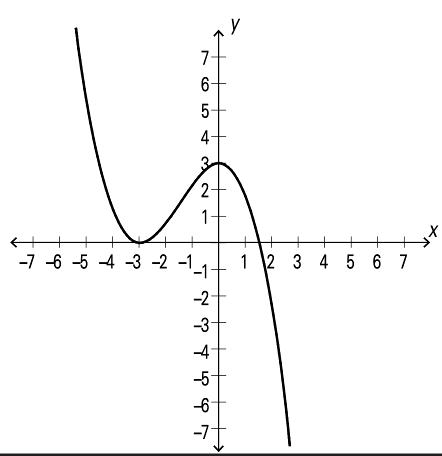
For quadratic functions, the *y*-coordinate of the vertex is an **absolute minimum (or maximum)** for the function since it is the least (or greatest) value of the function for all values of the domain.

Degree 3 polynomial functions do not have absolute minimum or maximum values. The *y*-coordinates of the vertices are local minimum (or maximum) values since they are the least (or greatest) value of the function when compared to the neighbourhood points on both sides of the vertex.

The quadratic, shown below, has an absolute minimum at the point (–2, 1).



The cubic, shown below, has a local minimum at the point (–3, 0) and a local maximum at the point (0, 3).



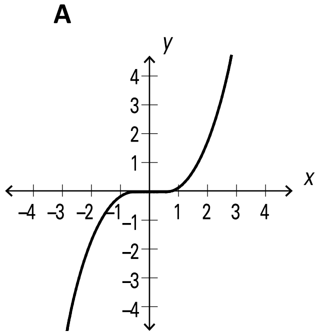
Cubic functions will be new to most students. Traditionally, polynomials with degree 3 or more are called higher-degree polynomials. Students will learn to identify the characteristics of a polynomial function with real coefficients, including the maximum number of *x*-intercepts (identifying coordinates from graphs and factored form of equations), the *y*-intercept, the domain, the range, the possible number of turning points or vertices (local maxima or minima), and the end behaviour.

For this outcome, students will discover how to make the connections between the equations of polynomial functions and their graphs, in order to predict where different features will occur.

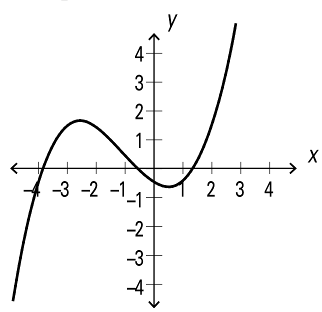
Equation	**Example**	**Type**	**Description**
$f(x) = a,$ $a \neq 0$	*y* = 5	Constant polynomial (degree 0)	Zero slope
$f(x) = ax + b,$ $a \neq 0$	*y* = 5*x* + 2	Linear polynomial (degree 1)	Constant slope; 1 *x*-intercept
$f(x) = ax^2 + bx + c,$ $a \neq 0$	$y = 5x^2 + 2x + 1$	Quadratic polynomial (degree 2)	Max. of 2 *x*-intercepts; Min. of 0 *x*-intercepts
$f(x) = ax^3 + bx^2 + cx + d,$ $a \neq 0$	$y = 5x^3 + 2x^2 + x + 6$	Cubic polynomial (degree 3)	Max. of 3 *x*-intercepts; Min. of 1 *x*-intercepts

Students should be able to identify the degree of any polynomial graph by looking at its shape. For example, all of the graphs shown here are those of third-degree polynomials, also known as cubics.

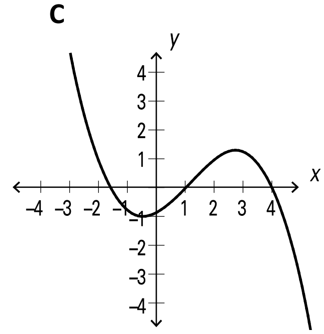
A



B



C



Graph A is the graph of the function $y = x^3$. Although it only crosses the *x*-axis once, it actually has a triple root of *x* = 0 (also known as a root of multiplicity 3). The graph of $y = x^3$ has no local maximum or minimum.

Graphs B and C are of the general cubic equation $y = ax^3 + bx^2 + cx + d$. Graph B shows a function in which *a* is positive, and Graph C, a function in which *a* is negative. Both have three roots.

Students should observe that the degree of a polynomial function determines the shape of the function. For a cubic function, the end behaviour is opposite on the left and right sides of the graph. If the leading coefficient of the cubic function is positive, then the graph falls to the left and rises to the right (similar to the graph of *y* = *x*). If the leading coefficient is negative, then the graph rises to the left and falls to the right (similar to *y* = –*x*).

For polynomials of degree 3 or less, students should summarize features such as the following:

- The graph of a polynomial function is continuous.
- The degree of the polynomial function determines the shape of the graph.
- The maximum number of *x*-intercepts: equal to the degree of the function.
- The minimum number of *x*-intercepts: zero for an even degree polynomial; one for an odd degree polynomial.

- There is one *y*-intercept for all polynomial functions.
- The graph of a polynomial function has only smooth turns.
- A function of degree *n* has a maximum of (*n* – 1) turns.
- A function of degree *n* has a minimum of 1 vertex (or turn) if it is even, 0 vertices (or turns) if it is odd.
- The end behaviour of a line or curve is the behaviour of the *y*-values as *x* becomes large in the positive (i.e., as$x \to \infty$) or negative direction (as$x \to -\infty$). Note that limit notation such as $\lim_{x\to\infty} f(x) \to \infty$ is not an expectation of this course.

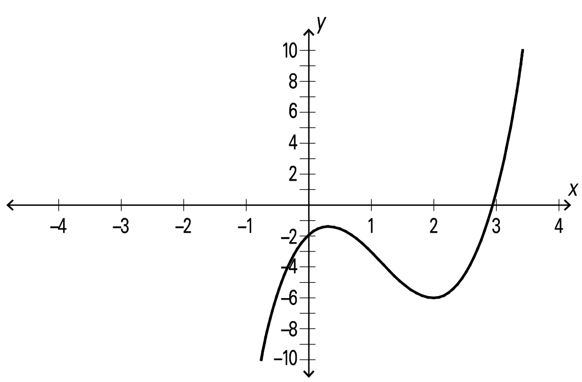
Students will not be expected to factor cubic polynomials to find the *x*-intercepts. However, they should be able to determine the *y*-intercept from the equation, and determine *x*-intercepts from the factored form of the equation.

When analyzing the graph, students will be expected to describe the *x*-intercepts as single, double, or triple roots.

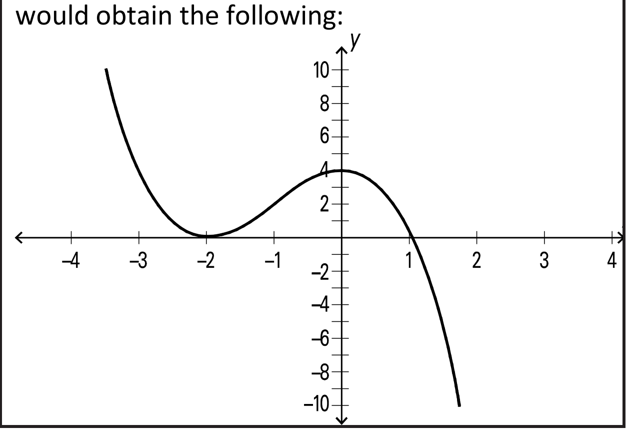
Students should be able to match a polynomial function with its corresponding graph and to justify their matches in relation to characteristics, such as end behaviour, turning points, *y*-intercept, and number of *x*-intercepts.

<table>
<tr>
<td>Given equation: $y = 2x^3 - 7x^2 + 4x - 2$

A student would know that this function would have a y-intercept of (0, –2) and that since the leading coefficient is positive, as $x \to \infty, y \to \infty$; and as $x \to -\infty, y \to -\infty$.

When students graph $y = 2x^3 - 7x^2 + 4x - 2$, they would obtain the following:
</td>
<td>Given equation: $y = -(x - 1)(x + 2)^2$

A student would know that this function would have a y-intercept of (0, 4); that the function would have a single root at (1, 0) and a double root at (–2, 0), and that since the leading coefficient is negative, as $x \to \infty, y \to -\infty$; and as $x \to -\infty, y \to \infty$.

When students graph $y = -(x - 1)(x + 2)^2$, they would obtain the following:
</td>
</tr>
</table>

Students should be able to explain how the degree of the function and the leading coefficient will indicate the end behaviour of the graph. They should also be able to describe how changing the constant term will affect the number of *x*-intercepts.

Students should determine a polynomial function (degree ≤ 3) that best fits a set of data and use the regression function to solve a problem. It should be noted that students are not expected to determine the equation of a regression line or curve without the use of technology.

In Mathematics 10, students were exposed to scatterplots and the correlation coefficient (RF08). This will be their first exposure to the concept of regression. Exponential, logarithmic, and sinusoidal regression equations will be addressed in RF02 and RF03.

Students need to be provided with an opportunity to become comfortable with using technology to find the regression equations. Graphing calculators, mobile devices, smartphone and tablet applications, computer programs, or online tools can be used to calculate and graph regression lines/curves.

Students should work with linear regression before moving on to a quadratic and a cubic regression. Real-life data is used to produce a table and create a scatter plot for the data. Students should be able to distinguish between independent and dependent variables. The scatter plot provides them with a visual representation of the data. It may be necessary to review with students a linear relation that can be expressed in slope-intercept form. They should identify the characteristics of the line of best fit and write the equation in *slope-intercept* form. Furthermore, students should be able to determine what the slope and y-intercept of a best fit line represents for a contextual problem.

Students should then be exposed to data where they determine the quadratic or cubic regression equation. Linear regression can be revisited when the different forms of regression are compared. It is important to encourage students to discuss the characteristics of the graph. If the curve appears parabolic and opens downward, for example, they should understand the leading coefficient of its equation should be negative.

Example:

A student standing on the top of a hill and shoots an arrow. The data that follows gives the height of the arrow versus the seconds since the arrow was shot.

time	0.2	0.6	1	1.6	1.8	2	2.4	2.7
height	62.5	68.8	73.5	77.6	78.2	78.4	77.6	76.0

Although some graphs can be plotted using tables of values, in most cases students should use a graphing calculator or similar tool to view the graph of a function or should be given a paper copy of the graph.

When working with examples, students will often be given the type of regression to apply (linear, quadratic, or cubic). When working with data in a specific context, students should be able to predict which type of polynomial function would best model a set of data. They can then use technology to run a regression to test their prediction.

It is important for students to observe that the polynomial regression results in an equation of a line (or curve) that balances the points on both sides of the line (or curve).

Note: If only four data points are available, then cubic regression will always yield $r = 1$ since using four points with the general equation $y = ax^3 + bx^2 + cx + d$ creates a 4 × 4 system of equations. That does not mean that a cubic model would be the best model. It is important for students to think about the end behaviour of the function and if it fits with the specific data.

It is important to provide students with a variety of problems that involves data that is best represented by graphs of polynomial functions.

The average gasoline price in Canada from 2005 to 2009 is shown. Using a cubic regression, ask students to predict the price of gasoline in the year 2020.

Numbers of years since 2005	**Price in cents/ litre**
0	92.3
1	87.7
2	101.8
3	114.1
4	94.5

Students can interpolate or extrapolate values by tracing along the line/curve of best fit or by substituting values into the equation of the regression function. Remind students that the regression equation is a model that best suits the data as a whole, rather than at any one point on the scatter plot. Hence, the predicted values calculated may not match experimental data.

Furthermore, caution needs to be used when extrapolating. Specifically, whenever a regression model is used to fit a group of data, the range of the data should be carefully observed. Attempting to use a regression equation to predict values outside of this range (extrapolation) is often inappropriate.

Present the following scenarios to students to give them a picture of how extrapolation may yield unrealistic answers.

- A linear model relates weight gain to the age for young children.
  - Applying such a model to adults, or even teenagers, would not be appropriate, since the relationship between age and weight gain is not consistent for all age groups.
- A cubic model relates the number of wins for a hockey team over time (in years).
  - If students extrapolate too far in the future, their model could yield more wins than is possible in a year.

Students will only be expected to estimate, outside the given data, examples that are mathematically appropriate. External factors that could affect predicted outcomes should also be analyzed. Suppose, for example, data was collected related to the number of shoppers at a local store in a year. If the data follows a cubic trend, the cubic regression could help students make predictions about the number of shoppers at the store on a given day. The model, however, does not predict whether the store decides to have a special sale on that day, or if there could have been increased competition, poor weather conditions, or any other number of factors.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

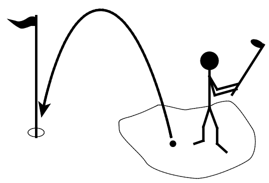
- What is a function? Give some examples of a linear function and a quadratic function.
- Determine the intercepts for the following functions:
  $$f(x) = 2x - 3, f(x) = x^2 + 5x + 6$$
- Sketch $f(x) = -(x + 1)(x - 2)$.
- Sketch $g(x) = -2x + 1$.
- What is the maximum and minimum number of *x*-intercepts for a quadratic?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Use graphing technology to determine any similarities and differences between polynomials such as the following:
  (a) $f(x) = 2x + 1$
  (b) $f(x) = x^2 + 1$
  (c) $f(x) = x^2 + 4x + 4$
  (d) $f(x) = (x + 3)^2 - 1$
  (e) $f(x) = 2x^3$
  (f) $f(x) = x^3 + 2x^2 - x - 2$
  Graph each of these with a negative leading coefficient.
- Which type of polynomial function has a maximum value?
- Is it possible for a cubic to have a minimum value? Explain.
- Which type of polynomial function always has a domain and range belonging to the real numbers?
- How many turning points are possible if the function is cubic? quadratic? linear?
- For each type of polynomial function, write an equation that has a *y*-intercept of –3.
  (a) constant
  (b) linear
  (c) quadratic
  (d) cubic

- Write an equation for a polynomial function that satisfies each set of characteristics:
  (a) having one turning point, y-intercept of 4
  (b) having three x-intercepts
  (c) having two turning points, y-intercept of –3
  (d) having a range of $y \geq 2$, y-intercept 2
  (e) increasing function, degree 1, y-intercept of 4
- Create a poster or a multimedia presentation summarizing the characteristics of polynomial functions of degree 3 or less.
- Sketch two possible graphs of polynomial functions that satisfy each set of characteristics:
  (a) two vertices, negative leading coefficient, constant term –5
  (b) degree two, one vertex that is a minimum, constant term –3
- This picture represents the parabolic path of a golf ball as it flies through the air.

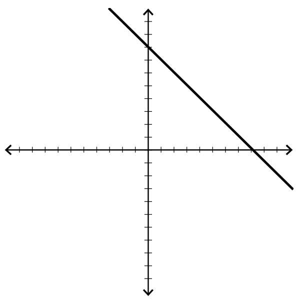


  (a) Describe how the height of the golf ball changes from the start to the finish of its path.
  (b) Sketch and explain a graph to illustrate your description.
  (c) When a golf ball travels through the air (goes up and then back down to the ground), do you think it maintains the same speed at all times? Explain.
  (d) Where in its flight is the speed of the ball the slowest? Explain.
  (e) After retrieving a ball from the roof of my house, I threw it up into the air towards the street. Sketch a picture of the flight of the ball.
    (i) How is it the same as the flight of the golf ball in question?
    (ii) How is it different?
- A box with no top is being made out of a 20 cm × 30 cm piece of cardboard by cutting equal squares of side length x from the corners and folding up the sides. The table of values below shows the volume of the box, $V(x)$ based on side length, x.

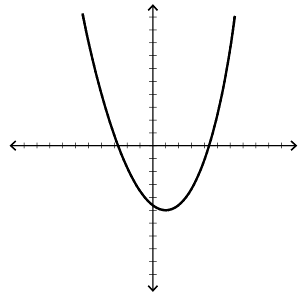
x	1	2	3	6	8
$V(x)$	504	832	1008	864	448

  (a) Graph the data and, based on the shape of the graph, determine the type of polynomial function (linear, quadratic, or cubic) that will best approximate the data.
  (b) Use technology to find the polynomial function that best models the situation.
  (c) Use the function to determine the volume of a box created by cutting out squares of side length.
  (d) Use the graph to determine the side length(s) that would result in a volume of 900 cm^3 by interpolating within the data set given.

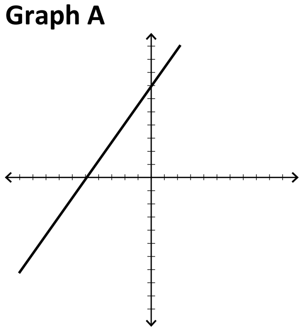
- Sketch the graphs of $f(x) = -(x-1)(x+2)(x-4)$ and $g(x) = -(x-1)(x+2)(x+2)$, then compare their shapes and the number of *x*-intercepts. (These window dimensions work well: *x*-min: –6, *x*-max: 6, *y*-min: –12, *y*-max: 12.)
- State a possible equation for each of the following graphs.



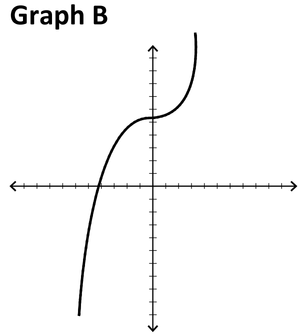
Graph A



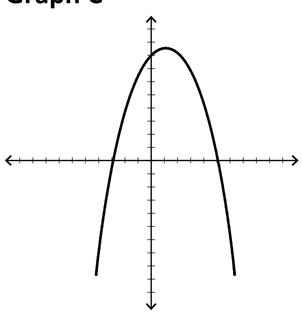
Graph B



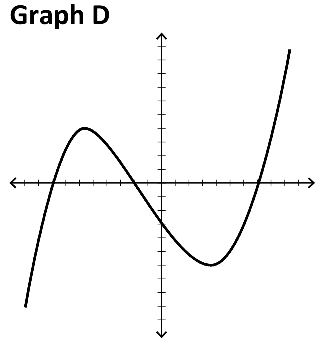
Graph C



Graph D

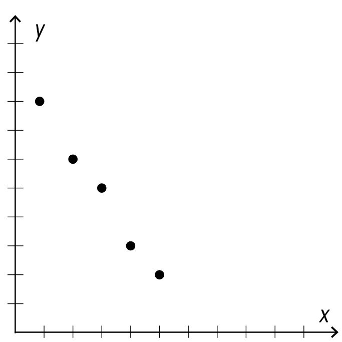


Graph E



Graph F

- The following scatter plot shows the change in temperature from 1 p.m. to 5 p.m. in Truro.



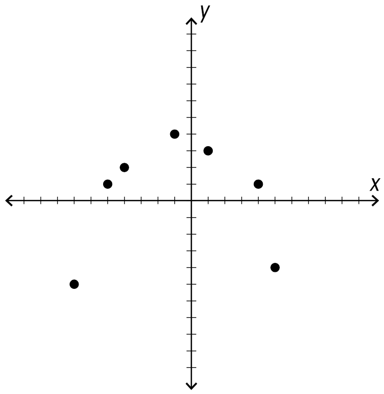
L1	**L2**
1	8
2	6
3	5
4	3
5	2
______	______

(a) Draw lines of best fit for the scatterplot below, share your results with another student and discuss any similarities and differences.

(b) Use graphing technology to determine the actual linear regression equation.

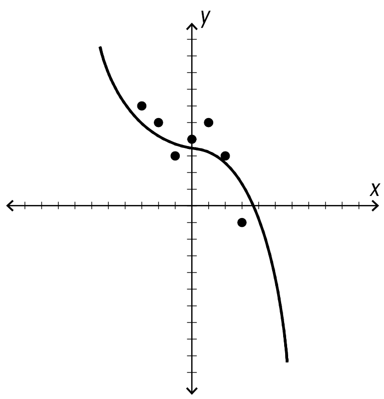
(c) What do the properties of the line (slope and y-intercept) represent in this particular situation?

- Using graphing technology, determine the equation of the quadratic regression function that models the data.



L1	L2
–7	–5
–5	1
–4	2
–1	4
1	3
4	1
5	–4

- The following shows the scatter plot and curve of best fit for a set of data. Use the graph or equation to estimate the value for y when $x = 3.5$.



L1	L2
–3	6
–2	5
–1	3
0	4
1	5
2	3
3	–1

CubicReg
$y = ax^3 + bx^2 + cx + d$
$a = -.0841750842$
$b = -.1608946609$
$c = -.2657527658$
$d = 4.155844156$

- Respond to the following email that Mary sent to her teacher.

To: p.lundrigan@gmail.com

Subject: formula for regression line…?

Hey! I have been working on a formula for the results of my physics project on the amount of weight that can be supported by the thickness of an elastic band. Here is the data I have collected:

Thickness of Elastic	Weight
1	1
5	9
10	20
15	34
20	54
25	87
30	141

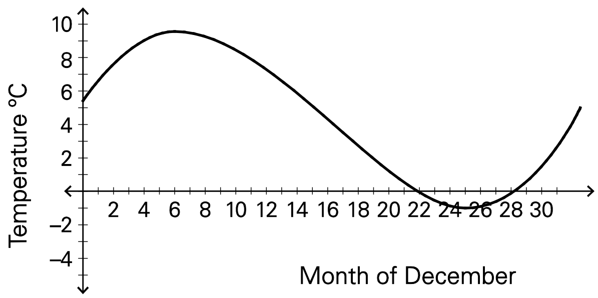
I did a quadratic regression and it looks like a pretty good fit.

The formula I determined was $w = 0.17t^2 - 0.81t + 5.89$ (w represents the weight supported [grams] and t represents the thickness of the band [centimetres])

Can you check if this is a good approximation or would a cubic regression be more accurate?

Thxs!

- The following graph was used to model the changes in temperature last December.



(a) What would your prediction be for the temperature on Jan 1?
(b) What was the coldest day of the month?
(c) What might the temperature have been on Nov. 27?
(d) Would you use this model to predict the temperature on Jan. 15? Explain your reasoning.

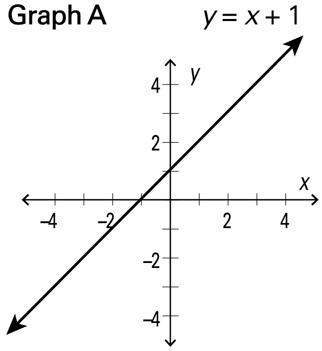
Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

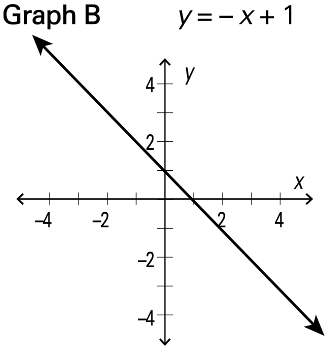
- A quick review of slope and equation of a line would be advisable at the beginning of this unit. Students could be presented with graphs of linear equations and be required to state the slope, intercepts, domain, and range.
- Discuss vertical lines with students. These lines could have no *y*-intercept or an infinite number of *y*-intercepts. However, they are not functions because they do not satisfy the vertical line test. This means they are not part of the family of polynomial functions.
- Teachers should begin by explaining the definition of end behaviour of the graph of a function as the behaviour of the y-values as x becomes large in the positive or negative direction. The following are two possibilities for oblique linear functions. Ask students to fill in the table and examine the relationship between the slope of a line and the end behaviour.

Graph A $y = x + 1$



Slope: positive
End Behaviour: increasing
As $x \to \infty, y \to \infty$
As $x \to -\infty, y \to -\infty$

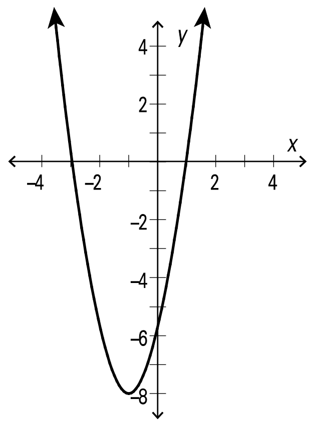
Graph B $y = -x + 1$



Slope: negative
End Behaviour: decreasing
As $x \to \infty, y \to -\infty$
As $x \to -\infty, y \to \infty$

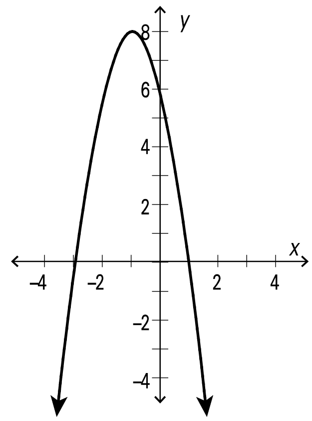
- Similarly, students could use the following graphs to explore the end behaviour of parabolas. Ask students to fill in the table and examine the relationship between the leading coefficient, range, and end behaviour.

Graph A $y = 2x^2 + 4x - 6$



Leading coefficient is positive.
Absolute minimum at vertex.
Range: $\{y \mid y \geq -8, y \in \mathrm{R}\}$
End Behaviour:
As $x \to \infty, y \to \infty$
As $x \to -\infty, y \to \infty$

Graph B $y = -2x^2 - 4x + 6$

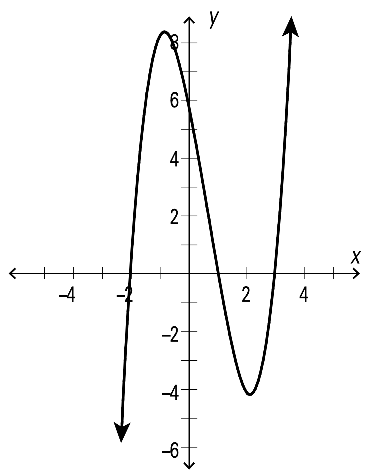


Leading coefficient is negative.
Absolute maximum at vertex.
Range: $\{y \mid y \leq 8, y \in \mathrm{R}\}$
End Behaviour:
As $x \to \infty, y \to -\infty$
As $x \to -\infty, y \to -\infty$

Students should conclude the graph has the same behaviour to the left and right.

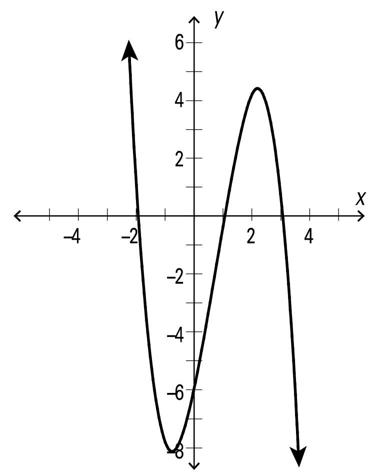
- When working with cubic polynomial functions, students should fill in the table and examine the relationship between the leading coefficient, range, and end behaviour.

Graph A $y = x^3 - 2x^2 - 5x + 6$



Leading coefficient is positive.
Local maximum and minimum at vertices.
Range: $\{y \mid y \in \mathbb{R}\}$
End Behaviour:
As $x \to \infty, y \to \infty$
As $x \to -\infty, y \to -\infty$

Graph B $y = -x^3 + 2x^2 + 5x - 6$



Leading coefficient is negative.
Local minimum and maximum at vertices.
Range: $\{y \mid y \in \mathbb{R}\}$
End Behaviour:
As $x \to \infty, y \to -\infty$
As $x \to -\infty, y \to \infty$

Students should notice a pattern as they analyze the end behaviour of cubic functions. That is, the graph has opposite behaviours to the right and left.

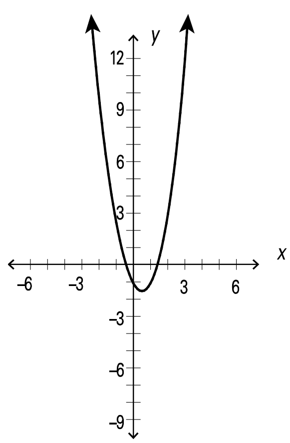
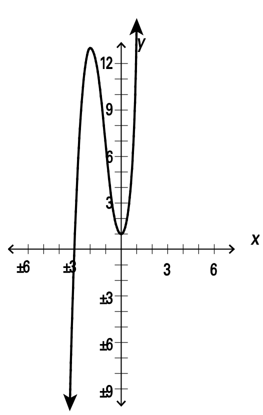
- Provide students with a variety of polynomial functions (of degree ≤ 3). Ask them to determine the following characteristics for each polynomial function:
  - *y*-intercept
  - maximum number of *x*-intercepts
  - end behavior
  - domain
  - range
  - number of turning points

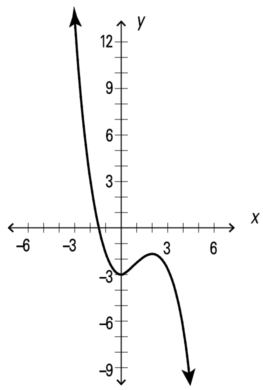
  Ask students to form three groups based on the shape of their graphs (linear, quadratic, or cubic). They should discuss the characteristics of their graphs, the similarities and the differences.

- Using graphing technology, ask students to investigate several equations of cubics and their corresponding graphs. It may be beneficial for students to initially graph $y = x^3$ and $y = -x^3$. They should then proceed to other variations such as $y = 2x^3 + 10x^2 - 2x - 8$ and $y = -x^3 + x^2 + 5x + 3$. Use prompts such as the following to promote discussion:
  - How does the sign of the leading coefficient in a polynomial function affect the end behaviour of the graph?
  - What is the relationship between the constant term in the polynomial function and the *y*-intercept of the graph?

- Have students create graphs or tables of values to represent relationships that are linear, quadratic, or cubic. Then have them exchange with a partner and ask the partner to determine which it is and explain why.
- Give one student in a pair the graph of a polynomial function. Ask the student to turn to their partner and describe, using the characteristics of the function, the graph they see. The partner will draw the graph based on the description from the student. Students should explain why two graphs could fit the description but still look different from each other.
- Have students investigate the maximum number of x-intercepts for linear ($y = ax + b$), quadratic ($y = ax^2 + bx + c$), and cubic ($y = ax^3 + bx^2 + cx + d$), functions using graphing technology.
- Have students describe and compare the key features of the graphs of the functions $y = x$, $y = x^2$, and $y = x^3$.
- To further develop their understanding of the properties of polynomial functions, provide students with a set of graphs and matching equations and ask them to match the graphs and equations and provide reasons for their choices.

  Their reasoning might be something like shown in the following table.

Graph	Equation	Reasoning
	$y = 2x^2 - 2x - 1$	▪ Degree 2 polynomial since one vertex and two x-intercepts. ▪ Y-intercept is –1, and constant term is also –1. ▪ Leading coefficient is positive and end behaviour is As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$
	$y = 3x^3 + 9x^2 + 1$	▪ Degree 3 polynomial since two vertices between one and three x-intercepts. ▪ Y-intercept is 1, and constant term is also 1. ▪ Leading coefficient is positive and end behaviour is As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$

Graph	Equation	Reasoning
	$y = \frac{1}{3}x^3 + x^2 - 3$	▪ Degree 3 polynomial since two vertices between one and three *x*-intercepts. ▪ *Y*-intercept is –3, and constant term is also –3. ▪ Leading coefficient is positive and end behaviour is As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$

- For the activity Match It Up, present students with a variety of index cards each containing information about equations, graphs, or a written description. Students move around the classroom trying to locate their matching cards.

  or

  Have groups of students create sets of cards, some with equations and others with the matching graphs. Have groups mix them up and challenge other groups to match them.
- Ask students to research and collect their own data on a subject of interest. They should create a scatter plot and describe a possible trend in the data. Ask students to determine an appropriate model using regression, pose a problem, and then use their model to solve the problem. They should present their findings to the class.

Suggested Models and Manipulatives

- ruler
- metre stick
- motion detector or CBR

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- continuous
- cubic function
- curve of best fit
- degree
- dependent variable
- end behaviour
- independent variable
- leading coefficient
- line of best fit
- polynomial function
- scatterplot
- standard form
- turning point or vertex
- *x*-intercept
- *y*-intercept

Resources/Notes

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 6.1 and 6.2, pp. 380–397
  - Sections 6.3 and 6.4, pp. 401–422

Internet

- "Explore the Quadratic Equation." *MathIsFun.com*: www.mathsisfun.com/algebra/quadratic-equation-graph.html (Use the interactive quadratic function grapher to explore relationships between quadratic equations and their graphed parabolas.)
- "Function Matching." *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2015): http://illuminations.nctm.org/ActivityDetail.aspx?ID=215 (Interactive graphing activity)

SCO RF02 Students will be expected to represent data, using exponential and logarithmic functions, to solve problems. [C, CN, PS, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF02.01 Describe, orally and in written form, the characteristics of exponential or logarithmic functions by analyzing their graphs.
RF02.02 Describe, orally and in written form, the characteristics of exponential or logarithmic functions by analyzing their equations.
RF02.03 Match equations in a given set to their corresponding graphs.
RF02.04 Graph data and determine the exponential or logarithmic function that best approximates the data.
RF02.05 Interpret the graph of an exponential or logarithmic function that models a situation, and explain the reasoning.
RF02.06 Solve, using technology, a contextual problem that involves data that is best represented by graphs of exponential or logarithmic functions, and explain the reasoning.

Scope and Sequence

Mathematics 11	**Mathematics 12**
–	**RF02** Students will be expected to represent data, using exponential and logarithmic functions, to solve problems.

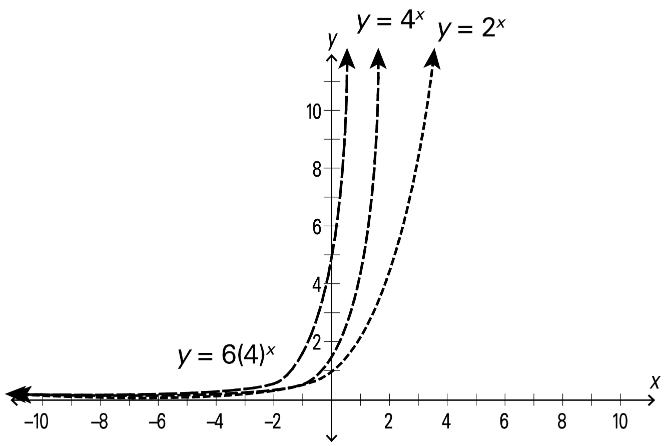
Background

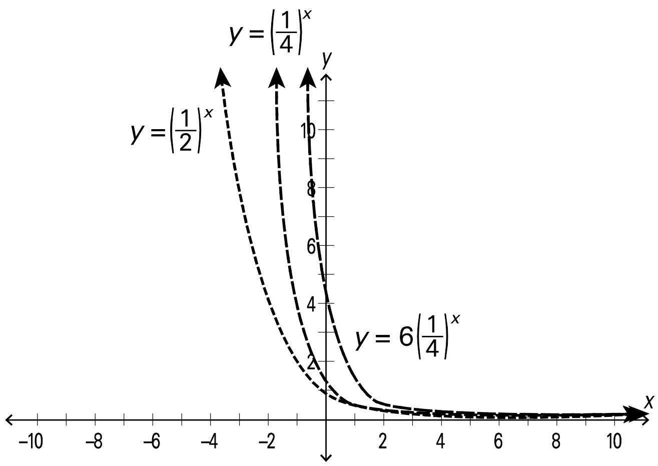
In Mathematics 10, students worked with linear relations expressed in slope-intercept form, general form, and slope-point form (RF06). In Mathematics 11, students solved problems that involved quadratic equations (RF02).

Students will now be exposed to exponential equations of the form $y = a(b)^x$, where $b > 1$ or $0 < b < 1$ and $a > 0$. They will also work with logarithmic functions and understand the relationship between exponential and logarithmic functions.

In this outcome, students are given the opportunity to explore data sets modelled by exponential and logarithmic functions. Although they will have heard the term **exponential growth and decay** with reference to topics such as population, compound interest, radioactivity, and value depreciation, this will be the first time for them to formally explore these types of functions.

Using graphing technology or a table of values, students should investigate the characteristics of an exponential function of the form $y = a(b)^x$ where $b > 0$, $b \neq 1$, and $a > 0$. Review of terms such as **end behaviour**, **domain**, **range**, and **intercepts** may be necessary.





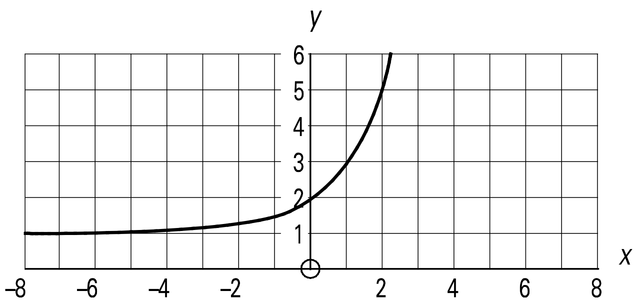
Using the above graphs as a visual representation, students should note the following characteristics of $y = a(b)^x$ where $b > 0$, $b \neq 1$, and $a > 0$:

- No *x*-intercepts; one *y*-intercept.
- Exponential functions have a restricted range bounded by the *x*-axis, but the domain consists of the real numbers.
- Can be increasing or decreasing.
- Some exponential functions increase/decrease at a faster rate than others.

An **asymptote** is a line whose distance to a given curve tends to zero. An asymptote may or may not intersect its associated curve. An asymptote can be horizontal, vertical, oblique, and even curved.

For the function $f(x) = \left(\frac{1}{2}\right)^x$, as $x \to \infty$ the values of $f(x) \to 0$ and, therefore, $f(x) = 0$ is a horizontal asymptote for the function $f(x) = \left(\frac{1}{2}\right)^x$.

Graphically, an asymptote can be observed. For example, the graph shown below appears to have a horizontal asymptote of $f(x) = 1$.



This is the first time that students have been exposed to the idea of an asymptote. For this unit students will only consider horizontal and vertical asymptotes.

Asymptotic behaviour is often discussed in courses such as biology. This would be a good opportunity to discuss the concept of asymptote and its connection to the range of an exponential function.

The intention of this outcome is to focus on the graphing of data sets. Students will understand that exponential and logarithmic functions are different from other polynomial graphs they have seen.

Students should have an opportunity to investigate the equation of an exponential function. They are aware that a polynomial of degree 1 is linear, degree 2 is quadratic, and degree 3 is cubic. Students should now be exposed to an equation where the power is a variable.

Students should investigate and describe how the parameters *a* and *b* in the exponential equation $f(x) = ab^x$, where $b > 0$, $b \neq 1$, and $a > 0$, affect the graph of the function. They can use graphing technology to create the graph, but it is also important to look for patterns in the table of values of the exponential function. Ensure students understand that when the *x*-values increase by 1, the quotient of the *y*-values will yield a constant ratio. The parameter associated with this ratio is *b.*

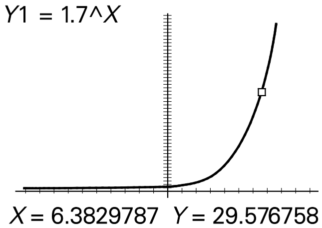
By graphing data sets and using a graphing calculator or similar tool, students will discover that exponential functions

- have no *x*-intercepts
- have a *y*-intercept at (0, *a*)
- are increasing functions when $b > 1$, and decreasing functions when $0 < b < 1$
- have a horizontal asymptote that is the *x*-axis

Students' understanding of how the parameters *a* and *b* can be used to predict characteristics of an exponential function can be consolidated when they match an exponential equation to its corresponding graph. When trying to identify the graph for $y = 25\left(\frac{1}{2}\right)^x$, for example, students should be looking for a decreasing exponential function where the *y*-intercept is 25.

Students could also use graphing technology to determine the solution for an exponential equation. With graphing technology, the solution can be found using the graph or a table of values.

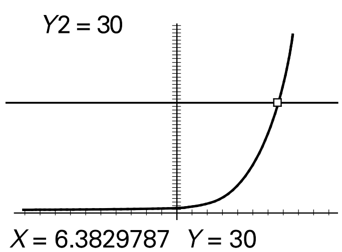
To determine the solution to the equation $1.7^x = 30$, for example, the graph of $y = 1.7^x$ can be used to determine the value of *x* that makes the value of the function approximately 30.



Students could also generate the table of values associated with the equation to find the value of *x* that makes the value of the function approximately 30.

X	Y_1
6	24.138
6.1	25.453
6.2	26.84
6.3	28.303
6.4	29.845
6.5	31.472
6.6	33.187
X = 6.4	

Alternatively, they can determine the intersection of the graphs of $y = 1.7^x$ and $y = 30$ to get the approximate solution.



Remind students that an exponential expression arises when a quantity changes by the same factor for each unit of time. For example, a population doubles every year, a bank account increases by 0.1% each month, or a mass of radioactive substance decreases by 50% every 462 years, they can expect an exponential expression. Many real-world phenomena can be modelled by functions that describe how things grow or decay as time passes.

An exponential growth function is an exponential function whose *y*-values increase as you move from left to right along the *x*-axis; for an exponential function of the form $y = ab^x$, exponential growth occurs when $a > 0$ and $b > 1$.

An exponential decay function is an exponential function whose *y*-values decrease as you move from left to right along the *x*-axis; for an exponential function of the form $y = ab^x$, exponential decay occurs when $a > 0$ and $0 < b < 1$.

Students should be introduced to the half-life exponential function, $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, and the meaning of the unknown variables. The value of h is called the half-life since it corresponds to the point on the graph of the function where the value of the function is half its initial value A_0.

Students should use their understanding of the meaning of the equation, where possible, to solve a problem where the exponential equation is given.

Example:

The population of trout growing in a lake can be modelled by the function $P(t) = 200(2)^{\frac{t}{5}}$ where $P(t)$ represents the number of trout and t represents the time in years after the initial count. How long will it take for there to be 6400 trout?

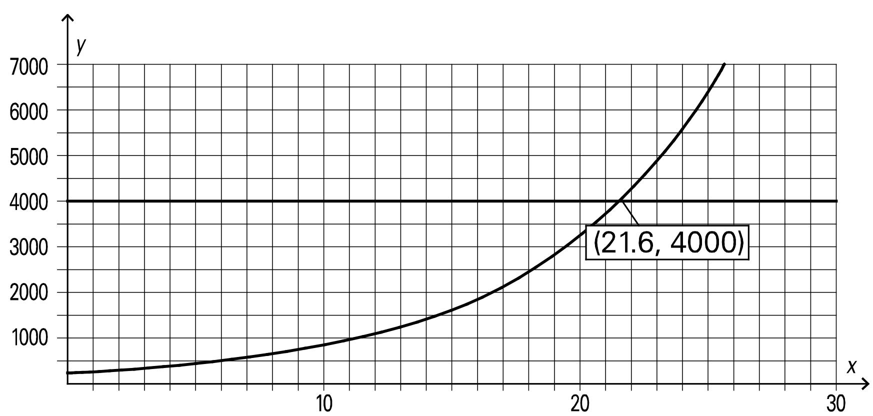
Students would be expected to understand that the population doubles every 5 years and is beginning at 200 (i.e., when $t = 0$, $P = 200$), so they could set up the table to find the solution to the posed problem.

t	0	5	10	15	20	**25**
P	200	400	800	1600	3200	**6400**

If the problem posed cannot be accurately answered by using a table such as the one above, students would graph the equation to answer the question.

Example:

The population of trout growing in a lake can be modelled by the function $P(t) = 200(2)^{\frac{t}{5}}$ where $P(t)$ represents the number of trout and t represents the time in years after the initial count. How long will it take for there to be 4000 trout?



Observing the intersection point, students would conclude that after 21.6 years, the trout population would reach 4000.

Students performed linear, quadratic, and cubic regressions when working with polynomial functions. They should continue to use technology to create a scatter plot and determine the equation of the exponential regression function that models the data yielding an equation in the form of $y = ab^x$.

Using graphing technology, students will learn to recognize exponential patterns in data sets. If a scatter plot of the data appears to be exponential, students will run an exponential regression to give an algebraic model of the data set.

In this unit, students will be introduced to the logarithmic function of the form $y = a \log_b x$, where $b > 0$, $b \neq 1$, $a > 0$, and a and b are real numbers. (Emphasis is on base e and base 10 logarithms.)

In Mathematics 10, students experienced inverse operations such as squaring and taking the square root (AN01). In this unit, they will be introduced to the idea of inverse functions.

An **inverse function** is a function that "reverses" another function. If for the function $f(x), f(4) = 16$, then for the inverse function, $g(x), g(16) = 4$.

Students will compare the graphs of exponential functions and logarithmic functions, and identify characteristics with emphasis on domain and range, intercepts, and end behaviour. They will compare exponential and logarithmic functions reflections of each other about the line $y = x$.

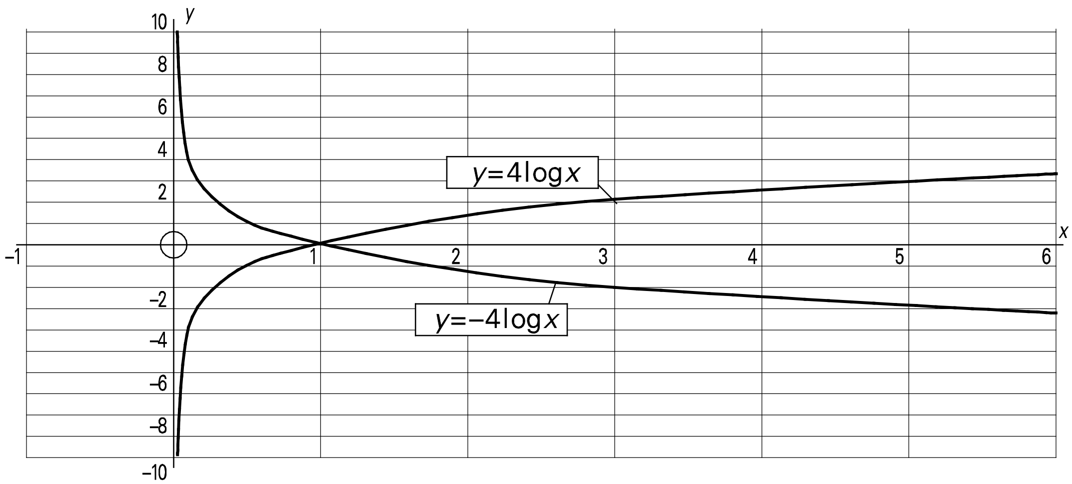
Students should notice the graph of the logarithmic curve ($y = \log_{10} x$) is a reflection of the exponential curve ($y = 10^x$). They should summarize the characteristics of the graphs using a table similar to the one shown.

	Exponential $y = 10^x$	**Logarithmic** $x = 10^y$ **or** $y = \log_{10}(x)$
Domain	$\{x \mid x \in \mathrm{R}\}$	$\{x \mid x > 0, x \in \mathrm{R}\}$
Range	$\{y \mid y > 0, y \in \mathrm{R}\}$	$\{y \mid y \in \mathrm{R}\}$
y-intercept	(0, 1)	No *y*-intercept
x-intercept	No *x*-intercept	(1, 0)
Increasing or decreasing	increasing	increasing
End behaviour	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to 0$	As $x \to \infty, y \to \infty$ As $x \to 0, y \to -\infty$

Students should then investigate the effect of changing the value of a by comparing logarithmic functions $y = \log_{10} x$ and $y = a \log_{10} x$, where $a \neq 0$. Use graphing technology to graph and analyze logarithmic functions, such as $y = \log_{10} x$, $y = 4 \log_{10} x$, and $y = -4 \log_{10} x$. Students should be able to answer the following:

- What is the impact on the graph of the function if $a > 0$? if $a < 0$?
- Does a affect the *x*-coordinate or the *y*-coordinate?
- Is this a vertical transformation or a horizontal transformation?
- Which point is easily identified from the graph?
- Which characteristics of the graphs of logarithmic functions differ from the characteristics of the graphs of exponential functions?

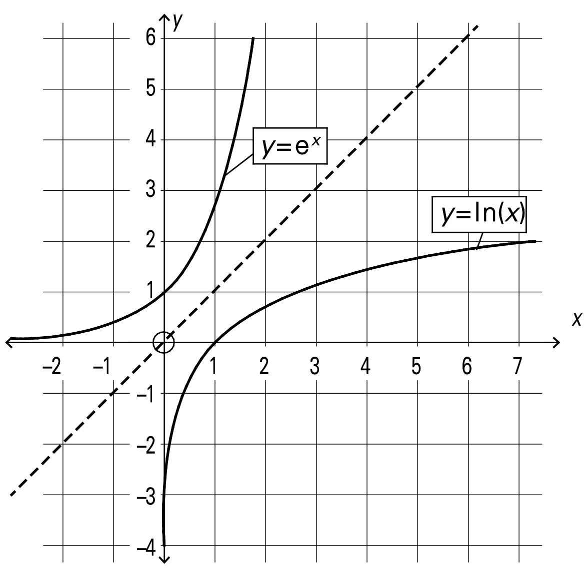
Students should understand that when $a > 0$, the y-values increase as the x-values increase. This is an increasing function. If $a < 0$, y-values decrease as the x-values increase. This is a decreasing function. Students should understand that logarithmic functions do not have a y-intercept but do have a restricted domain (i.e., $x > 0$).



Students should explicitly graph logarithmic functions with base ten and base e.

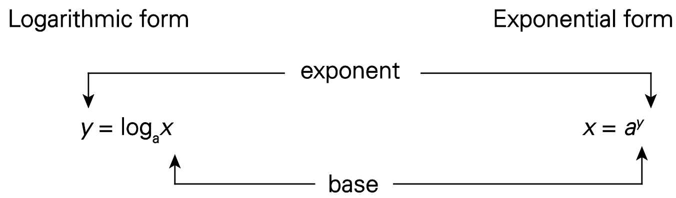
It may be beneficial for students to first investigate the characteristics of logarithmic functions with base ten. Using graphing technology, ask students to graph $y = 10^x$ and the line $y = x$.

As students explore further, they should compare the graphs of $y = a^x$, $y = \log_a x$, and the inverse graph of $y = a^x$ in which x and y are switched, $x = a^y$. They will discover that $x = a^y$ is the same graph as $y = \log_a x$, and that both are the reflection of $y = a^x$ across the $y = x$ line.



Students should express a logarithmic equation as an exponential equation and vice versa. They evaluate a logarithmic expression such as log(100) by writing the expression in exponential form. For example, log (100) = 2 since $10^2 = 100$. It is important for students to understand that when evaluating a log, they are finding the exponent, given the base and the argument.

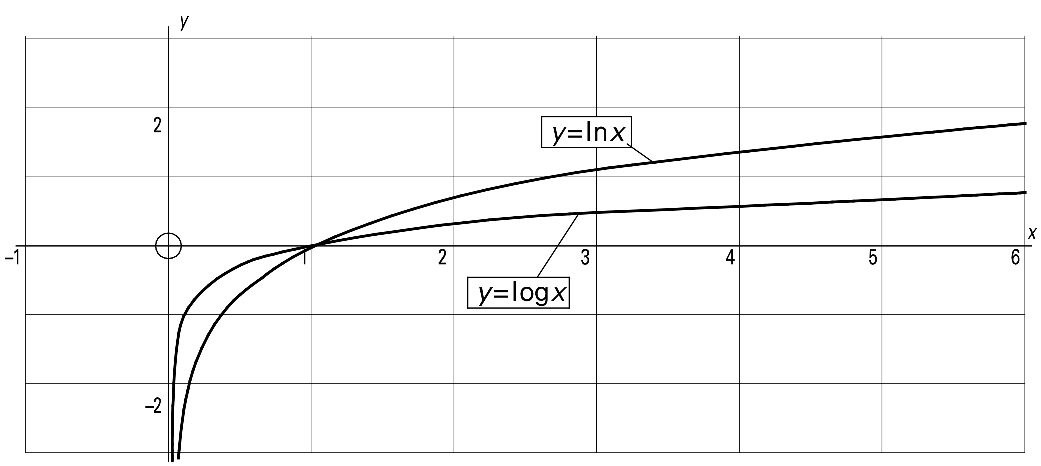
In other words, students should be able to switch between exponential and logarithmic forms.



Common logarithms have a base of 10 and can be written with or without the base as $y = \log_{10} x$ or simply as $y = \log x$. Students should be familiar with this convention.

Similarly, natural logarithms have a base of e and can be written with the as $y = \log_e x$ or base or in its more common abbreviated form simply as $y = \ln x$. Students should also be familiar with this convention.

Students should compare the natural logarithmic function $y = \log_e x$ (i.e., $y = \ln x$) and the exponential function $y = e^x$. Discuss with students that the value of e is an irrational number. Using graphing technology, they should observe that the graph of $y = \ln x$ is a reflection of the graph of $y = e^x$ about the line $y = x$. The visual aid should also help students understand that changing the base from $b = 10$ to $b = e$ has no effect on the characteristics of the graphs.



Students will work in more detail with natural logarithms when they perform logarithmic regressions on a set of data yielding an equation in the form of $y = a + b \ln x$.

Students should match equations of exponential and logarithmic functions in a given set to their corresponding graphs. As students distinguish between logarithmic functions of the form $y = a \log x$ or $y = a \ln x$ and exponential functions $y = a(b)^x$, they should reflect on the following features:

- An exponential function has a horizontal asymptote.
- A logarithmic function has a vertical asymptote.
- The value of a in the logarithmic function, $y = a \log x$ or $y = a \ln x$, is used to determine if the function is increasing or decreasing.
- The value of b in exponential function, $y = a(b)^x$, is used to determine if the function is increasing or decreasing.
- The value of a in the exponential function, $y = a(b)^x$, is used to determine the y-intercept.

From the equation of exponential and logarithmic functions, students should be able to determine if the function is increasing or decreasing, the end behaviour, the asymptotes, the *x*- or *y*-intercept, and the domain and range.

When using regression on a data set, it is important to ask students why they think a particular model would be a good fit for the particular data set, even if they are given the type of regression to perform on a set of data.

As students work through problems, ask them to reflect on the following:

- The domain of a logarithmic function is restricted to the set of positive real numbers.
- Logarithmic regressions are mostly used for phenomena that grow quickly at first and then slow down over time, but the growth continues to increase without bound (e.g., the length of cod fish over time).
- Exponential regressions are typically used on phenomena where the growth begins slowly and then increases very rapidly as time increases (e.g., bacteria growth).

Students should solve problems involving logarithmic scales such as the Richter scale (used to measure the magnitude of an earthquake), the pH scale (used to measure the acidity of a solution), and the decibel scale (used to measure sound level). In pH investigations students will not be expected to develop formulas, but will be given the formula.

- Students may be familiar with pH scales in science and chemistry courses. The pH scale is used to measure the acidity of a solution. The pH of a solution is determined using the equation $p(x) = -\log x$ where x is the concentration of hydrogen ions measured in moles per litre (mol/L). This scale ranges from 0 to 14, with the lower numbers being acidic and the higher numbers being basic. A value where the pH = 7 is considered neutral. The scale is a logarithmic scale with one unit of increase in pH resulting in a ten-fold decrease in acidity. Another way to consider this would be a one unit increase in pH results in a ten-fold increase in basicity.
- The magnitude of an earthquake, y, can be determined using $y = \log x$, where x is the amplitude of the vibrations measured using a seismograph. An increase of one unit in magnitude results in a ten-fold increase in the amplitude. This topic lends itself to the incorporation of current events with the inclusion and comparison of a variety of earthquakes.
- Sound levels are measured in decibels using the function $\beta = 10(\log I + 12)$, where β is the sound level in decibels (dB) and I is the sound intensity measured in watts per metre squared (W/m^2).

Students should be exposed to problems where they are asked to compare the intensity of different earthquakes, compare the sound intensity of different events, or compare the acidity of different solutions.

This could be a good opportunity for students to measure audio volume in the environment around them. They can use an application for a smartphone, for example, to show the approximate decibel level wherever they and their smartphone are located. Although quite accurate, the application is mainly a tool for detecting noise levels in casual settings.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Using laws of exponents, what is the value of the variable?
  (a) $3^x = 9$
  (b) $5^x = 1$
  (c) $2^x = \frac{1}{8}$
- Evaluate $4(3)^2$.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Describe how $y = x^2$ is different from $y = 2^x$.
- Explain why, as $x \to \infty$ for the function $y = abx, f(x) \to 0$ when $0 < b < 1$.
- Explain why, as $x \to \infty$ for the function I = $abx, f(x) \to \infty$ when $0 < b < 1$.
- Complete the following charts and discuss the patterns that you notice.

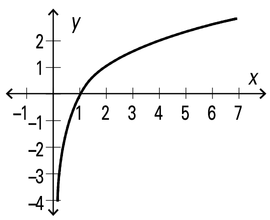
	$y = 2^x$	$y = 3^x$	$y = 5(3)^x$
y-intercept			
number of *x*-intercepts			
end behaviour			
domain			
range			

	$y = \left(\frac{1}{2}\right)^x$	$y = \left(\frac{1}{4}\right)^x$	$y = 3\left(\frac{1}{4}\right)^x$
y-intercept			
number of *x*-intercepts			
end behaviour			
domain			
range			

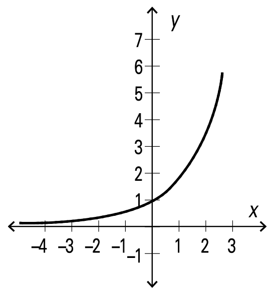
- Match the following equations with the graphs below

(a) $y = \left(\frac{1}{2}\right)^x$ (b) $y = 2^x$ (c) $y = -2^x$ (d) $y = \log_2 x$

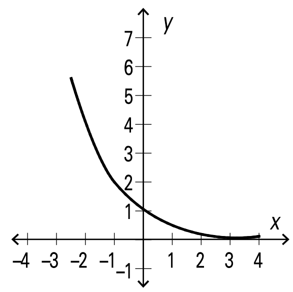
(i)



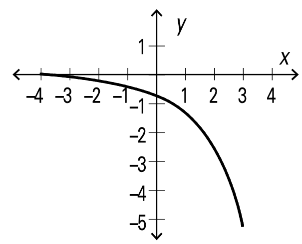
(ii)



(iii)



(iv)



- Complete the following table using $y = 8\left(\frac{2}{3}\right)^x$.

$y = 8\left(\frac{2}{3}\right)^x$	**True**	**False**	**Why I think so**
▪ The *y*-intercept is 1.			
▪ The graph has one *x*-intercept.			
▪ The range is $\{y \mid y > 0, y \in \mathrm{R}\}$.			
▪ The domain is $\{x \mid x > 8, x \in \mathrm{R}\}$.			
▪ This is a decreasing exponential function.			

- Using a table of values, graph $y = (-2)x$ and $y = (1)x$. Explain why the graphs do not represent exponential functions.
- When Jose and Terri carried out a coffee cooling experiment, they obtained the results given in the table below. Here *t* stands for the time in minutes since the experiment began and *T* for the temperature of the water in degrees Celsius. Room temperature was 20°C.

t	0	2	4	6	8	10	12	14	16	18	20
T	89.0	83.2	78.0	73.7	70.4	67.4	64.8	62.6	60.0	58.0	55.8

(a) Graph the data. What do you think will happen to the water temperature if you wait long enough?

(b) Add another row to the table showing the difference between the water temperature and room temperature. How would you test whether an exponential model would fit these numbers?

(c) The first three numbers do not appear to fit the same pattern as the rest. Can you suggest a reason for this?

(d) Find an exponential function that fits the revised data.

(e) According to this model, what temperature would you expect the water to be after 30 minutes?

- Examine the following tables and indicate which one(s) are suggesting an exponential relationship. Explain your thinking.

(a)

x	y
0	0.093
1	0.1875
2	0.375
3	0.75
4	1.5
5	3

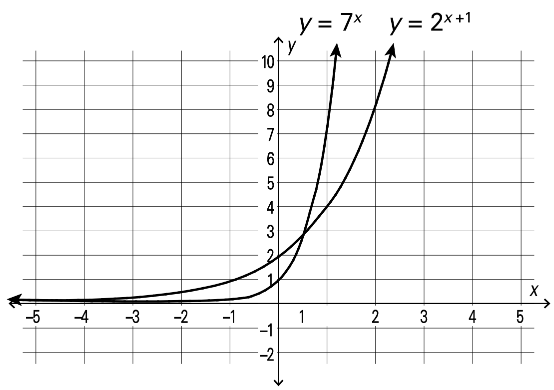
(b)

x	y
–7	20 000
–6	200
–5	2
–4	0.02
–3	0.0002
–2	0.000002

(c)

x	y
–6	–3456
–4	–1504
–3	–828
–2	–352
1	–124

- The graph shown below provides the solution to an exponential equation.



(a) What is the exponential equation?

(b) What is the solution to this exponential equation?

- Small rural water systems are often contaminated with bacteria by animals. Suppose that a water tank is infested with a colony of 14 000 E. coli bacteria. In this tank, the colony doubles in number every four days.

(a) Explain how you would know that $A(t) = 14000(2)^{\frac{t}{4}}$ could be used to represent the size of this colony of bacteria.

(b) Graphically, determine the value of *t* when *A*(*t*) = 224 000.

(c) What does your answer mean in this context? Explain.

- The half-life of a radioactive isotope is 30 hours. How long will it take for a sample of 1792 mg to decay to 56 mg?

- The following table shows the average undergraduate university tuition fees paid by full-time Canadian students.

School Year	Tuition Fee
2007–2008	$4558
2008–2009	$4747
2009–2010	$4942
2010–2011	$5146
2011–2012	$5366

(a) Using graphing technology, construct a scatter plot to display the data.

(b) Does your graph appear to have an exponential or a logarithmic curve pattern?

(c) Use regression to define a function that models the data.

(d) Estimate the tuition fee for the school year 2015–2016.

- Kyle invested his summer earnings of $5000 at 8% interest, paid annually. Graph the growth of the investment for six years using "time (years)" as the domain and "value of the investment" as the range. Using the graph of $P(t) = 5000(1 + 0.08)^t$, answer the following questions.

  (a) What does the shape of the graph tell you about the type of growth? Why is the data discrete?

  (b) What do the y-intercept and ratio represent for the investment?

  (c) What is the value of the investment after 10 years?

- Mary evaluated log(–3.24) on her calculator and an error message was displayed. Explain why an error message occurred.
- Explain, using the graph of $y = \log x$, why you cannot evaluate log(–3) and log(0).
- Match the following equations with their graphs.

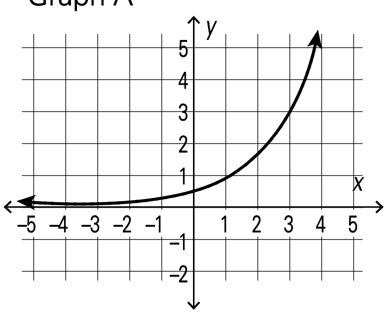
Equation 1: $y = 3\left(\frac{1}{2}\right)^x$

Equation 2: $y = \frac{1}{3}(2)^x$

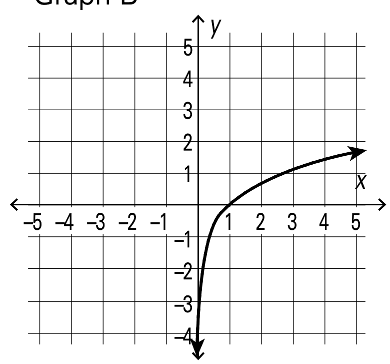
Equation 3: $y = \ln x$

Equation 4: $y = -2\ln x$

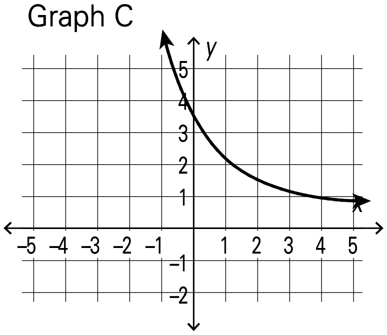
Graph A



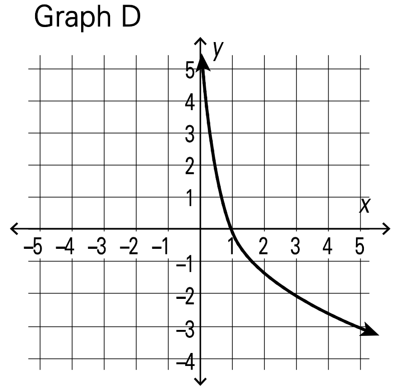
Graph B



Graph C



Graph D



- Answer the following questions using the table below.

Location and Date	Magnitude
Chernobyl, 1987	4
Haiti, January 12, 2012	7
Northern Italy, May 20, 2012	6

(a) How many times as intense was the earthquake in Haiti compared to the one in Chernobyl?

(b) How many times as intense was the earthquake in Haiti compared to the one in Northern Italy?

(c) How many times as intense was the earthquake in Northern Italy compared to the one in Chernobyl?

(d) If a recent earthquake was half as intense as the one in Haiti, what would be the approximate magnitude?

- After taking a cough suppressant, the amount, A, in mg, remaining in the body is given by $A = 10(0.85)^t$, where t is given in hours.

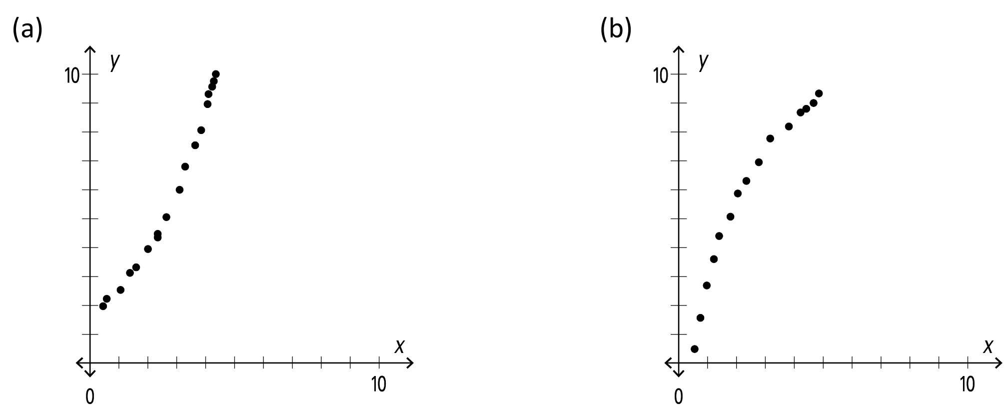
(a) What was the initial amount taken?

(b) What percent of the drug leaves the body each hour?

(c) How much of the drug is left in the body six hours after the dose is administered?

(d) How long is it until only 1 mg of the drug remains in the body?

- The half-life of a radioactive sample is three years. If the initial mass of the sample is 67 g, how long will it take for the sample to reach 7 g?
- Kelly invests $5000 with a bank. The value of her investment can be determined using the formula $y = 5000(1.06)^t$, where y is the value of the investment at time t, in years. Using technology, determine approximately how many years will it take for Kelly's investment to reach a value of $20,000?
- Which scatter plot appears to model an exponential function and which models a logarithmic function. Explain your reasoning.



- Using graphing technology, create a scatterplot of this data and determine the equation of the logarithmic regression for the data.

x	0.5	0.7	0.9	1.0	1.2	1.4	1.8	2.0	2.3	2.7	3.2	3.8
y	0.5	1.6	2.7	3.1	3.7	4.4	5.1	5.8	6.4	7.0	7.7	8.3

  Compare the *y*-values given by the model with the actual *y*-values in the table and explain why this model is or is not a good fit.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Ask students to compare, for example, $y = 2^x$ and $y = 5(2)^x$. Similarly, students can compare $y = \left(\frac{1}{2}\right)^x$ and $y = 3\left(\frac{1}{2}\right)^x$.

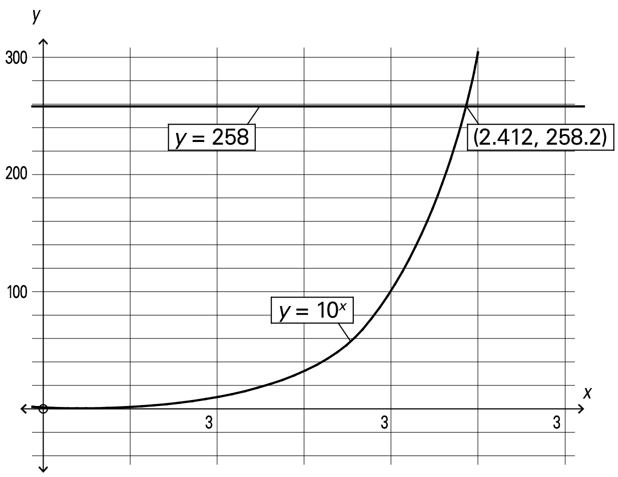
  Use the following prompts to promote student discussion.
  - What is the relationship between the *y*-intercept and the parameter *a* in $y = a(b)^x$?
  - How can the *y*-intercept be determined algebraically using the equation?
  - What happens to the exponential function if $b > 1$? What happens to the *y*-values as you move from left to right on the *x*-axis?
  - What happens to the exponential function if $0 < b < 1$? What happens to the *y*-values as you move from left to right on the *x*-axis?

  Why does the graph have no *x*-intercepts?
  - Is the domain or range affected by changing the parameters *a* or *b* in the equation $y = ab^x$?

  This activity helps students to understand that the *y*-intercept is (0, *a)* and all exponential functions of this form have the same end behaviour, domain, and range. The exponential function is increasing if $a > 0$ and $b > 1$ and the exponential function is decreasing if $a > 0$ and $0 < b < 1$.

  This would be a good opportunity to investigate the graph of $y = ab^x$ where $b = 1$ and where $b < 0$. Students should observe the following:
  - When $b = 1$, a horizontal line is produced.
  - When *b* is negative, if integer values of *x* are chosen, the *y*-values oscillate between positive and negative values; for rational values of *x*, non-real values of *y* may be obtained.

- Consider solving an equation such as $10^x = 258$ using several different methods:

Method 1	**Method 2**	**Method 3**
Guess and check $10^2 = 100$ $10^3 = 1000$ $10^{2.5} = 316.22$ $10^{2.2} = 158.5$ $10^{2.4} = 251.2$ $10^{2.41} = 257.0$ $10^{2.42} = 263.0$ Therefore, $x \doteq 2.41$.	Graphing and intersection point  Therefore, $x \doteq 2.412$.	Changing to logarithmic form $10^x = 258$ $\log_{10}(258) = x$ $x = 2.41162$ Therefore, $x = 2.41162$.

- The average household expenses in Nova Scotia for the past 10 years would create a data set that would likely be approximately exponential. Examples of household expenditures include food, shelter, transportation, health care, personal care, education, and recreation.

  Note: Students may find it beneficial to rewrite the variable to be t = years since 2000 and then $H(t)$ = household expenditure t years after 2000.

  Guide students through the process using the following directions:

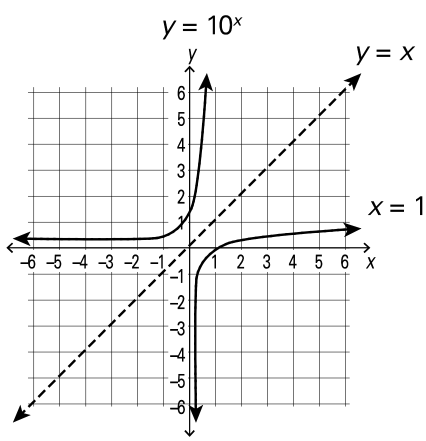
  – Construct a scatter plot to display the data.
  – What type of curve pattern does your data seem to have? Explain your reasoning.
  – Determine the equation of the regression function that models the data.
  – Graph the curve of best fit.
  – According to this model, what will be the household expenditure for the year 2020? Are there any factors that might cause the actual amount to be different from the amount projected by this model?

- Explore the inverse of $y = 10^x$ using a graphing utility.

  Example:

  – On a graphing calculator, graph $y_1 = 10^x$.
  – On the same graph, using a window of [–2.35, 2.35] by [–2, 2] graph $y_2 = \log x$. What do you notice?
  – On the graphing calculator, look at the TABLE for the two functions. Especially look at the y-values when $x = 1$ and $x = 10$.

- Have students explore characteristics of exponential functions as they work through the following steps.
  - Graph $y = 2^x$ and explain why the graph (from left to right) curves slowly at first, then much more quickly.
  - Graph $y = 3^x$ and compare its growth rate with that of $y = 2^x$.
  - Work in pairs, each with a graphic calculator to explore the behaviour of the graphs of $y = 2^x$, $y = 3^x$, and $y = e^x$ using the window settings with $-10 \leq x \leq 2$ and $-1 \leq y \leq 10$.
    - > Describe the curving behaviour and explore the behaviour at the extreme left and extreme right.
    - > Explore the behaviour of these three graphs using the window settings $-5 \leq x \leq -1$ and $-0.1 \leq y \leq 1$.
    - > Change the domain to $-10 \leq x \leq -5$ and the range to $-0.01 \leq y \leq 0.01$ and then discuss the behaviour of the graphs as they approach the *x*-axis.
  - Do you think any of the two graphs will ever intersect the *x*-axis? Explain your answer.
- Ask students to participate in a Find Your Partner activity. Create pairs of cards with exponential equations and matching graphs. Distribute the cards amongst the students and have them find their partner by matching the equation to its corresponding graph. Once they have found their partner, students should explain why they chose each other.
- Teachers should ask the following questions to help students make the connection between exponential and logarithmic functions.
  - How can a reflection of $y = 10^x$ be drawn about the line $y = x$? Why is it necessary to interchange the *x* and *y*-variables?
  - Use the table of values of $y = 10^x$ to generate the table for $x = 10^y$.How are the *x*-and *y*-values affected?
  - What are the coordinates of the new function?
  - What are the similarities and differences between the two functions with reference to domain, range, intercepts, and end behaviour?



$y = 10^x$	$x = 10^y$
$(-2, \frac{1}{100})$	$(\frac{1}{100}, -2)$
$(-1, \frac{1}{10})$	$(\frac{1}{10}, -1)$
(0, 1)	(1, 0)
(1, 10)	(10, 1)
(2, 100)	(100, 2)

- Have students use their calculators to complete the following chart and then make a statement about the relationship between exponents and logarithms based on their observations.

(a) $10^0 =$
(b) $10^1 =$
(c) $10^2 =$
(d) $10^3 =$
(e) $10^{-1} =$
(f) $10^{-2} =$
(g) $10^{-3} =$
(h) $10^{\frac{1}{2}} =$
(i) log(1) =
(j) log(10) =
(k) log(100) =
(l) log(1000) =
(m) $log\left(\frac{1}{10}\right) =$
(n) $log\left(\frac{1}{100}\right) =$
(o) $log\left(\frac{1}{1000}\right) =$
(p) $log\left(\sqrt{10}\right) =$

- Ask students to participate in the activity Speed Match. Place desks in pairs and distribute one cue card to each student containing a logarithmic graph or a logarithmic equation. Students will then be given a time frame to ask their partner questions to determine if they have found their match. The person with the graph will ask questions, and their partner may only answer *yes* or *no*. When the time is up, the person asking the questions will move on to the next person. This will continue until they find their match. (**Variation:** The person with the equation can ask the questions.)
- Teachers should remind students that logarithms are a different form of an exponential statement. The statement $10^2 = 100$, for example, can be written as log(100) = 2. Ask students what they notice about the base of the exponent and the base of the logarithm. Ask students to think about $\log_b x = y$ as, What exponent is needed so that, $b^y = x$.
- *Quiz-Quiz-Trade:* Each student is given a card with a problem. The answer is written on the back of the card. In groups of two, partner A asks the question and partner B answers. They switch roles and repeat. Students move around the classroom until every student has had a chance to solve all the problems.

Suggested Models and Manipulatives

- coins
- die

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- argument
- base
- end behaviour
- exponent
- exponential decay
- exponential function
- exponential growth
- initial amount
- logarithmic function
- parameter
- ratio
- *x*-intercept
- *y*-intercept

Resources/Notes

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 7.1, 7.2, and 7.3, pp. 436–468
  - Sections 7.4 and 7.5, pp. 474–499

SCO RF03 Students will be expected to represent data, using sinusoidal functions, to solve problems. [C, CN, PS, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF03.01 Demonstrate an understanding of angles expressed in degrees and radians.
RF03.02 Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their graphs.
RF03.03 Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their equations.
RF03.04 Match equations in a given set to their corresponding graphs.
RF03.05 Graph data and determine the sinusoidal function that best approximates the data.
RF03.06 Interpret the graph of a sinusoidal function that models a situation, and explain the reasoning.
RF03.07 Solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning.

Scope and Sequence

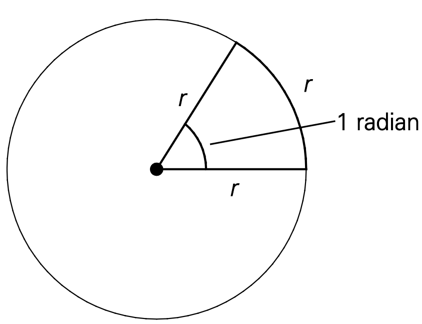
Mathematics 11	**Mathematics 12**
G03 Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case.	**RF03** Students will be expected to represent data, using sinusoidal functions, to solve problems.

Background

In Mathematics 10, students used the primary trigonometric ratios and the Pythagorean theorem to solve right triangle problems (M04). In Mathematics 11, they solved problems that involved the cosine law and the sine law, including the ambiguous case (G03).

Students have, up to this point, described angle size using degree measure. In this unit they will learn about radian measure as well as being able to convert between radians and degrees.

A radian is the measure of the central angle of a circle that cuts an arc that is the same length as the radius of the circle.



Since the circumference of a circle is $2\pi r$, there are 2π radians in a complete revolution. Therefore, a complete revolution can be described as both are 2π radians and as 360°.

Students should be able to state that 2π radians = 360° and then to solve this equation for one radian, resulting in 1 radian ≈ 57.3°. They should then manipulate the relationship further to get a sense for the different angles. If 2π radians = 360°, for example, then π radians = 180°, $\frac{\pi}{2}$ radians = 90°, $\frac{\pi}{3}$ radians = 60°, $\frac{\pi}{4}$ radians = 45°, $\frac{\pi}{6}$ radians = 30°.

It is important that students understand that when they are stating the size of an angle, they must include the degree sign if they mean degrees. It is convention that radian measure will be assumed if there is no degree sign indicated.

Students should be able to change a degree measure to radian measure and each radian measure to degree measure.

Example:

Method 1 Using the fact that 180° = π radians and solving a proportion, $\frac{\pi \text{ radians}}{180^\circ} = \frac{R}{D}$.	
Determine an approximate radian measure for 160°. $\frac{\pi \text{ radians}}{180^\circ} = \frac{R}{160^\circ}$ $\left(\frac{\pi \text{ radians}}{180^\circ}\right)160^\circ = R$ $R \approx 2.79 \text{ radians}$	Determine an approximate degree measure of 0.7 radians. $\frac{\pi \text{ radians}}{180^\circ} = \frac{0.7 \text{ radians}}{D}$ $\left(\frac{\pi \text{ radians}}{180^\circ}\right)D = 0.7 \text{ radians}$ $D = \frac{0.7 \text{ radians}}{\left(\frac{\pi \text{ radians}}{180^\circ}\right)} \approx 40.1^\circ$

Method 2 Using the fact that 180° = π radians and solving for 1 degree, $1^\circ = \frac{\pi}{180^\circ}$.	**Method 2** Using the fact that 180° = π radians and solving for 1 degree, 1 radian = $\frac{180^\circ}{\pi}$.
Determine an approximate radian measure for 160°. $1^\circ = \frac{\pi}{180^\circ}$ $160^\circ = 160\left(\frac{\pi}{180^\circ}\right)$ $160^\circ \approx 2.79 \text{ radians}$	Determine an approximate degree measure of 0.7 radians. 1 radian = $\frac{180^\circ}{\pi}$ 0.7 radian = $0.7\left(\frac{180^\circ}{\pi}\right)$ 0.7 radian ≈ 40.1°

Representing sinusoidal functions graphically will be new to them. In this unit, students describe the characteristics of sinusoidal functions by analyzing the graph and its corresponding equation: $y = a\,sin\,b\,(x - c) + d$ or $y = a\,cos\,b\,(x - c) + d$. They should also determine the equation of the sinusoidal regression function that models a set of data.

As with the previous two outcomes, the intention in this outcome is to have students gain a general understanding of a function and the applications it models. In this case, students will explore the characteristics of sine functions and how they model real-life situations. They will also learn to recognize sinusoidal graphs and the data sets on which they are based.

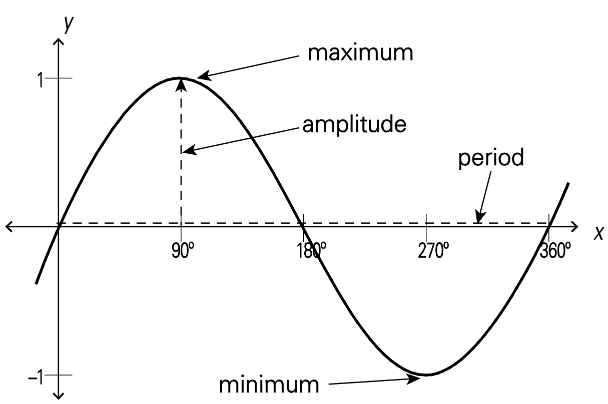
Sine functions are **periodic**, meaning they repeat over a specific period. They can be used to describe oscillating events such as the tides coming in and out over time, the height of a person on a Ferris wheel going up and down, or the height of a point on a rolling object as it rolls along a horizontal distance. A sine and cosine graphs can be described as **sinusoidal curves**.

Sinusoidal curves should be introduced slowly and include discussions about angles with measures greater than 180°. Students will have encountered angles larger than 180° in Mathematics 11, but this will have been in the context of right, obtuse, straight, and reflex angles. Their work with trigonometry previous to this has involved right angle trigonometry, sine law, cosine law, or geometry. They will not be familiar with angles in standard position, quadrantal angles, or negative angles. As students work with sine functions, both as graphs and as equations, they should become comfortable with the fact that $sin(x) = \frac{1}{2}$ for many different values of *x*, some negative, some positive.

The **sinusoidal axis** of a sine curve (also called the midline) is the horizontal central axis of the curve, halfway between the maximum and minimum values. $\left(y = \frac{\max + \min}{2}\right)$.

The **amplitude** is the maximum vertical distance of the graph of a sinusoidal function, above and below the sinusoidal axis of the curve measured as half the distance between the minimum and maximum values, $y = \frac{\max - \min}{2}$.

The **period** of the graph is the horizontal distance for one cycle of the graph. It can also be defined as the horizontal distance between two corresponding points on the graph.



Draw attention to the five key points associated with each graph, since these points help us determine the characteristics of the graph. In addition to the characteristics listed, students should also identify local maximums and minimums, domain, range, and intercepts.

The emphasis for this outcome should be on the determination and comparison of these characteristics from given graphs or equations. Students are expected to obtain the amplitude, period, and sinusoidal axis from a graph and from an equation. They are not expected to obtain the equation from a graph or the graph from an equation without a graphing utility. Students are expected to be able to determine a good window to use based on their understanding of amplitude, period, and sinusoidal axis.

It would be appropriate to discuss the reasons radian measure is used with sinusoidal functions. The advantage of radian measure is that it is directly related to the radius of the circle (which is also the amplitude). This means that the units on the *x*- and *y*-axis are consistent and the graph of the sine curve will have its true shape, without vertical exaggeration. If degrees and length are used for the *x*- and *y*-axes respectively, then the shape of the two graphs may change depending on the scale.

As students explore applications of sine functions, they will see functions that are transformations of $y = sin\,x$. Students will have some familiarity with vertical shifts, horizontal shifts, and vertical reflections from their study of quadratic functions in the form $y = ax^2 + bx + c$ in Mathematics 11. However, they will not have experience with horizontal stretches.

Students can explore the effect of changing the value of the parameters *a*, *b*, *c*, or *d* in a sine function. For $y = a\,sin[b(x - c)] + d$, it would be beneficial if students investigated the parameters separately so they can describe how each one affects the graph. For the purpose of this activity, students should be familiar with working with both radians and degrees.

Students should note that

- *a* represents the amplitude. Amplitude is $|a|$.
- *b* affects the length of the period. Period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.
- *c* is the value of the horizontal shift (translation), right if $c > 0$, or left if $c < 0$.
- *d* is the value of the vertical shift (translation) up or down, and also shifts the sinusoidal axis, up if $d > 0$, or down if $d < 0$.

This exploration should extend only to the level needed to understand characteristics of a sinusoidal function from graphs or equations that are given. Applications of these functions to model real-life situations should also be used to increase understanding of the effects of the parameters *a*, *b*, *c*, and *d*.

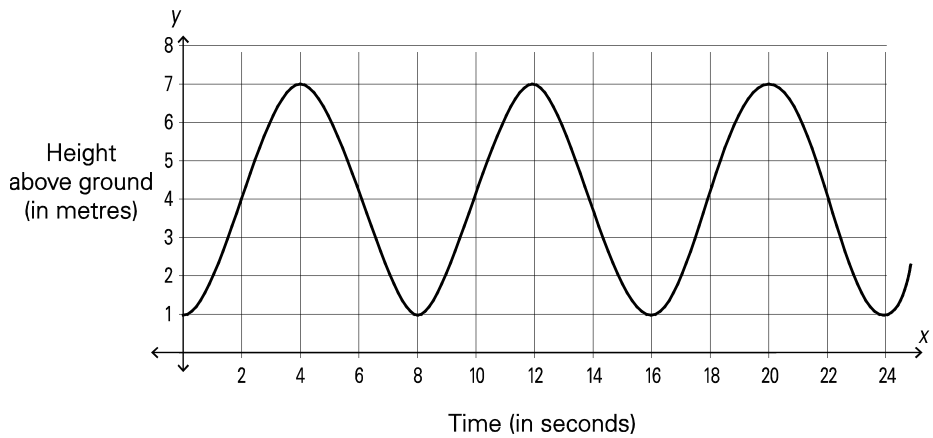
Once students have been exposed to all the different parameters, provide them with a variety of equations where they have to state the amplitude, equation of the midline, range, and period, as well as the horizontal translation. Encourage them to confirm their description by sketching the graph using graphing technology.

Provide students with a variety of graphs where some graphs are periodic but not sinusoidal and others are periodic and sinusoidal. Comparing the graphs, students should be able to conclude that all sinusoidal functions are periodic but not all periodic functions are sinusoidal.

Sinusoidal functions can be used as models to solve problems that involve repeating or periodic behaviour. Provide students with various graphs that model a situation and ask them to identify the characteristics of a sinusoidal function and relate its significance within the context of the problem.

Teachers could use the following example.

- While riding on a Ferris wheel, Mason's height above the ground in terms of time can be represented by the following graph.



In addition to identifying the period, range, amplitude, intercepts, and the equation of the sinusoidal axis or midline, students are expected to explain what the values represent within the context of this problem.

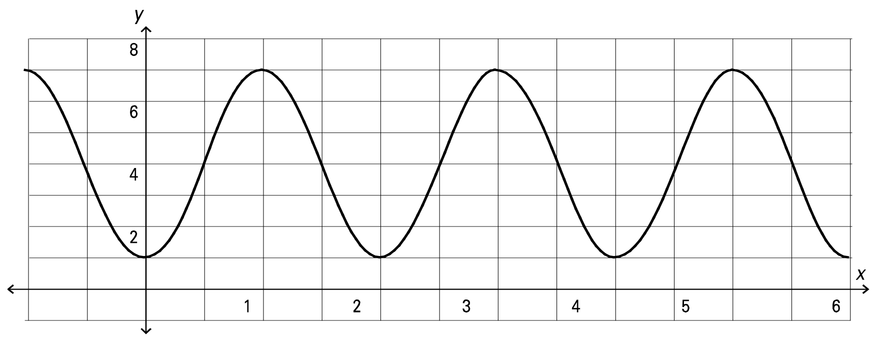
Students have been previously exposed to finding a polynomial, exponential, and logarithmic regression equation. They should now progress to a sinusoidal regression in the form of $y = a\,sin(\,bx + c) + d$ in radian measure.

In this unit, students have worked with the sine function in terms of amplitude, period, horizontal, and vertical translation. Examples such as sound waves and height of tides can be modelled with sinusoidal functions. Ask students to create a scatterplot of the data and explain why a sinusoidal function can be used to model the data. They should then determine and graph the sinusoidal regression function.

Students should be able to understand that two sinusoidal equations, written differently represent the same graph.

Example:

Marie graphed the equation $y = 3\,sin\,\pi\,(x - 0.5) + 4$ and obtained the following graph:



She then picked the points (0, 1), (0.5, 4), (1, 7), (1.5, 4), and (2, 1) from this graph and used sinusoidal regression to obtain

SinReg
y = a*sin(bx+c)+d
a = 3
b = 3.141592654
c = –1.570796327
d = 4

$$y = 3\,sin(\,3.14x - 1.57) + 4$$

Show that $y = 3\,sin(\,3.14x - 1.57) + 4$ and $y = 3\,sin\,\pi\,(x - 0.5) + 4$ are equivalent equations.

Remind students to check their graph to make sure it is a reasonable fit to the data. This reinforces the concept that regression equations are best-fit equations of available data, not perfect models.

Regression analysis can be performed using a variety of technologies, such as graphing calculators and their emulators and graphing applications available for smartphones, tablets, or computers.

Students should analyze the graph to determine the maximum and minimum values, the amplitude, and the period. They should also interpolate or extrapolate values that can be predicted by reading values from the graph. Students can also use the equation of the sinusoidal regression function to, graphically, solve for values of the independent variable (i.e., given *x* evaluate for *y*).

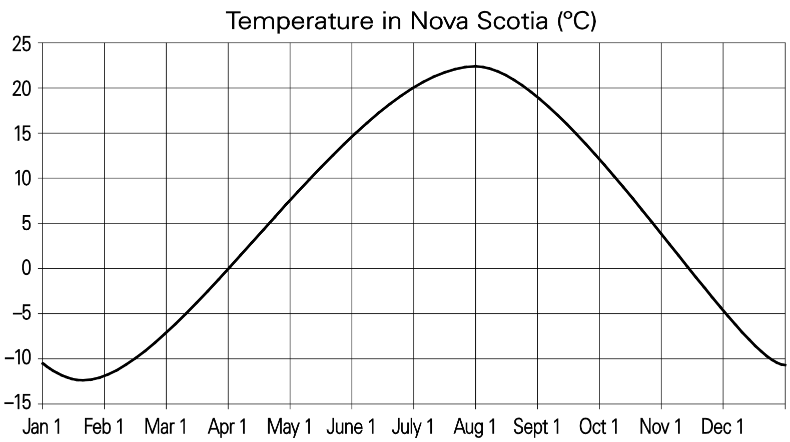
Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

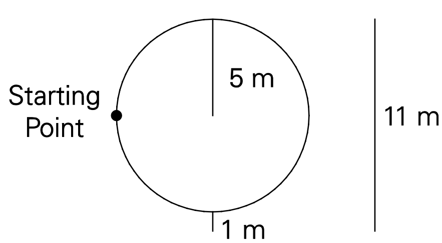
- Given the graph shown here, describe what you could predict about the temperatures in Nova Scotia.



WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

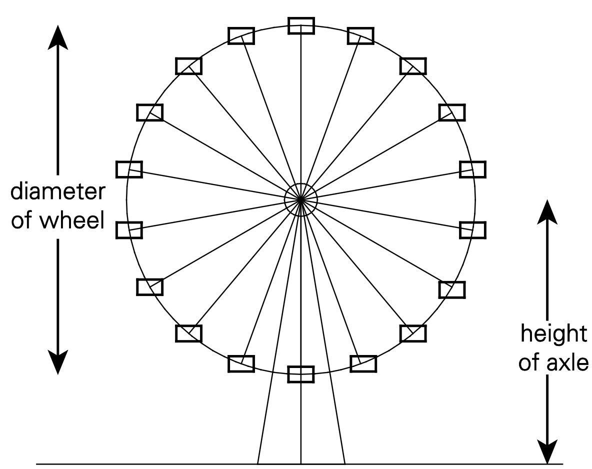
- A Ferris wheel has a maximum height of 11 m, a radius of 5 m (which allows a 1.0 m clearance at the bottom) and rotates once every 40 seconds. As the ride begins, you are half-way to the top of the wheel.
  - (a) Sketch a graph that shows height above the ground as a function of time using a sine function.
  - (b) What is the lowest height you go as the wheel turns? Explain why this must be a positive number.
  - (c) How high up will you be after 2.5 minutes?



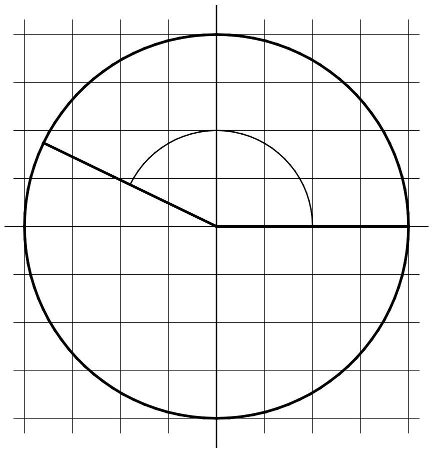
- Plot the following points and use regression to determine the values for *a*, *b*, *c*, and *d* for the function, $y = a\,sin(bx + c) + d$. State the vertical distance between the maximum and local minimum.

x	0	1	2	3	4	5	6	7	8
y	–1.976	–1.794	–1.5	–1.206	–1.024	–1.024	–1.206	–1.5	–1.796

- A spring oscillates in height according to the function $y = \frac{1}{2} sin(\,180t - 40) + 5$ where *t* represents the time in seconds and *y* represents the height in metres.
  (a) How do you know the period is in radians?
  (b) What is the amplitude of the oscillation?
  (c) What is the period of oscillation?
  (d) When is the midline?
  (e) What is the maximum height it reaches?
  (f) At what times, for 0 < *t* < 10, is the object at its minimum height?
- Tommy has a tree swing near the river in his backyard. The swing is a single rope hanging from a tree branch. When Tommy swings, he goes back and forth across the shore of the river. One day his mother (who was taking an adult mathematics course) decided to model his motion using her stopwatch. She finds that, after 2 seconds, Tommy is at one end of his swing, 4 m from the shoreline while over land. After 6 seconds, he reaches the other end of his swing, 5.2 m from the shoreline while over the water.
  (a) Sketch a graph of this sinusoidal function.
  (b) Use regression to obtain the equation expressing distance from the shore versus time.
  (c) Predict the distance when
    (i) time is 6.8 s
    (ii) time is 15 s
    (ii) time is 30 s
  (d) Where was Tommy when his mother started the watch?
- Explain why you know that the height of the passenger on this wheel as the wheel moves around at a constant rate could be described by a sine or a cosine function.

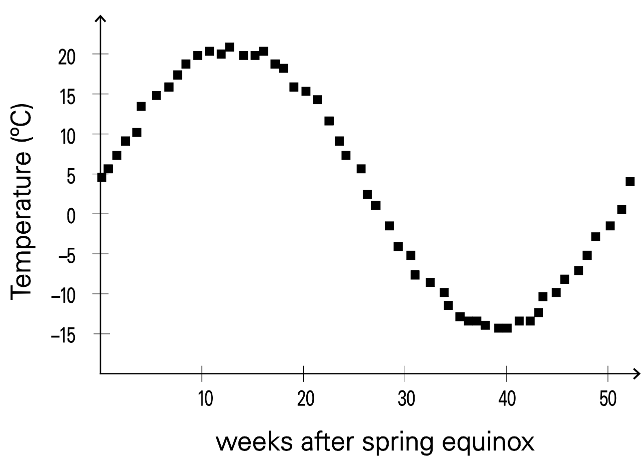


- Determine the approximate value for each conversion.
  - (a) 0.4 radians to degrees
  - (b) 4.5 radians to degrees
  - (c) 150° to radians
  - (d) 470° to radians
- Estimate the size of the angle shown below in radians?



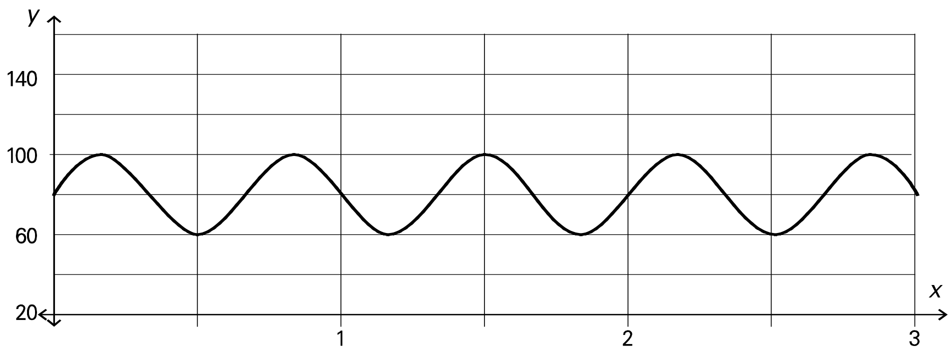
Explain the strategies you used to estimate this angle.

- A student approximated 45° to be 0.8 radians. Use this measure to estimate the missing values.
  - (a) 90° = ? radians
  - (b) ?° = 4 radians
  - (c) 135° = ? radians
  - (d) ?° = 3.2 radians
- Write the sinusoidal equation that could be used to describe the data shown below.

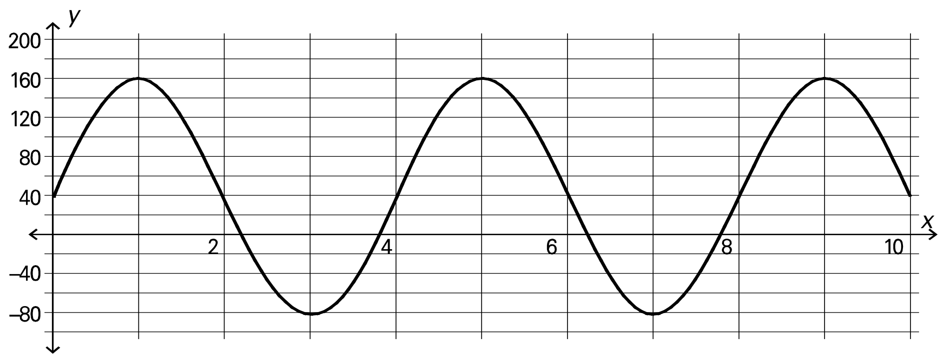


- State the amplitude, period, and sinusoidal axis for the following graphs.

(a)

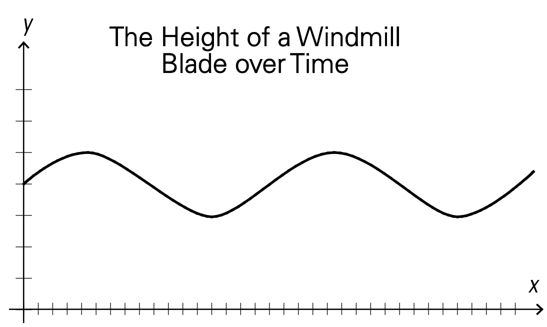


(b)

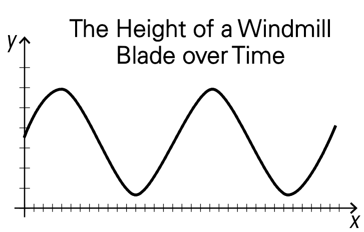


- A company is experimenting with a new type of windmill. The graph below shows the path of a blade on the original windmill over time. Describe what has changed and what has stayed the same in the new models. (**Note:** The scale is the same on all graphs.)

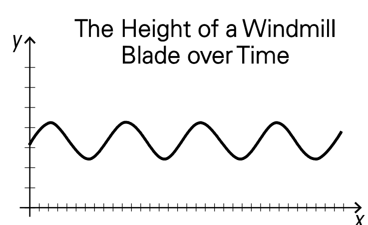
Original model



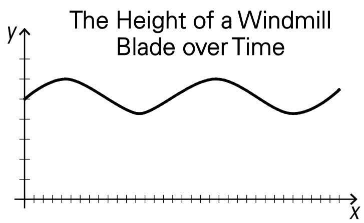
(a)



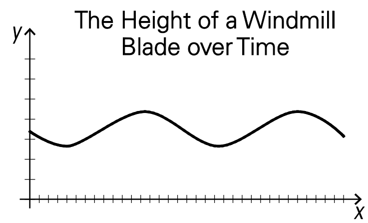
(b)



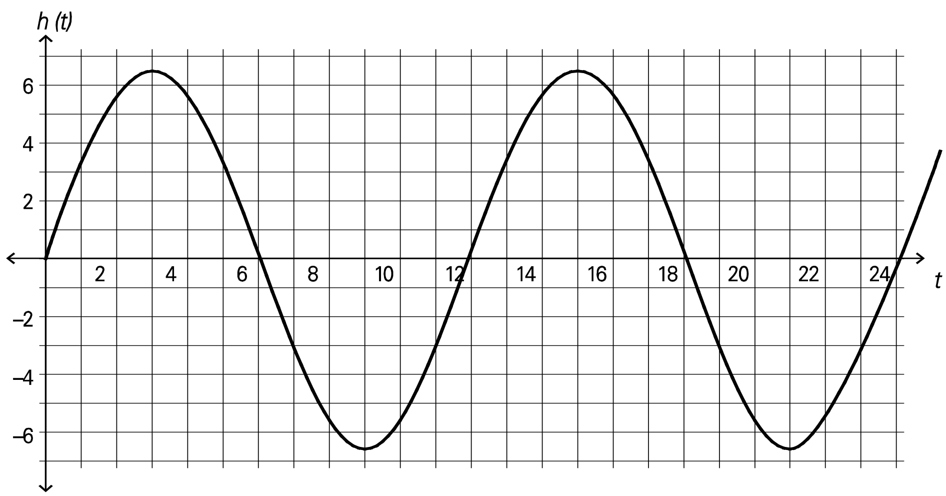
(c)



(d)



- The following graph represents the rise and fall of sea level in part of the Bay of Fundy, where *t* is the time, in hours, and *h*(*t*) represents the height relative to the mean sea level:



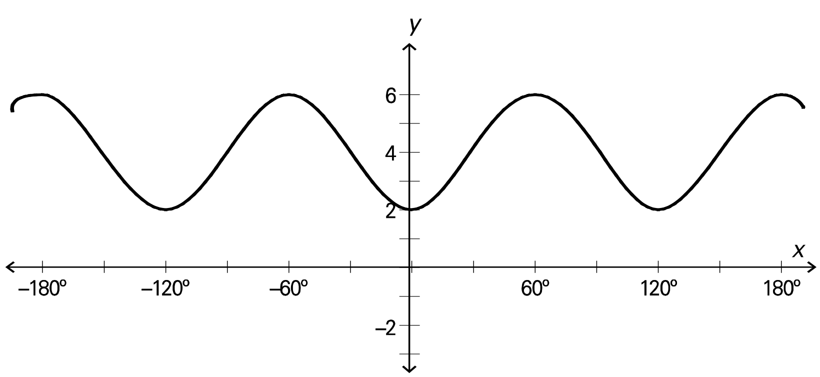
(a) What is the range of the tide levels?

(b) What does the equation of the midline represent in the graph?

(c) What is the period of the graph?

(d) The equation of the sinusoidal function is represented by $h(t) = 6.5\, sin\left(\frac{\pi}{6}t\right)$. Calculate the period from the equation and compare it to your answer in (c).

- The equation of the graph below is $y = 2\, sin\, 3\,(x - 30°) + 4$.



(a) If you were not given this equation, explain how you would find the values of the parameters *a*, *b*, *c*, and *d* for the equation $y = a\, sin\, b\,(x - c) + d$.

(b) Which value(s) did you find first? Why?

(c) Which value was the most difficult to determine?

(d) Which values would change if this is a transformed cosine curve?

- The temperature of an air-conditioned home on a hot day can be modelled using the function $T(x) = 1.5(cos\,1\,5°t) + 20$, where t is the time in minutes after the air conditioner turns on and $T(x)$ is the temperature in degrees Celsius.

  (a) What are the maximum and minimum temperatures in the home?

  (b) What is the temperature 10 minutes after the air conditioner has been turned on?

  (c) What is the period of the function? How would you interpret this value in this context?

  (d) Give three different times when the temperature is 20.5 degrees Celsius.

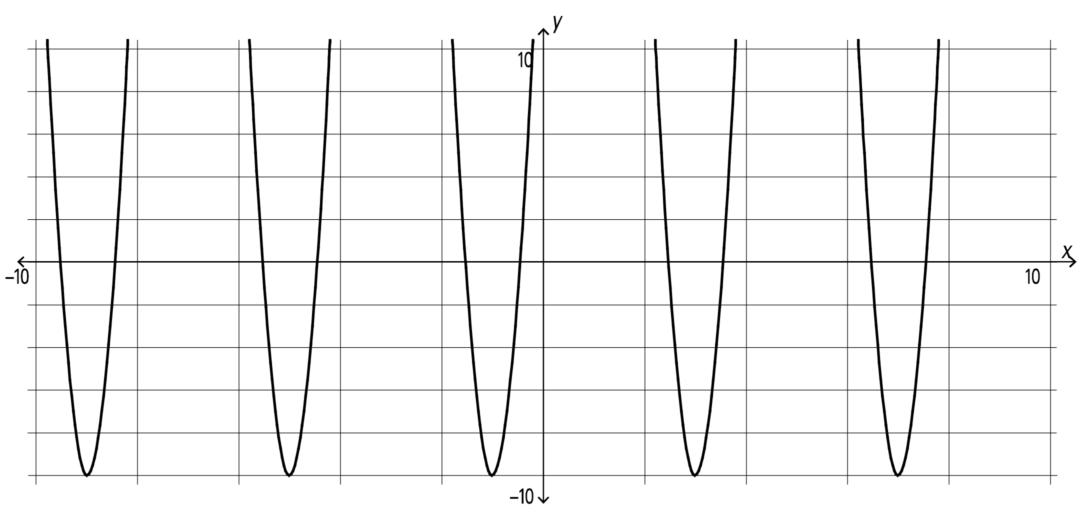
- The depth of water $d(t)$, in metres, in a seaport can be approximated by the function $d(t) = 2.5\,sin\,0\,.\,164\pi(t - 1.5) + 13.4$, where t is the time in hours.

  (a) What window would be appropriate for graphing this equation on the calculator?

  (b) What is the period of the tide? What information does the period give you?

  (c) A cruise ship needs a water depth of at least 12 m to dock safely. For how many hours per tide cycle can the ship dock safely?

- When Ashley graphed the equation $y = 30\,sin\left(\frac{\pi}{2}\right) + 20$, her calculator display showed the following:

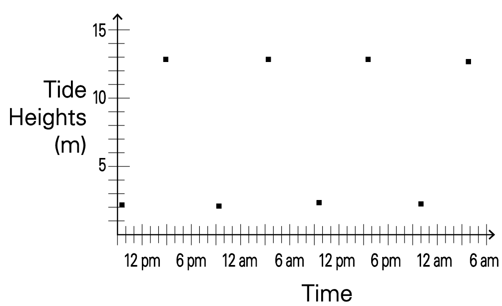


Identify how Ashley should adjust the window to see the sinusoidal graph.

- A marine biologist recorded the tide times in Hantsport for February 7 and 8, 2015 in the chart below.

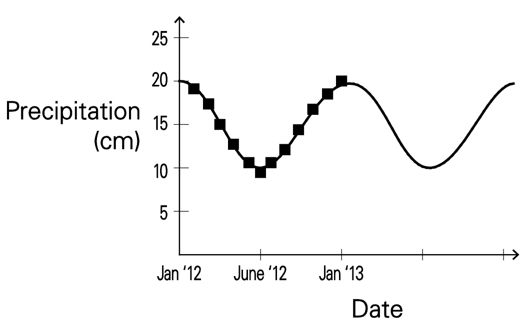
Time (hour:minutes)	**Height** (metres)
9:06 a.m.	2.16
2:59 p.m.	12.80
9:24 p.m.	2.01
3:20 a.m.	12.62
9:42 a.m.	2.31
3:37 p.m.	12.6
10:00 p.m.	2.27
3:58 a.m.	12.55

She recorded the data in her calculator as below.



Use a curve of best fit to estimate the water height after 15 hours of observation. Can this model be used to predict the tide height over the next two days? The next week? The next month? Explain.

- A meteorologist recorded the average precipitation in a location for 2012 and created a sinusoidal regression for the data.



SinReg
$y = a*\sin(bx+c)+d$
$a = 4.990586412$
$b = 0.5036758921$
$c = 1.582560398$
$d = 14.90530136$

(a) Use the graph to predict the amount of precipitation in August 2023.

(b) Use the equation to predict the amount of precipitation in March 2057.

(c) Consider what some problems might be with the use of a sinusoidal regression to predict future precipitation.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Have students create sinusoidal data sets by rolling and measuring the height of a point on the circumference of cylindrical objects with different diameters, as well as the distance that the cylinder has been rolled along a straight line.

  Students could complete several charts such as the one below and plot the data they obtain. Recommend before students begin this exercise that

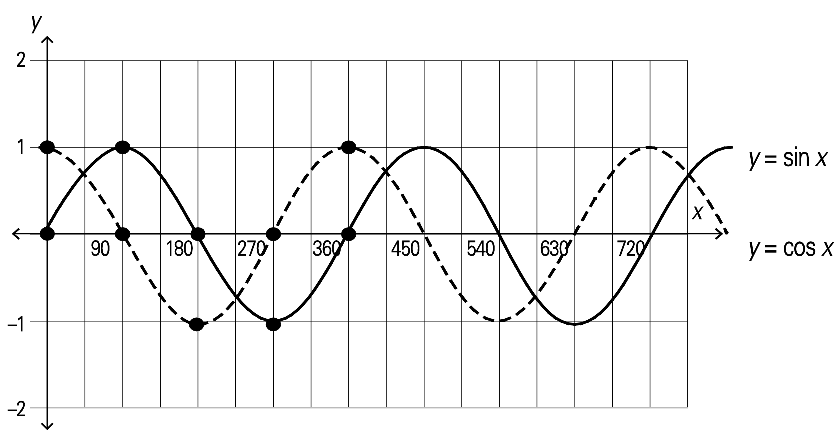
  - they record points that would represent the maximum and minimum heights of the point as well as at least two other points for each full rotation of the cylinder
  - they complete a minimum of two full rotations of the cylinder

Diameter of cylinder: 9 cm									
Distance (cm) rolled along straight line.	0	7	14	21	28	35	42	49	56
Height (cm) of point on circumference of circle.	0	4.5	9	4.5	0	4.5	9	4.5	0

  You can use this activity to introduce sinusoidal graphs as well as the terms **period**, **midline**, and **amplitude**. It is important that these terms be linked to the specific characteristics of the cylindrical container.

- Create a model of the coordinate grid on the classroom floor. Use a metre stick, which can be moved physically, to represent the radius of the circle. Ask students to record and graph the distance, in metres, from the tip of the metre stick to the *x*-axis over 360°. These values will produce the sine curve. Ask students to repeat the process measuring the distance, in metres, to the *y*-axis over 360°. These values produce the cosine curve.
- The graphs of the functions, $y = sin\,x$ and $y = cos\,x$ are shown below.

  Ask students to compare and contrast these graphs.



Students should notice that

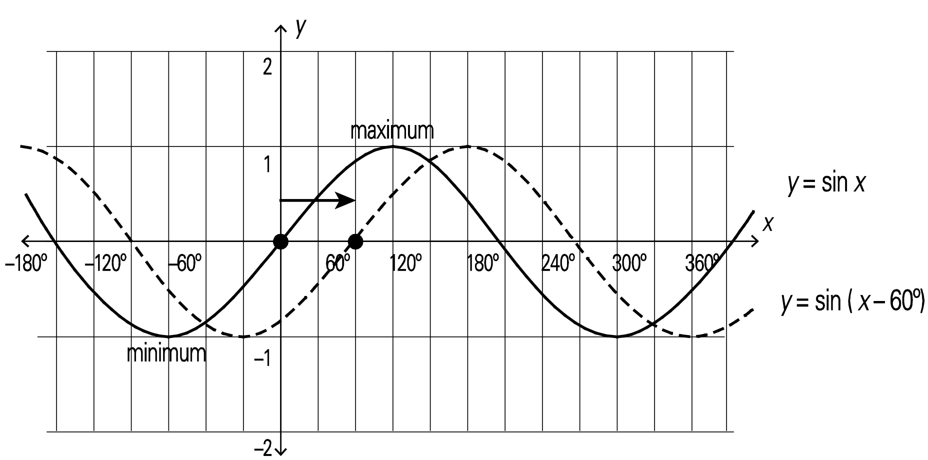
- they are periodic, continuous, have a domain, $\{x|x \in R\}$, and a range, $\{y|-1 \leq y \leq 1, y \in R\}$ (They both have a maximum of 1, and a minimum of –1, an amplitude of 1, and a period of 360° or 2π radians.)
- the equation $y = sin\, x$ has a *y*-intercept of (0, 0) and the equation $y = cos\, x$ has a *y*-intercept of (0, 1)
- for cosine, one complete wave can be seen from the maximum point (0°, 1) to the next maximum point at (360°, 1)
- for sine, one complete wave can be seen from an *x*-intercept at (0°, 0) to the *x*-intercept at (360°, 0).

It is important to ask students if there are other points that can be used to show a complete wave for either the sine or cosine function. *Students should conclude that both curves have a period of 360° (or 2π radians), amplitude of 1, and a midline defined by the equation y = 0.*

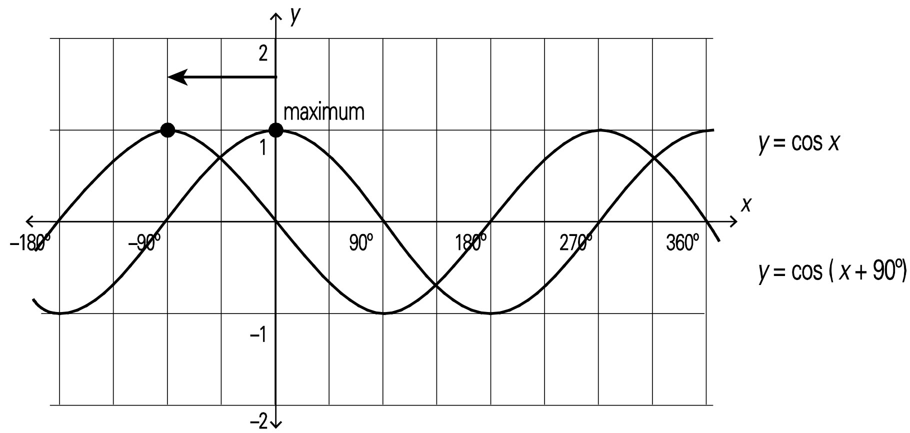
Using the graphs as a visual aid, students should observe the horizontal shift between the two functions. They should notice that the graph of $y = cos\, \theta$ is related to the graph of $y = sin\, \theta$ by a shift of 90°, or $\frac{\pi}{2}$ radians to the left.

- To introduce radian measure for angles:
  - Ask students to think about the different ways they can measure things. (Length can be measured in centimetres or in yards, temperature can be measured in degrees Celsius or degrees Fahrenheit.)
  - Ask students to think about the different ways to measure angles. (They may refer to degree measure or use the concept of turns.)
  - Introduce students to radian measure as an alternative way to express the size of an angle. (This will be their first time being exposed to the concept of radian measure.)
    - Ask students to visualize a circle where the radius equals one unit.
    - Teachers can use the following questions to help students connect the concepts of degrees and radians:
- What is the circumference of a circle?
- What is a complete revolution of a circle?
- Why must the two equations be equal to each other?
- Use interactive software to demonstrate the graphical impact of changes in the coefficients (*a*, *b*, *c*, and *d*) from the equation $y = a\, sin[b(x - c)] + d$.
  - Using graphing technology, students should examine the effects of manipulating the value of *a* by comparing the sinusoidal functions $y = sin\, x$ and $y = a\, sin\, x$, where *a* > 0. For example, as they compare the graphs of $y = 2\, sin\, x$, $y = 5\, sin\, x$, and $y = 0.5\, sin\, x$ to the graph of $y = sin\, x$, use the following questions to promote student discussion:
    - What happens to the amplitude if *a* > 0?
    - Is the shape of the graph affected by the parameter *a*?
    - How is the range affected by the parameter *a*?
    - Will the value of *a* affect the cosine graph in the same way that it affects the sine graph? Why or why not?

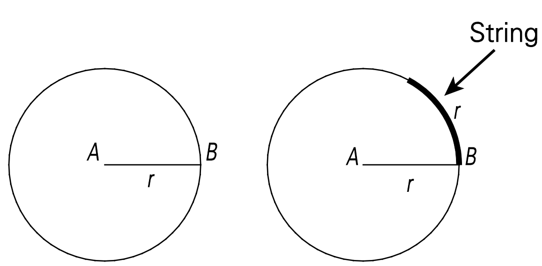
- Similarly, students should examine the value of *d* by comparing the sinusoidal function $y = sin\, x$ and $y = sin(x) + d$. Ask them to compare the graphs of $y = sin(x) + 2, y = sin(x) - 3$ to the graph of $y = sin\, x$ and answer the following questions:
  - > How does each graph change when compared to $y = sin\, x$?
  - > How is the value of *d* related to the equation of the midline?
  - > Is the shape of the graph or the location of the graph affected by the parameter *d*?
  - > Is the period affected by changing the value of *d*?
  - > Will the value of *d* affect the cosine graph in the same way that it affects the sine graph? Why or why not?
- Students should manipulate parameter *b* by comparing the sinusoidal functions $y = sin\, x$ and $y = sin\, b\, x$, where *b* > 0. They could describe how the graph of $y = sin\, 2\, x$ and $y = sin\, 0.5x$ compares to the graph of $y = sin\, x$. Ask students what effect varying *b* has on the period. They should notice the period of $y = sin\, 2\, x$ is 180° while the period of $y = sin\, 0.5x$ is 720°. Students may need guidance to make the connection that the period of a sinusoidal function is $\frac{360°}{b}$ or $\frac{2\pi}{b}$. Provide students with an opportunity to graph the cosine function in a similar way so they can conclude that parameter *b* affects both functions in the same way.
- When manipulating the value of *c*, students should identify key points located on the sine and cosine graph. They are aware that the graph of the cosine and sine function have different *y*-intercepts. In order to determine the horizontal shift (or phase shift), they have to observe if the key point has shifted left or right. Ask students to first compare the functions $y = sin\, x$ to $y = sin(x - c)$. They could describe how the graph of $y = sin(x - 60°)$ and $y = sin(x + 30°)$ compares to the graph of $y = sin\, x$. One of the key points of $y = sin\, x$ is the point (0, 0). This point intersects the midline at *x* = 0, going from a minimum to a maximum. The graph of $y = sin(x - 60°)$, for example, intersects the midline at *x* = 60° resulting in a horizontal shift of 60° to the right.



- When determining the horizontal shift for the cosine function, one of the key points students can consider is the point (0, 1), which is a maximum point on the graph. Ask students to compare the functions $y = cos\, x$ to $y = cos(x - c)$. They could describe how the graph of $y = cos(x + 90°)$ and $y = cos(x - 30°)$ compares to the graph of $y = cos\, x$. The graph of $y = cos(x + 90°)$, for example, has a maximum point at (–90°, 1), which results from a horizontal shift of 90° to the left.



- In your classroom, post a labelled sine and cosine graph with their corresponding equations.
- Students sometimes do not understand the difference parentheses make in the meaning of horizontal and vertical translations. When comparing $y = cos(x + 2)$ to $y = cos\, x + 2$, they should observe that $y = cos(x + 2)$ results in a horizontal translation while $y = cos\, x + 2$ results in a vertical translation. It may be beneficial if students write $y = cos\, x + 2$ as $y = cos(x) + 2$. This would reinforce their understanding that $y = cos(x) + 2$ does not result in a horizontal translation (i.e., $y = cos(x + 0) + 2$ but rather a vertical translation of two units upward.
- Students should match a sinusoidal graph with its corresponding equation. Using the equations given, they will work through the process of elimination until they reach one that satisfies the given graph. Encourage students to explain their reasoning as to why they are eliminating choices as they work through the solution.
- Students should investigate environmental data that is periodic in nature and present a scatterplot and sinusoidal graph of best fit for the data. They should use the data to interpolate and extrapolate during the presentation. Students should extend their findings to make other generalizations about that environment.
- The following activity could be used as an alternative method for developing the concept of a radian. Present the class with a visual of a large circle. This could be one within the classroom or a circle on the gymnasium floor. Use a piece of rope (or string) to represent the radius (*r*) of the circle. A radius is a straight line segment and the arc length is curved because it is part of the circumference, but they have the same length. Point out to students that one radian is the angle made by taking the radius and wrapping it along the edge of the circle. Ask them to predict how many pieces of rope, of this length, it would take to represent the circumference of the circle.



Students should understand that the circumference is a little more than six radius lengths (approximately 6.25 pieces of rope [of length *r*] are needed for one complete circle). They can, therefore, estimate that 1 radian ≈ 60°. Discuss with students if the size of the radius of a circle has an effect on the size of one radian.

- Ask students to provide their opinion on the advantages and disadvantages of measuring in degrees versus measuring in radians.

Suggested Models and Manipulatives

- compass
- pipe cleaners
- protractor
- ruler
- unit circle template

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- amplitude
- central angle
- frequency
- midline
- period
- periodic function
- radian
- sinusoidal function
- sinusoidal regression

Resources

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Sections 8.1 and 8.2, pp. 514–525
  - Section 8.3, pp. 527–542
  - Section 8.4, pp. 546–561
  - Section 8.5, pp. 563–577

Mathematics Research Project
10–15 hours

GCO: Students will be expected to develop an appreciation of the role of mathematics in society.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO MRP01 Students will be expected to research and give a presentation on a topic that involves the application of mathematics. [C, CN, ME, PS, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

MRP01.01 Collect primary or secondary data (statistical or informational) related to the topic.
MRP01.02 Assess the accuracy, reliability, and relevance of the primary or secondary data.
MRP01.03 Make a statement and justify the statement based on your data.
MRP01.04 Identify controversial issues, if any, and present multiple sides of the issues with supporting data.
MRP01.05 Organize and present the research project, with or without technology.

Scope and Sequence

Mathematics 11	**Mathematics 12**
S01 Students will be expected to demonstrate an understanding of normal distribution, including standard deviation and *z*-scores. **S02** Students will be expected to interpret statistical data, using confidence intervals, confidence levels, and margin of error. **S03** Students will be expected to critically analyze society's use of inferential statistics.	**R01** Students will be expected to research and give a presentation on a topic that involves the application of mathematics.

Background

This research project is intended to be completed over time and not be done in isolation or during a single block of time. Teachers might wish to spread the project over one term, with some class periods being designated to complete the project introduction while other periods are checkpoints, each with a particular expectation. Additionally, class time could be allowed for finalization, preparing, and presenting.

A research project can be a very important part of mathematics education. Besides the greatly increased learning intensity that comes from personal involvement with a project, and the chance to show universities, colleges, and potential employers the ability to initiate and carry out a complex task, it gives students an introduction to mathematics as it is—a living and developing intellectual discipline where progress is achieved by the interplay of individual creativity and collective knowledge.

A major research project must successfully pass through several stages. Over the next few pages, these stages are highlighted, together with some ideas that students may find useful when working through such a project.

Creating an Action Plan

The following is an outline for an action plan, with a list of stages, a suggested time, and space for students to include a probable time to complete each stage.

		Example #1	**Example #2**	**Example #3**	**Example #4**
Stage 1	Selecting the Research Topic	Global Warming	Tim Hortons / Canadian Tire / Winners (or some other store) locations	Movement of people from Nova Scotia to Western Canada.	Post-secondary training and the job market for working in the (health) field.
Stage 2	Creating the Research Question or Statement	How does it appear that average temperatures and/or rainfall are changing in Nova Scotia?	How are locations of XXXX stores linked to population density and transportation routes?	How is the number of people leaving Nova Scotia to find work elsewhere changing?	What program of study will best prepare someone for finding a job in the (health care) profession?
Stage 3	Carrying out the Research	Collect temperature and rainfall data from the Nova Scotia database.	Collect data about the location of XXXX stores in Nova Scotia.	Collect data about people leaving Nova Scotia to work elsewhere (number, education, ages, amount of time, etc.)	Collect data about several available jobs and the training required for those jobs in the (health care) profession.
Stage 4	Analyzing the Data	Try plotting data and using various regression models to analyze data.	Consider plotting population density and transportation routes to draw conclusions and predict where a new location might be opened or a current location shut down.	Plot the data, calculate measures of central tendency, measures of dispersion, etc.	Plot cost of training (time and money) against job expectation (salary or security) to see any patterns.
Stage 5	The Final Product and Presentation				

		Example #1	Example #2	Example #3	Example #4
Stage 6	Peer Critiquing of Research Projects				

Completing this action plan, with deadlines for each stage, will help students organize their time and provide goals and deadlines that they can manage. Students can adjust the time that they will spend on each stage to match the scope of their projects. For example, a project based on **primary data** (data that they collect) will usually require more time than a project based on **secondary data** (data that other people have collected and published). A student will also need to consider the student's personal situation — the issues that each student deals with that may interfere with the completion of the project. Examples of these issues may include

- a part-time job
- after-school sports and activities
- regular homework
- assignments for other courses
- tests in other courses
- time they spend with friends
- family commitments
- access to research sources and technology

STAGE 1: SELECTING THE RESEARCH TOPIC

To decide what to research, students can start by thinking about a subject and then consider specific topics. Some examples of subjects and topics may be as follows:

Subject	Topic
Entertainment	▪ effects of new electronic devices ▪ file sharing
Health care	▪ doctor and/or nurse shortages ▪ funding
Post-secondary education	▪ entry requirements ▪ graduate success
History of Western and Northern Canada	▪ relations among First Nations ▪ immigration

It is important to take the time to consider several topics carefully before selecting a topic for a research project. The following questions will help students determine if a topic that is being considered is suitable. (Note that much raw data has been collected and is available at statcan.gc.ca.)

Does the topic interest the student?
Students will be more successful if they choose a topic that interests them. They will be more motivated to do research, and they will be more attentive while doing the research. As well, they will care more about the conclusions they make.

Is the topic practical to research?
If students decide to use first-hand data, can the data be generated in the time available, with the resources available? If students decide to use second-hand data, are there multiple sources of data? Are the sources reliable, and can they be accessed in a timely manner?

Is there an important issue related to the topic?
Students should think about the issues related to the topic chosen. If there are many viewpoints on an issue, they may be able to gather data that support some viewpoints but not others. The data they collect from all viewpoints should enable them to come to a reasoned conclusion.

Will the audience appreciate the presentation?
The topic should be interesting to the intended audience. Students should avoid topics that may offend some members of their audience.

Stage 2: Creating the Research Question or Statement

A well-written research question or statement clarifies exactly what the project is designed to do. It should have the following characteristics:

- The research topic is easily identifiable.
- The purpose of the research is clear.
- The question or statement is focused. The people who are listening to or reading the question or statement will know what the student is going to be researching.

A good question or statement requires thought and planning. Below are three examples of initial questions or statements and how they were improved.

Unacceptable Question or Statement	**Why?**	**Acceptable Question or Statement**
Is mathematics used in computer technology?	Too general.	What role has mathematics played in the development of computer animation?
Water is a shared resource.	Too general.	Homes, farms, ranches, and businesses east of the Rockies all use runoff water. When there is a shortage, that water must be shared.
Do driver education programs help teenagers parallel park?	Too specific, unless the student is generating data.	Do driver education programs reduce the incidence of parking accidents?

The following checklist can be used to determine if the research question or statement is effective.

- Does the question or statement clearly identify the main objective of the research? After the question or statement is read to someone, can they tell what the student will be researching?
- Is the student confident that the question or statement will lead to sufficient data to reach a conclusion?
- Is the question or statement interesting? Does it make the student want to learn more?
- Is the topic that the student chose purely factual, or is that student likely to encounter an issue with different points of view?

STAGE 3: CARRYING OUT THE RESEARCH

As students continue with their projects, they will need to conduct research and collect data (secondary or primary). The strategies that follow will help them in their data collection.

There are two types of data that students will need to consider—primary and secondary. Primary data are data that students collect themselves using surveys, interviews, and direct observations. Secondary data are data that students obtain through other sources, such as online publications, journals, magazines, and newspapers.

Both primary and secondary data have their advantages and disadvantages. Primary data provide specific information about the research question or statement, but may take time to collect and process. Secondary data are usually easier to obtain and can be analyzed in less time. However, because the data was originally gathered for other purposes, students may need to sift through it to find exactly what they are looking for.

The type of data chosen can depend on many factors, including the research question, the skills of students, and available time and resources. Based on these and other factors, students may choose to use primary data, secondary data, or both.

When collecting primary data, students must ensure the following:

- For surveys, the sample size must be reasonably large, and the random sampling technique must be well designed.
- For surveys and interviews, the questionnaires must be designed to avoid bias.
- For experiments and studies, the data must be free from measurement bias.
- The data must be compiled accurately.

When collecting secondary data, students should explore a variety of resources, such as

- textbooks and other reference books
- scientific and historical journals and other expert publications
- the Internet
- library databases

After collecting the secondary data, students must ensure that the source of the data is reliable.

- If the data is from a report, determine what the author's credentials are, how recent the data is, and whether other researchers have cited the same data.
- Be aware that data collection is often funded by an organization with an interest in the outcome or with an agenda that it is trying to further. Knowing which organization has funded the data collection may help students decide how reliable the data is, or what type of bias may have influenced the collection or presentation of the data.
- If the data is from the Internet, check it against the following criteria:
  - Authority: the credentials of the author should be provided
  - Accuracy: the domain of the web address may help students determine the accuracy
  - Currency: the information is probably being accurately managed if pages on a site are updated regularly and links are valid

STAGE 4: ANALYZING THE DATA

Statistical tools can help a student analyze and interpret the data that is collected. Students need to think carefully about which statistical tools to use and when to use them, because other people will be scrutinizing the data. A summary of relevant tools follows.

Measures of central tendency will give information about which values are representative of the entire set of data. Selecting which measure of central tendency (mean, median, or mode) to use depends on the distribution of the data. As the researcher, students must decide which measure most accurately describes the tendencies of the population. The following criteria should be considered when deciding upon which measure of central tendency best describes a set of data.

- Outliers affect the mean the most. If the data include outliers, students should use the median to avoid misrepresenting the data. If students choose to use the mean, the outliers should be removed before calculating the mean.
- If the distribution of the data are not symmetrical, but instead strongly skewed, the median may best represent the set of data.
- If the distribution of the data are roughly symmetrical, the mean and median will be close, so either may be appropriate to use.
- If the data are not numeric (e.g., colour), or if the frequency of the data is more important than the values, use the mode.

Measures of dispersion, such as the range and the standard deviation, will give students information about the distribution of data in a set. The range of a set of data changes considerably because of outliers. The disadvantage of using the range is that it does not show where most of the data in a set lies – it only shows the spread between the highest and the lowest values. The range is an informative tool that can be used to supplement other measures, such as standard deviation, but it is rarely used as the only measure of dispersion.

Standard deviation is the measure of dispersion that is most commonly used in statistical analysis when the mean is used to calculate central tendency. It measures the spread relative to the mean for most of the data in the set. Outliers can affect standard deviation significantly. Standard deviation is a very useful measure of dispersion for symmetrical distributions with no outliers. Standard deviation helps with comparing the spread of two sets of data that have approximately the same mean. For example, the set of data with the smaller standard deviation has a narrower spread of measurement around the mean and, therefore, has comparatively fewer high or low scores than a set of data with a higher standard deviation.

When working with several sets of data that approximate normal distributions, z-scores can be used to compare the data values. A z-score table enables students to find the area under a normal distribution curve with a mean of zero and a standard deviation of one.

When analyzing the results of a survey, students may need to interpret and explain the significance of some additional statistics. Most surveys and polls draw their conclusions from a sample of a larger group.

The margin of error and the confidence level indicate how well a sample represents a larger group. For example, a survey may have a margin of error of plus or minus 3% at a 95% level of confidence. This means that if the survey were conducted 100 times, the data would be within 3% points above or below the reported results in 95 of the 100 surveys.

The size of the sample that is used for a poll affects the margin of error. If students are collecting data, they must consider the size of the sample that is needed for a desired margin of error.

Identifying Controversial Issues

While working on a research project, students may uncover some issues on which people disagree. To decide on how to present an issue fairly, they should consider some questions to ask as the research proceeds.

What is the issue about?

Students should identify which type of controversy has been uncovered. Almost all controversy revolves around one or more of the following:

- Values: What is best?
- Information: What is the truth? What is a reasonable interpretation?
- Concepts: What does this mean? What are the implications?

What positions are being taken on the issue?

Students should determine what is being said and whether there is reasonable support for the claims being made. Some questions to ask are as follows:

- Would you like that done to you?
- Is the claim based on a value that is generally shared?
- Is there adequate information?
- Are the claims in the information accurate?
- Are those taking various positions on the issue all using the same term definitions?

What is being assumed?

Faulty assumptions reduce legitimacy. The student can ask,

- What are the assumptions behind an argument?
- Is the position based on prejudice or an attitude contrary to universally held human values, such as those set out in the United Nations', The Universal Declaration of Human Rights?
- Is the person who is presenting a position or an opinion an insider or an outsider?

What are the interests of those taking positions?

Students should try to determine the motivations of those taking positions on the issue. What are their reasons for taking their positions? The degree to which the parties involved are acting in self-interest could affect the legitimacy of their opinions.

Stage 5: The Final Product and Presentation

The final presentation should be more than just a factual written report of the information gathered from the research. To make the most of students' hard work, they should select a format for the final presentation that suits their strengths, as well as the topic. To make the presentation interesting, a format should be chosen that suits the student's style. Some examples are as follows:

- A report on an experiment or an investigation
- A short story, musical performance, or play
- A web page
- A slide show, multimedia presentation, or video
- A debate
- An advertising campaign or pamphlet
- A demonstration or the teaching of a lesson

Sometimes, it is also effective to give the audience an executive summary of the presentation. This is a one-page summary of the presentation that includes the research question and the conclusions that were made.

Before giving their presentations, students can use these questions to decide if their presentation will be effective.

- Did I define my topic well? What is the best way to define my topic?
- Is my presentation focused? Will my classmates find it focused?
- Did I organize my information effectively? Is it obvious that I am following a plan in my presentation?
- Am I satisfied with my presentation? What might make it more effective?
- What unanswered questions might my audience have?

Stage 6: Peer Critiquing of Research Projects

After students have completed their research for the question or statement being studied, and their report and presentation have been delivered, it is time to see and hear the research projects developed by other students. However, rather than being passive observers, students should have an important role—to provide feedback to their peers about their projects and presentations. Critiquing a project does not involve commenting on what might have been or should have been. It involves reacting to what is seen and heard. In particular, students should pay attention to

- strengths and weaknesses of the presentation
- problems or concerns with the presentation

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Your friend is unclear what the term **bias** means. Develop an example to help explain the term.
- A friend tells you that she has heard that there are lots of jobs for pharmacists. How could you find out if this is true? How could you determine if this is a job that might be right for you to consider?

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Create an organizer, such as the flow chart on the next page, to help organize the research project and carry out the plan.
  - Step 1: Develop the project plan.
    - > Write a statement describing what questions you wish to answer.
    - > Describe how you will collect data.
  - Step 2: Continue to develop the project plan.
    - > Describe how you will display the data.
    - > Describe how you will analyze the data.
    - > Describe how you will present your findings.
  - Step 3: Complete the project according to the plan.
    - > Display the data.
    - > Analyze the data.
    - > Draw a conclusion or make a prediction.
    - > Evaluate the research results.
  - Step 4: Present your findings.
- Demonstrate what you have learned from presentations of others by completing a questionnaire that focuses on the highlights of a presentation.
- Collect summaries from students who have presented their research projects. Construct a question from each presentation that could be asked on a test.
- Keep a journal entry describing what you have learned from each of the projects that have been presented in class.

Planning for Instruction

Consider the following sample instructional strategies when planning lessons.

- Approximately 10–15 hours of class time should be devoted to this research project. Teachers should allow time for
  - students to present the results of their research to other students and for the student audience to respond to each presentation (Presentation and response time of an average of 8 –12 minutes per student group should be planned. If students are working collaboratively on this project, it is expected that each would be responsible for gathering certain information and, thus, could be held responsible for the oral presentation that deals with that part of the project.)
  - initial discussion of various topics to be explored and an assignment of topics to groups
  - discussing the expectations and developing, with student input, assessment rubrics for the students' presentations and how students will be assessed during the unit
  - brainstorming, topic-webbing, developing action plans and timelines, and conferencing
  - peer evaluation
- Statistics Canada (http://statcan.gc.ca) provides an excellent source of data.
- It is critical to give students class time to complete some of the necessary steps, such as brainstorming, writing an outline, analyzing data, making decisions, and using technology. The time will be most productive if, at the end of each class work session, tasks are assigned to be completed prior to the next meeting.
- Students should brainstorm possible questions, ideas, and/or issues relating to their topic.
- Encourage students to create and submit an action plan or graphic organizer outlining their tasks and time frames.
- Ask students questions periodically to ensure that their work is progressing and to give them support. Observe their attitudes, contributions, time on task, and teamwork. Some possible questions are as follows:
  - Where are you in the research process?
  - What are the different roles of the individuals in your group?
  - How are you collecting data or using data already collected?
  - What are you doing next?
  - Are you experiencing any problems with the research information?
  - Are you having any technological difficulties?
  - What resources are you using?
  - How are you presenting your result(s)?
  - What questions do you have?
- Ask students about the presentation of the project. To make the presentation interesting, encourage students to choose a format that suits the student's style or the group's style. Will students use a multimedia presentation, such as a video or a slide show? Will it be in the form of a podcast? Will students create their own website or blog to present their project? Some students may prefer to present without the use of technology and use, for example, a poster board, a pamphlet, a debate, an advertising campaign, or a demonstration.

Students must distribute summaries of the topic they are presenting to the class. This means that students should summarize what they learned so that other students can read over the summary, see a couple of examples, and have a good foundation in the new topic.

Peer evaluation should occur during presentations and from each group member on the group effort. This could include a reflective journal entry on each student's experience within the group, as well as experience gained from seeing and hearing other groups.

- A rubric, such as the one shown below, should be used to assess the project, and students should be aware of the criteria before they start their project. Encourage them to participate in the development of the rubric and to work out the appropriate categories and criteria for specific tasks. This involvement will improve student motivation, interest, and performance in the project. Content, organization, sources, and layout are critical components used to evaluate projects. Illustrations, images, and graphics are also important features that should be included in assessment. Remind students that there are many different ways to deliver a project. Therefore, the rubric may have to be modified to fit the format of the presentation.

Rubric for Major Project Presentation **Instructor/Self/Peer Names:** ______________________

Criteria	Level 1	Level 2	Level 3	Level 4
Activates previous knowledge	Uses questions that are ineffective in activating previous knowledge.	Uses questions that lack clarity and are not sufficient to activate previous knowledge.	Uses questions or problems that are clearly activating previous knowledge.	Uses questions or problems that are clear, precise, and effective in activating previous knowledge.
Explanation of concepts and problem solving	Provides incomplete or inaccurate explanations or justifications that lack clarity or logical thought, using minimal words, pictures, and symbols.	Provides partial explanations or justifications that show some clarity and logical thought, using simple words, pictures, and symbols.	Provides complete, clear, and logical explanations/ justifications using appropriate words, pictures, and symbols.	Provides thorough, clear, and insightful explanations/ justifications using a range of words, pictures, and symbols.
Organization	Presented content that was unfocused, poorly organized, showed little thought or effort, and lacked supporting evidence.	Presented content that failed to maintain a consistent focus, showed minimal organization and effort, and lacked an adequate amount of supporting evidence.	Presented most of the content with a logical progression of ideas and supporting evidence.	Presented content clearly and concisely with a logical progression of ideas and effective supporting evidence.

Criteria	Level 1	Level 2	Level 3	Level 4
Use of mathematical representations	Uses representations that exhibit minimal clarity and accuracy and are ineffective in communicating.	Uses representations that lack clarity and accuracy, though not sufficient to impede communication.	Uses representations that are sufficiently clear and accurate to communicate.	Uses representations that are clear, precise, and effective in communicating.
Connections	Makes no connections to students' social, cultural, and school experiences.	Makes limited connections to students' social, cultural, and school experiences.	Makes connections to students' social, cultural, and school experiences.	Makes many connections to students' social, cultural, and school experiences.
Assessment for/of learning	Uses questions that are ineffective in assessment for/ of learning.	Uses questions that lack clarity and are not sufficient to aid assessment for/of learning.	Uses questions or problems that are clearly assessment for/ of learning.	Uses questions or problems that are clear, precise, and effective in assessment for/ of learning.
Other comments:				

- It may be helpful for students to discuss categories and criteria that may be included in a rubric. Suggestions are shown below.

	Excellent	**Good**	**Competent**	**Needs Improvement**
Reasoning	Justifies all mathematical statements in an efficient and accurate manner, and draws valid conclusions.	Justifies most mathematical statements accurately, and draws valid conclusions.	Justifies some of the mathematical statements accurately, and draws valid conclusions.	Does not justify mathematical statements accurately, and does not draw valid conclusions.
Connections	Discusses, in depth, how mathematical concepts interconnect and build on each other.	Discusses how mathematics concepts interconnect and build on each other.	Discusses superficially how mathematics concepts interconnect and build on each other.	Does not discuss the inter-connection of mathematical concepts.

Top level	▪ Contains a complete report with clear, coherent, unambiguous, and elegant explanations. ▪ Includes clear and simple diagrams, charts, graphs, etc. ▪ Communicates effectively to an identified audience. ▪ Shows understanding of the financial mathematics topic, processes, and thinking that indicate careful and thorough consideration. ▪ Identifies all the important elements of the topic. ▪ Includes examples and counter-examples where appropriate. ▪ Gives strong supporting arguments.
Second level	▪ Contains good solid report with some of the characteristics above. ▪ Explains less elegantly and less completely than desired. ▪ Does not go beyond the requirements of the project (or topic).
Third level	▪ Contains a complete report, but the explanation is vague in places. ▪ Presents arguments, but they are incomplete at times. ▪ Includes diagrams, but some are inappropriate, unclear, or misplaced. ▪ Indicates some understanding of mathematical ideas, but not expressed clearly enough.
Fourth level	▪ Omits significant parts. ▪ Has major errors. ▪ Uses inappropriate strategies.

- See the books in Resources/Notes for more examples of rubrics for evaluating projects and open-ended activities.

Suggested Models and Manipulatives

- *z*-score charts

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- confidence intervals
- margin of error
- mean
- measures of central tendency
- measures of dispersion
- median
- mode
- normal distribution
- outliers
- population
- primary data
- sample
- secondary data
- standard deviation
- symmetrical distribution
- *z*-score

Resources

Internet

- "Statistics Canada," *Government of Canada* (Government of Canada 2015): http:statcan.gc.ca.

Print

- *Foundations of Mathematics 12* (Canavan-McGrath et al. 2012)
  - Chapter 1: Financial Mathematics: Investing Money
    - Project Connection: Creating an Action Plan, pp. 74–75
  - Chapter 2: Financial Mathematics: Borrowing Money
    - Project Connection: Selecting Your Research Topic, pp. 138–139
  - Chapter 3: Set Theory and Logic
    - Project Connection: Creating Your Research Question or Statement, pp. 222–223
  - Chapter 4: Counting Methods
    - Project Connection: Carrying Out Your Research, pp. 296–297
  - Chapter 5: Probability
    - Project Connection: Analyzing Your Data, pp. 370–371
  - Chapter 6: Polynomial Functions
    - Project Connection: Identifying Controversial Issues, pp. 430–431
  - Chapter 7: Exponential and Logarithmic Functions
    - Project Connection: The Final Product and Presentation, pp. 508–509
  - Chapter 8: Sinusoidal Functions
    - Project Connection: Peer Critiquing of Research Projects, pp. 584–585
- *Financial Mathematics* (Etienne and Kalwarowsky, McGraw-Hill 2013)
  - p. 88 (Part of Mathematics 10 supplement)
  - Teacher's Resource, including reproducibles
    - BLM FM–16 Project Plan
    - BLM FM–17 Project Rubric
    - BLM FM–18 Presentation Planner
    - BLM FM–19 Presentation Peer Assessment
    - BLM FM–20 Presentation Self-Assessment
    - BLM FM–21 BLM Answers

Supplementary Print

- Assessment Alternatives in Mathematics (Stenmark 1989)
- Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions (Stenmark 1991)
- *How to Evaluate Progress in Problem Solving* (Randall, Randall, Lester, and O'Daffer 1987)

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